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General composite implicit iteration process for a finite family of asymptotically pseudo-contractive mappings

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Abstract

In this paper, a modified general composite implicit iteration process is used to study the convergence of a finite family of asymptotically nonexpansive mappings. Weak and strong convergence theorems have been proved, in the framework of a Banach space.

MSC: 47H09; 47H10

Keywords: implicit iteration process; asymptotically pseudo-contractive mapping; common fixed points

1 Introduction

Let K be a nonempty subset of a real Banach space E and let $J: E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|; \|x\| = \|f\|\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

It is well known that if E^* is strictly convex, then J is single valued.

In the sequel, we shall denote the single valued normalized duality mapping by j.

Let K be a nonempty subset of E. A mapping $T: K \to K$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that for all $x, y \in K$, we have $\|Tx - Ty\| \le L\|x - y\|$. It is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in K$. T is called *asymptotically nonexpansive* [1] if there exists a sequence $\{h_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that $\|T^n x - T^n y\| \le h_n\|x - y\|$, for all integers $n \ge 1$ and all $x, y \in K$.

A mapping T is said to be *pseudo-contractive* [2, 3], if there exists $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2$, for all $x, y \in K$. T is called *strongly pseudo-contractive*, if there exists a constant $\beta \in (0, 1)$, $j(x - y) \in J(x - y)$ such that $\langle Tx - Ty, j(x - y)\rangle \le \beta\|x - y\|^2$, for all $x, y \in K$. It is said to be asymptotically pseudo-contractive [4] if there exists a sequence $\{h_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \le h_n\|x - y\|^2, \quad \forall x, y \in K, \forall n \ge 1. \tag{1.1}$$

It follows from Kato [5] that

$$\|x - y\| \le \|x - y + r[(h_n I - T^n)x - (h_n I - T^n)y]\|, \quad \forall x, y \in K, \forall n \ge 1, r > 0. \tag{1.2}$$

We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in K : x = Tx\}$.

It follows from the definition that if T is asymptotically nonexpansive, then for all $j(x - y) \in J(x - y)$,

$$\langle T^n x - T^n y, j(x-y)\rangle = \|x-y\| \|T^n x - T^n y\| \leq h_n \|x-y\|^2.$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudo-contractive.

It can be observed from the definition that an asymptotically nonexpansive mapping is uniformly L-Lipschitzian, where $L = \sup_{n\geq 1}\{h_n\}$.

Now consider an example of non-Lipschitzian mapping due to Rhoades [6]. Define a mapping $T: [0,1] \to [0,1]$ by the formula $Tx = \{1 - x^{\frac{2}{3}}\}^{\frac{3}{2}}$, for $x \in [0,1]$. Schu [4] used this example to show that the class of asymptotically nonexpansive mappings is a subclass of the class of pseudo-contractive mappings. Since T is not Lipschitzian, it cannot be asymptotically nonexpansive. Also T^2 is the identity mapping and T is monotonically decreasing, and it follows that

$$|x-y| \left|T^n x - T^n y\right| = |x-y|^2 \quad \text{for all } n = 2m, m \in \mathbb{N}$$

and

$$\begin{aligned}(x-y)\left(T^n x - T^n y\right) &= (x-y)(Tx - Ty) \\ &\leq 0 \\ &\leq |x-y|^2 \quad \text{for all } n = 2m-1, m \in \mathbb{N}.\end{aligned}$$

Hence T is asymptotically pseudo-contractive mapping with constant sequence $\{1\}$.

The iterative approximation problems for a nonexpansive mapping, an asymptotically nonexpansive mapping, and an asymptotically pseudo-contractive mapping were studied extensively by Browder [7], Kirk [8], Goebel and Kirk [1], Schu [4], Xu [9, 10], Liu [11] in the setting of Hilbert space or uniformly convex Banach space.

In 2001, Xu and Ori [12] introduced the following implicit iteration process for a finite family of nonexpansive self-mappings in Hilbert space:

$$\begin{cases} x_0 \in K \quad \text{arbitrary}, \\ x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ be a sequence in $(0,1)$ and $T_n = T_{n \bmod N}$. They proved in [12] that the sequence $\{x_n\}$ converges weakly to a common fixed point of T_n, $n = 1, 2, \ldots, N$.

Later on Osilike and Akuchu [13], and Chen *et al.* [14] extended the iteration process (1.3) to a finite family of asymptotically pseudo-contractive mapping and a finite family of continuous pseudo-contractive self-mapping, respectively. Zhou and Chang [15] studied the convergence of a modified implicit iteration process to the common fixed point of a finite family of asymptotically nonexpansive mappings. Then Su and Li [16], and Su and Qin [17] introduced the composite implicit iteration process and the general iteration algorithm, respectively, which properly include the implicit iteration process. Recently, Beg and Thakur [18] introduced a modified general composite implicit iteration process for a finite family of random asymptotically nonexpansive mapping and proved strong convergence theorems.

The purpose of this paper is to consider a finite family $\{T_i\}_{i=1}^N$ of asymptotically pseudo-contractive mappings and to establish convergence results in Banach spaces based on the modified general composite implicit iteration:

For $x_0 \in K$, construct a sequence $\{x_n\}$ by

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T_{i(n)}^{k(n)} y_n, \\ y_n &= r_n x_n + s_n x_{n-1} + t_n T_{i(n)}^{k(n)} x_n + w_n T_{i(n)}^{k(n)} x_{n-1} \end{aligned} \tag{1.4}$$

for each $n \ge 1$, which can be written as $n = (k(n)-1)N + i(n)$, where $i(n) = 1, 2, \ldots, N$ and $k(n) \ge 1$ is a positive integer, with $k(n) \to \infty$ as $n \to \infty$. The sequences $\{\alpha_n\}$, $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{w_n\}$ are in $(0,1)$ such that $r_n + s_n + t_n + w_n = 1$ for all $n \ge 1$.

2 Preliminaries

In what follows we shall use the following results.

Lemma 2.1 [19] *Let E be a Banach space, K be a nonempty closed convex subset of E, and $T: K \to K$ be a continuous and strong pseudo-contraction. Then T has a unique fixed point.*

Lemma 2.2 [20] *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \le (1+b_n)a_n + c_n \quad \text{for all } n \ge n_0,$$

where n_0 is some nonnegative integer, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$.

Then

(i) $\lim_{n\to\infty} a_n$ *exists;*

(ii) *if, in addition, there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$ such that $a_{n_i} \to 0$, then $a_n \to 0$ as $n \to \infty$.*

Lemma 2.3 [21] *Let E be a uniformly convex Banach space and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a,b]$ is a real sequence and $\{x_n\}$, $\{y_n\}$ are two sequences in E. Then the conditions*

$$\limsup_{n\to\infty} \|x_n\| \le d, \qquad \limsup_{n\to\infty} \|y_n\| \le d \quad \text{and} \quad \lim_{n\to\infty} \big\| t_n x_n + (1-t_n) y_n \big\| = d$$

imply that $\lim_{n\to\infty} \|x_n - y_n\| = 0$, where $d \ge 0$ is some constant.

Lemma 2.4 [22] *Let E be a reflexive smooth Banach space with a weakly sequential continuous duality mapping J. Let K be a nonempty bounded and closed convex subset of E and $T: K \to K$ be a uniformly L-Lipschitzian and asymptotical pseudo-contraction. Then $I - T$ is demiclosed at zero, where I is the identical mapping.*

We shall denote weak convergence by $\rightharpoonup$ and strong convergence by $\to$.

A Banach space E is said to satisfy Opial's condition if for any sequence $\{x_n\} \in E$, $x_n \rightharpoonup x$ as $n \to \infty$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in E \text{ with } x \neq y.$$

We know that a Banach space with a sequentially continuous duality mapping satisfies Opial's condition (for details, see [23]).

3 The main results

Throughout this section, E is a uniformly convex Banach space, K a nonempty closed convex subset of E. $\mathbb{N}$ denotes the set of natural numbers and $I = \{1, 2, \dots, N\}$, the set of the first N natural numbers. T_i $(i \in I)$ are N uniformly Lipschitzian asymptotically pseudo-contractive self-mappings on K. Let $\mathcal{F} = \bigcap_{i\in I} F(T_i) \neq \emptyset$.

Since T_i $(i \in I)$ are uniformly Lipschitzian, there exist constants $L_i > 0$ such that $\|T_i^n x - T_i^n y\| \le L_i\|x - y\|$, for all $x, y \in K$, $n \in \mathbb{N}$ and $i \in I$. Also, since T_i $(i \in I)$ are asymptotically pseudo-contractive; therefore there exist sequences $\{h_n^{(i)}\}$ such that $\langle T_i^n x - T_i^n y, j(x - y)\rangle \le h_n^{(i)}\|x - y\|^2$ for all $x, y \in K$ and $i \in I$.

Take $L = \max_{i\in I}(L_i)$ and $h_n = \max_{i\in I}(h_n^{(i)})$.

Before presenting the main results, we first show that the proposed iteration (1.4) is well defined.

Let T be uniformly Lipschitzian asymptotically pseudo-contractive mapping. For every fixed $u \in K$ and $\alpha \in (\frac{L+L^2}{L+L^2+1}, 1)$, define a mapping $S_n \colon K \to K$ by the formula

$$\begin{aligned} S_n x &= \alpha u + (1-\alpha)T^n a, \\ a &= rx + su + tT^n x + wT^n u \text{ for all } x \in K, \end{aligned} \tag{3.1}$$

where $\alpha, r, s, t, w \in (0,1)$, with $(1-\alpha)(L + L^2) < 1$.

Then, for all $x, y \in K$, $j(x - y) \in J(x - y)$, we have

$$\begin{aligned} S_n y &= \alpha u + (1-\alpha)T^n b, \\ b &= ry + su + tT^n y + wT^n u \text{ for all } x \in K. \end{aligned} \tag{3.2}$$

Now

$$\begin{aligned} \langle T^n a - T^n b, j(x-y)\rangle &= \|T^n a - T^n b\|\,\|x-y\| \\ &\le L\|a-b\|\,\|x-y\| \\ &= L\|r(x-y) + t(T^n x - T^n y)\|\,\|x-y\| \\ &\le L(r\|x-y\| + tL\|x-y\|)\|x-y\| \\ &= (Lr + tL^2)\|x-y\|^2 \\ &\le (L + L^2)\|x-y\|^2, \end{aligned}$$

so

$$\begin{aligned} \langle S_n x - S_n y, j(x-y)\rangle &= (1-\alpha)\langle T^n a - T^n b, j(x-y)\rangle \\ &\le (1-\alpha)(L + L^2)\|x-y\|^2. \end{aligned}$$

Since $(1-\alpha)(L+L^2) \in (0,1)$, S_n is strongly pseudo-contractive, which is also continuous, by Lemma 2.1, S_n has a unique fixed point $x^* \in K$, *i.e.*

$$\begin{aligned} &S_n x^* = \alpha u + (1-\alpha)T^n a, \\ &\quad a = rx^* + su + tT^n x^* + wT^n u \text{ for all } x \in K. \end{aligned} \tag{3.3}$$

Thus the implicit iteration (1.4) is defined in K for a finite family $\{T_i\}$ of uniformly Lipschitzian asymptotically pseudo-contractive self-mappings on K, provided $\alpha_n \in (\alpha, 1)$, where $\alpha = \frac{L+L^2}{L+L^2+1}$, for all $n \in \mathbb{N}$, $L = \max_{i\in I}(L_i)$.

Lemma 3.1 *Let E, K, and T_i ($i \in I$) be as defined above and let $\{x_n\}$ be the sequence defined by* (1.4), *where $\{\alpha_n\}$ is a sequence of real numbers such that $0 < \alpha < \alpha_n \le \beta < 1$ for $\alpha = \frac{L+L^2}{L+L^2+1}$ and β is some constant and satisfying the conditions $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$ and $\lim_{n\to\infty} \frac{h_n-1}{1-\alpha_n} = 0$. Let $b > 0$ be a real number such that $t_n + w_n \le b/L < 1$. Then*

(i) $\lim_{n\to\infty} \|x_n - p\|$ *exists, for all* $p \in \mathcal{F}$,

(ii) $\lim_{n\to\infty} d(x_n, \mathcal{F})$ *exists, where* $d(x_n, \mathcal{F}) = \inf_{p\in\mathcal{F}} \|x_n - p\|$,

(iii) $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$, $\forall l \in I$.

Proof Let $p \in \mathcal{F}$. Using (1.4), we have

$$\begin{aligned} \|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\ &\le \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle T_{i(n)}^{k(n)} y_n - T_{i(n)}^{k(n)} x_n, j(x_n - p)\rangle \\ &\quad + (1-\alpha_n)h_{k(n)}\|x_n - p\|^2 \\ &= \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n)L\|y_n - x_n\|\|x_n - p\| \\ &\quad + (1-\alpha_n)h_{k(n)}\|x_n - p\|^2. \end{aligned} \tag{3.4}$$

Using (1.4), we obtain

$$\begin{aligned} \|y_n - x_n\| &= \left\|s_n(x_{n-1} - x_n) + t_n\left(T_{i(n)}^{k(n)} x_n - x_n\right) + w_n\left(T_{i(n)}^{k(n)} x_{n-1} - x_n\right)\right\| \\ &\le s_n\|x_{n-1} - p\| + s_n\|x_n - p\| + t_n L\|x_n - p\| + t_n\|x_n - p\| \\ &\quad + w_n L\|x_{n-1} - p\| + w_n\|x_n - p\|. \end{aligned} \tag{3.5}$$

Substituting (3.5) in (3.4), we get

$$\begin{aligned} \|x_n - p\|^2 &\le \left(\alpha_n + (1-\alpha_n)L(s_n + w_n L)\right)\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + (1-\alpha_n)\left[(s_n + t_n + w_n + t_n L)L + h_{k(n)}\right]\|x_n - p\|^2 \\ &\le \left(\alpha_n + (1-\alpha_n)(1+L)L\right)\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + (1-\alpha_n)\left[(1+L)L + h_{k(n)}\right]\|x_n - p\|^2 \\ &\le \left(\alpha_n + (1-\alpha_n)(1+L)L\right)\|x_{n-1} - p\|\|x_n - p\| \\ &\quad + \left[(1-\alpha_n)(1+L)L + (1-\alpha_n + \mu_{k(n)})\right]\|x_n - p\|^2, \end{aligned} \tag{3.6}$$

where $\mu_{k(n)} = h_{k(n)} - 1$ for all $n \ge 1$, by condition $\sum_{n=1}^{\infty}(h_{k(n)} - 1) < \infty$, we have $\sum_{n=1}^{\infty} \mu_{k(n)} < \infty$.

Therefore, we have

$$\begin{aligned}\|x_n - p\| &\le \frac{(\alpha_n + (1-\alpha_n)(1+L)L)}{\alpha_n - \mu_{k(n)} - (1-\alpha_n)(1+L)L}\|x_{n-1} - p\| \\ &\le \left[1 + \frac{\mu_{k(n)} + 2(1-\alpha_n)(1+L)L}{\alpha_n - \mu_{k(n)} - (1-\alpha_n)(1+L)L}\right]\|x_{n-1} - p\| \\ &\le \left[1 + \frac{\mu_{k(n)} + 2(1-\alpha_n)(1+L)L}{1 - (1-\alpha_n + \mu_{k(n)} + (1-\alpha_n)(1+L)L)}\right]\|x_{n-1} - p\|. \end{aligned} \tag{3.7}$$

Since $\lim_{n\to\infty}\frac{h_{k(n)}-1}{1-\alpha_n} = \lim_{n\to\infty}\frac{\mu_{k(n)}}{1-\alpha_n} = 0$, there exists a M such that $\frac{\mu_{k(n)}}{1-\alpha_n} < M$.

Now, we consider the second term on the right side of (3.7). We have

$$\bigl(1 - \alpha_n + \mu_{k(n)} + (1-\alpha_n)(1+L)L\bigr) \le (1-\alpha_n)\bigl[1 + M + (1+L)L\bigr].$$

By condition $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, we have $\lim_{n\to\infty}(1-\alpha_n) = 0$, then there exists a natural number N_1 such that if $n > N_1$, then

$$1 - \bigl(1 - \alpha_n + \mu_{k(n)} + (1-\alpha_n)(1+L)L\bigr) \ge \frac{1}{2}.$$

Therefore, it follows from (3.7) that

$$\begin{aligned}\|x_n - p\| &\le \bigl[1 + 2\bigl\{\mu_{k(n)} + 2(1-\alpha_n)(1+L)L\bigr\}\bigr]\|x_{n-1} - p\| \\ &= (1+\sigma_n)\|x_{n-1} - p\|, \end{aligned} \tag{3.8}$$

where $\sigma_n = 2\{\mu_{k(n)} + 2(1-\alpha_n)(1+L)L\}$.

Taking the infimum over $p \in \mathcal{F}$, we have

$$d(x_n, \mathcal{F}) \le (1+\sigma_n)d(x_{n-1}, \mathcal{F}). \tag{3.9}$$

Since $\sum_{n=1}^{\infty}\mu_{k(n)} < \infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n) < \infty$, we have

$$\sum_{n=1}^{\infty}\sigma_n < \infty.$$

Thus, by Lemma 2.2, $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exist.

Without loss of generality, we assume

$$\lim_{n\to\infty}\|x_n - p\| = d^1. \tag{3.10}$$

Set $\nu_{k(n)} = \frac{h_{k(n)}-1}{h_{k(n)}}$, and from (1.2), we have

$$\begin{aligned}\|x_n - p\| &\le \left\|x_n - p + \frac{1-\alpha_n}{2\alpha_n h_{k(n)}}\bigl[\bigl(h_{k(n)}I - T_{i(n)}^{k(n)}\bigr)x_n - \bigl(h_{k(n)}I - T_{i(n)}^{k(n)}\bigr)p\bigr]\right\| \\ &\le \left\|x_n - p + \frac{1-\alpha_n}{2\alpha_n}\bigl[\alpha_n\bigl(x_{n-1} - T_{i(n)}^{k(n)}x_n\bigr) + (1-\alpha_n)\bigl(T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n\bigr)\bigr]\right\|\end{aligned}$$

$$\begin{aligned}
&+\left(\frac{1-\alpha_n}{2\alpha_n}\right)\left(\frac{h_{k(n)}-1}{h_{k(n)}}\right)\left\|T_{i(n)}^{k(n)}x_n-p\right\| \\
&=\left\|x_n-p+\frac{1-\alpha_n}{2}\left(x_{n-1}-T_{i(n)}^{k(n)}x_n\right)+\frac{(1-\alpha_n)^2}{2\alpha_n}\left(T_{i(n)}^{k(n)}y_n-T_{i(n)}^{k(n)}x_n\right)\right\| \\
&+\left(\frac{1-\alpha_n}{2\alpha_n}\right)v_{k(n)}\left\|T_{i(n)}^{k(n)}x_n-p\right\| \\
&\le\left\|x_n-p+\frac{1}{2}(x_{n-1}-x_n)\right\|+\left(\frac{1-\alpha_n}{2\alpha_n}\right)v_{k(n)}\left\|T_{i(n)}^{k(n)}x_n-p\right\| \\
&+\frac{(1-\alpha_n)^2}{2\alpha_n}L\|y_n-x_n\| \\
&\le\left\|\frac{1}{2}(x_n-p)+\frac{1}{2}(x_{n-1}-p)\right\|+\left(\frac{1-\alpha_n}{2\alpha_n}\right)v_{k(n)}\left\|T_{i(n)}^{k(n)}x_n-p\right\| \\
&+\frac{(1-\alpha_n)^2}{2\alpha_n}L\|y_n-x_n\|.
\end{aligned}$$

Thus

$$\begin{aligned}
\liminf_{n\to\infty}\|x_n-p\|\le{}&\liminf_{n\to\infty}\left\|\frac{1}{2}(x_n-p)+\frac{1}{2}(x_{n-1}-p)\right\| \\
&+\liminf_{n\to\infty}\left(\frac{1-\alpha_n}{2\alpha_n}\right)v_{k(n)}\left\|T_{i(n)}^{k(n)}x_n-p\right\| \\
&+\liminf_{n\to\infty}\frac{(1-\alpha_n)^2}{2\alpha_n}L\|y_n-x_n\|.
\end{aligned}$$

Since $v_{k(n)}=\frac{h_{k(n)}-1}{h_{k(n)}}\in(0,1)$, we have $\lim_{n\to\infty}v_{k(n)}=0$ and from $\sum_{n=1}^{\infty}(1-\alpha_n)<\infty$, we have $\lim_{n\to\infty}(1-\alpha_n)=0$ and using (3.10), we have

$$\liminf_{n\to\infty}\left\|\frac{1}{2}(x_n-p)+\frac{1}{2}(x_{n-1}-p)\right\|\ge d^1. \tag{3.11}$$

On the other hand, we obtain

$$\limsup_{n\to\infty}\left\|\frac{1}{2}(x_n-p)+\frac{1}{2}(x_{n-1}-p)\right\|\le\limsup_{n\to\infty}\left[\frac{1}{2}\|x_n-p\|+\frac{1}{2}\|x_{n-1}-p\|\right]=d^1, \tag{3.12}$$

from (3.11) and (3.12), we have

$$\lim_{n\to\infty}\left\|\frac{1}{2}(x_n-p)+\frac{1}{2}(x_{n-1}-p)\right\|=d^1.$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0. \tag{3.13}$$

Thus, for any $i\in I$, we have

$$\lim_{n\to\infty}\|x_n-x_{n+i}\|=0. \tag{3.14}$$

Since $0 < \alpha < \alpha_n \leq \beta < 1$ and from (1.4) and (3.13), we get

$$\begin{aligned}\lim_{n\to\infty}\left\|x_n - T_{i(n)}^{k(n)}y_n\right\| &= \lim_{n\to\infty}\frac{\alpha_n}{1-\alpha_n}\|x_n - x_{n-1}\| \\ &\leq \frac{1}{1-\beta}\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0. \end{aligned}\tag{3.15}$$

On the other hand, from (3.13) and (3.15)

$$\lim_{n\to\infty}\left\|x_{n-1} - T_{i(n)}^{k(n)}y_n\right\| \leq \lim_{n\to\infty}\|x_{n-1} - x_n\| + \lim_{n\to\infty}\left\|x_n - T_{i(n)}^{k(n)}y_n\right\| = 0. \tag{3.16}$$

Now,

$$\begin{aligned}\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\| + \left\|T_{i(n)}^{k(n)}y_n - T_{i(n)}^{k(n)}x_n\right\| \\ &\leq (1+L)\|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\| + L\|y_n - x_{n-1}\|. \end{aligned}\tag{3.17}$$

Again, by using (1.4), we obtain

$$\begin{aligned}\|y_n - x_{n-1}\| &\leq \left\|r_n x_n + s_n x_{n-1} + t_n T_{i(n)}^{k(n)}x_n + w_n T_{i(n)}^{k(n)}x_{n-1} - x_{n-1}\right\| \\ &\leq t_n\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| + w_n\left\|T_{i(n)}^{k(n)}x_{n-1} - x_n\right\| + (r_n + t_n + w_n)\|x_n - x_{n-1}\| \\ &\leq (t_n + w_n)\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| + (r_n + t_n + w_n + w_n L)\|x_n - x_{n-1}\|. \end{aligned}\tag{3.18}$$

Substituting (3.18) into (3.17), we get

$$\begin{aligned}\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| \leq{}& (1+L)\|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\| + L(t_n + w_n)\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| \\ &+ L(r_n + t_n + w_n + w_n L)\|x_n - x_{n-1}\|. \end{aligned}$$

Since $t_n + w_n \leq b/L < 1$, the above inequality gives

$$(1-b)\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| \leq \left[1 + L(1 + r_n + t_n + w_n + w_n L)\right]\|x_n - x_{n-1}\| + \left\|T_{i(n)}^{k(n)}y_n - x_{n-1}\right\|.$$

Then from (3.13), (3.16), and the above inequality, we have

$$\lim_{n\to\infty}\left\|T_{i(n)}^{k(n)}x_n - x_n\right\| = 0. \tag{3.19}$$

From (3.13), (3.18), and (3.19), we get

$$\lim_{n\to\infty}\|y_n - x_{n-1}\| = 0. \tag{3.20}$$

On the other hand, from (3.13) and (3.20) we have

$$\lim_{n\to\infty}\|y_n - x_n\| \leq \lim_{n\to\infty}\|y_n - x_{n-1}\| + \lim_{n\to\infty}\|x_{n-1} - x_n\| = 0. \tag{3.21}$$

Since for any positive integer $n > N$, we can write $n = (k(n)-1)N + i(n)$, $i(n) \in I$.

Let $\mathcal{A}_n = \|T_{i(n)}^{k(n)} y_n - x_{n-1}\|$, then from (3.16), we have $\mathcal{A}_n \to 0$. Also,

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\le \bigl\|x_{n-1} - T_{i(n)}^{k(n)} y_n\bigr\| + \bigl\|T_{i(n)}^{k(n)} y_n - T_n x_n\bigr\| \\
&= \mathcal{A}_n + \bigl\|T_{i(n)}^{k(n)} y_n - T_{i(n)} x_n\bigr\| \le \mathcal{A}_n + L\bigl\|T_{i(n)}^{k(n)-1} y_n - x_n\bigr\| \\
&\le \mathcal{A}_n + L\bigl\{\bigl\|T_{i(n)}^{k(n)-1} y_n - T_{i(n-N)}^{k(n)-1} x_{n-N}\bigr\| \\
&\quad + \bigl\|T_{i(n-N)}^{k(n)-1} x_{n-N} - T_{i(n-N)}^{k(n)-1} y_{n-N}\bigr\| \\
&\quad + \bigl\|T_{i(n-N)}^{k(n)-1} y_{n-N} - x_{(n-N)-1}\bigr\| + \|x_{(n-N)-1} - x_n\|\bigr\}.
\end{aligned} \tag{3.22}$$

Since for each $n > N$, $n = (n-N) \pmod N$ and $n = (k(n)-1)N + i(n)$, $n - N = ((k(n)-1)-1)N + i(n) = (k(n-N)-1)N + i(n-N)$, *i.e.*

$$k(n-N) = k(n) - 1 \quad \text{and} \quad i(n-N) = i(n).$$

Therefore from (3.22), we have

$$\begin{aligned}
\|x_{n-1} - T_n x_n\| &\le \mathcal{A}_n + L\bigl\{\bigl\|T_{i(n)}^{k(n)-1} y_n - T_{i(n)}^{k(n)-1} x_{n-N}\bigr\| \\
&\quad + \bigl\|T_{i(n-N)}^{k(n-N)} x_{n-N} - T_{i(n-N)}^{k(n-N)} y_{n-N}\bigr\| \\
&\quad + \bigl\|T_{i(n-N)}^{k(n-N)} y_{n-N} - x_{(n-N)-1}\bigr\| + \|x_{(n-N)-1} - x_n\|\bigr\} \\
&\le \mathcal{A}_n + L\bigl\{L\|y_n - x_{n-N}\| + L\|x_{n-N} - y_{n-N}\| \\
&\quad + \mathcal{A}_{n-N} + \|x_{(n-N)-1} - x_n\|\bigr\} \\
&\le \mathcal{A}_n + L^2\bigl(\|y_n - x_n\| + \|x_n - x_{n-N}\| + \|x_{n-N} - y_{n-N}\|\bigr) \\
&\quad + L\bigl(\mathcal{A}_{n-N} + \|x_{(n-N)-1} - x_n\|\bigr).
\end{aligned} \tag{3.23}$$

From (3.14), (3.21), and $\mathcal{A}_n \to 0$, we have

$$\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0. \tag{3.24}$$

It follows from (3.13) and (3.24) that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| \le \lim_{n\to\infty} \bigl\{\|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|\bigr\} = 0. \tag{3.25}$$

Consequently, for any $i \in I$, from (3.14), (3.25), we obtain

$$\begin{aligned}
\|x_n - T_{n+i} x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| + \|T_{n+i} x_{n+i} - T_{n+i} x_n\| \\
&\le (1+L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i} x_{n+i}\| \to 0,
\end{aligned}$$

as $n \to \infty$. This implies that the sequence

$$\bigcup_{i=1}^{N} \bigl\{\|x_n - T_{n+i} x_n\|\bigr\}_{n=1}^{\infty} \to 0, \quad \text{as } n \to \infty.$$

Since for each $l = 1, 2, \dots, N$, $\{\|x_n - T_l x_n\|\}$ is a subsequence of $\bigcup_{i=1}^{N}\{\|x_n - T_{n+i}x_n\|\}$, therefore, we have

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in I. \tag{3.26}$$

This completes the proof. □

3.1 Strong convergence theorems

First, we prove necessary and sufficient conditions for the strong convergence of the modified general composite implicit iteration process to a common fixed point of a finite family of asymptotically pseudo-contractive mappings.

Theorem 3.1 *Let E, K, and T_i ($i \in I$) be as defined above and $\{\alpha_n\}$ be a sequence of real numbers as in Lemma* 3.1. *Then the sequence $\{x_n\}$ generated by* (1.4) *converges strongly to a member of $\mathcal{F}$ if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Proof The necessity of the condition is obvious. Thus, we will only prove the sufficiency.

Let $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Then from (ii) in Lemma 3.1, we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in K. For any given $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, there exists a natural number n_1 such that $d(x_n, \mathcal{F}) < \varepsilon/4$ when $n \geq n_1$.

Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$, we have $\|x_n - p\| < M'$, for all $n \geq 1$ and some positive number M'.

Furthermore $\sum_{n=1}^{\infty} \sigma_n < \infty$ implies that there exists a positive integer n_2 such that $\sum_{j=n}^{\infty} \sigma_j < \varepsilon/4M'$ for all $n \geq n_2$. Let $N' = \max\{n_1, n_2\}$. It follows from (3.8) that

$$\|x_n - p\| \leq \|x_{n-1} - p\| + M'\sigma_n.$$

Now, for all $n, m \geq N'$ and for all $p \in \mathcal{F}$, we have

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - p\| + \|x_m - p\| \\ &\leq \|x_{N'} - p\| + M' \sum_{j=N'+1}^{n} \sigma_j + \|x_{N'} - p\| + M' \sum_{j=N'+1}^{m} \sigma_j \\ &\leq 2\|x_{N'} - p\| + 2M' \sum_{j=N'}^{\infty} \sigma_j.\end{aligned}$$

Taking the infimum over all $p \in \mathcal{F}$, we obtain

$$\|x_n - x_m\| \leq 2d\big(x_{N'}, \mathcal{F}\big) + 2M' \sum_{j=N'}^{\infty} \sigma_j < \varepsilon.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since E is complete, therefore $\{x_n\}$ is convergent.

Suppose $\lim_{n\to\infty} x_n = q$.

Since K is closed, we get $q \in K$, then $\{x_n\}$ converges strongly to q.

It remains to show that $q \in \mathcal{F}$.

Notice that

$$\left|d(q,\mathcal{F}) - d(x_n,\mathcal{F})\right| \le \|q - x_n\|, \quad \forall n \in \mathbb{N},$$

since $\lim_{n\to\infty} x_n = q$ and $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$, we obtain $q \in \mathcal{F}$.

This completes the proof. □

Corollary 3.1 *Suppose that the conditions are the same as in Theorem* 3.1. *Then the sequence* $\{x_n\}$ *generated by* (1.4) *converges strongly to* $u \in \mathcal{F}$ *if and only if* $\{x_n\}$ *has a subsequence* $\{x_{n_j}\}$ *which converges strongly to* $u \in \mathcal{F}$.

A mapping $T\colon K \to K$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [24] on K if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$, with $f(0) = 0$ and $f(r) > r$, for all $r \in (0,\infty)$, such that for all $x \in K$,

$$\|x - Tx\| \ge f\big(d\big(x, F(T)\big)\big).$$

A family $\{T_i\}_{i=1}^N$ of N self-mappings of K with $\mathcal{F} = \bigcap_{i\in I} F(T_i) \neq \emptyset$ is said to satisfy

(1) *condition* (B) on K [25] if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0,\infty)$ such that for all $x \in K$ such that

$$\max_{1\le l\le N}\big\{\|x - T_l x\|\big\} \ge f\big(d(x,\mathcal{F})\big);$$

(2) *condition* $(\overline{\mathrm{C}})$ on K [26] if there is a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > r$ for all $r \in (0,\infty)$ such that for all $x \in K$ such that

$$\big\{\|x - T_l x\|\big\} \ge f\big(d(x,\mathcal{F})\big)$$

for at least one T_l, $l = 1,2,\ldots,N$ or, in other words, at least one of the T_l's satisfies *condition* (A).

Condition (B) reduces to condition (A) when all but one of the T_l's are identities. Also condition (B) and condition $(\overline{\mathrm{C}})$ are equivalent (see [26]).

Senter and Dotson [24] established a relation between *condition* (A) and demicompactness that the *condition* (A) is weaker than demicompactness for a nonexpansive mapping T defined on a bounded set. Every compact operator is demicompact. Since every completely continuous mapping $T\colon K \to K$ is continuous and demicompact, it satisfies *condition* (A).

Therefore in the next result, instead of complete continuity of mappings $T_1, T_2, \ldots, T_N$, we use condition $(\overline{\mathrm{C}})$.

Theorem 3.2 *Let* E *and* K *be as defined above,* T_i $(i \in I)$ *be* N *asymptotically pseudocontractive mappings as defined above and satisfying* condition $(\overline{\mathrm{C}})$ *and* $\{\alpha_n\}$ *be a sequence of real numbers as in Lemma* 3.1. *Then the sequence* $\{x_n\}$ *generated by* (1.4) *converges strongly to a member of* $\mathcal{F}$.

Proof By Lemma 3.1, we see that $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} d(x_n,\mathcal{F})$ exist.

Let one of the T_i's, say T_l, $l \in I$, satisfy condition (A).

By Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$. Therefore we have $\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. By the nature of f and the fact that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists, we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Theorem 3.1, we find that $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$.

This completes the proof. □

A mapping $T: K \to K$ is said to be *semicompact*, if the sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$, as $n \to \infty$, has a convergent subsequence.

Theorem 3.3 *Let E and K be as defined above, and let T_i ($i \in I$) be N asymptotically pseudo-contractive mappings as defined above such that one of the mappings in $\{T_i\}_{i=1}^N$ is semicompact, and let $\{\alpha_n\}$ be a sequence of real numbers as in Lemma* 3.1. *Then the sequence $\{x_n\}$ generated by* (1.4) *converges strongly to a member of $\mathcal{F}$.*

Proof Without loss of generality, we may assume that T_s is semicompact for some fixed $s \in \{1, 2, \dots, N\}$. Then by Lemma 3.1, we have $\lim_{n\to\infty} \|x_n - T_s x_n\| = 0$. So by definition of semicompactness, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to $x^* \in K$. Now again by Lemma 3.1, we have

$$\lim_{n_j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0$$

for all $l \in I$. By continuity of T_l, we have $T_l x_{n_j} \to T_l x^*$ for all $l \in I$.

Thus $\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = \|x^* - T_l x^*\| = 0$ for all $l \in I$. This implies that $x^* \in \mathcal{F}$. Also, $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By Theorem 3.1, we find that $\{x_n\}$ converges strongly to a common fixed point in $\mathcal{F}$. □

3.2 Weak convergence theorem

Theorem 3.4 *Let E be a uniformly convex and smooth Banach space which admits a weakly sequentially continuous duality mapping, and let K and T_i ($i \in I$) be as defined above and $\{\alpha_n\}$ be a sequence of real numbers as in Lemma* 3.1. *Then the sequence $\{x_n\}$ generated by* (1.4) *converges weakly to a member of $\mathcal{F}$.*

Proof Since $\{x_n\}$ is a bounded sequence in K, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$. Hence from Lemma 3.1, we have

$$\lim_{n\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0, \quad \forall l \in I.$$

By Lemma 2.4, we find that $(I - T_l)$ is demiclosed at zero, *i.e.* $(I - T_l)q = 0$, so that $q \in F(T_l)$. By the arbitrariness of $l \in I$, we know that $q \in \mathcal{F} = \bigcap_{l\in I} F(T_l)$.

Next we prove that $\{x_n\}$ converges weakly to q.

If $\{x_n\}$ has another subsequence $\{x_{n_j}\}$ which converges weakly to $q_1 \neq q$, then we must have $q_1 \in \mathcal{F}$, and since $\lim_{n\to\infty} \|x_n - q_1\|$ exists and since the Banach space E has a weakly sequentially duality mapping, it satisfies Opial's condition, and it follows from a standard argument that $q_1 = q$. Thus $\{x_n\}$ converges weakly to $q \in \mathcal{F}$. □

Remark 3.1 Our results improve and generalize the corresponding results of Browder [7], Kirk [8], Goebel and Kirk [1], Schu [4], Xu [9, 10], Liu [11], Zhou and Chang [15], Osilike [27], Osilike and Akuchu [13], Su and Li [16], Su and Qin [17], and many others.

Let K be a nonempty subset of a real Banach space E. Let D be a nonempty bounded subset of K. The set-measure of noncompactness of D, $\gamma(D)$, is defined as

$$\gamma(D) = \inf\{d > 0 : D \text{ can be covered by a finite number of sets of diameter} \leq d\}.$$

The ball-measure of compactness of D, $\chi(D)$, is defined as

$$\chi(D) = \inf\{r > 0 : D \text{ can be covered by a finite family of balls with centers in } E \text{ and radius } r\}.$$

A bounded continuous mapping $T : K \to E$ is called

(1) k-set-contractive if $\gamma(T(D)) \leq k\gamma(D)$, for each bounded subset D of K and some constant $k \geq 0$;
(2) k-set-condensing if $\gamma(T(D)) < \gamma(D)$, for each bounded subset D of K with $\gamma(D) > 0$;
(3) k-ball-contractive if $\chi(T(D)) \leq k\chi(D)$, for each bounded subset D of K and some constant $k \geq 0$;
(4) k-ball-condensing if $\chi(T(D)) < \chi(D)$, for each bounded subset D of K with $\chi(D) > 0$.

A mapping $T : K \to E$ is called

(5) compact if $\mathrm{cl}(T(A))$ is compact whenever $A \subset K$ is bounded;
(6) completely continuous if it maps weakly convergence sequences into strongly convergent sequences;
(7) a generalized contraction if for each $x \in K$ there exists $k(x) < 1$ such that $\|Tx - Ty\| \leq k(x)\|x - y\|$ for all $y \in K$;
(8) a mapping $T : E \to E$ is called uniformly strictly contractive (relative to E) if for each $x \in E$ there exists $k(x) < 1$ such that $\|Tx - Ty\| \leq k(x)\|x - y\|$ for all $y \in K$. Every k-set-contractive mapping with $k < 1$ is set-condensing and also every compact mapping is set-condensing.

Let K be a nonempty closed bounded subset of E and $T : K \to E$ a continuous mapping. Then

(a) T is strictly semicontractive if there exists a continuous mapping $V : E \times E \to E$ with $T(x) = V(x, x)$ for $x \in E$ such that for each $x \in E$, $V(\cdot, x)$ is a k-contraction with $k < 1$ and $V(x, \cdot)$ is compact;
(b) T is of strictly semicontractive type if there exists a continuous mapping $V : K \times K \to E$ with $T(x) = V(x, x)$, for $x \in K$ such that for each $x \in K$, $V(\cdot, x)$ is a k-contraction with some $k < 1$ independent of x and $x \mapsto V(\cdot, x)$ is compact from K into the space of continuous mapping of K into E with the uniform metric;
(c) T is of strongly semicontractive type relative to X if there exists a mapping $V : E \times K \to E$ with $T(x) = V(x, x)$, for $x \in K$ such that $x \in K$, $V(\cdot, x)$ is uniformly strictly contractive on K relative to E and $V(x, \cdot)$ is a completely continuous from K to E, uniformly for $x \in K$.

For details refer to [28–30].

Let K be a nonempty closed convex bounded subset of a uniformly convex Banach space E. Suppose $T : K \to K$. Then T is semicompact if T satisfies any one of the following conditions [25, Proposition 3.4]:

(i) T is either set-condensing or ball-condensing (or compact);
(ii) T is a generalized contraction;

(iii) T is uniformly strictly contractive;
(iv) T is strictly semicontractive;
(v) T is of strictly semicontractive type;
(vi) T is of strongly semicontractive type.

Remark 3.2 In view of the above, it is possible to replace the semicompactness assumption in Theorem 3.3 with any of the contractive assumptions (i)-(vi).

We now give an example of asymptotically pseudo-contractive mapping with nonempty fixed point set.

Example 3.1 [31] Let $E = \mathbb{R} = (-\infty, \infty)$ with usual norm and $K = [0,1]$ and define $T : K \to K$ by

$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{9} & \text{if } x = 1, \\ x - \frac{1}{3^{n+1}} & \text{if } \frac{1}{3^{n+1}} \leq x < \frac{1}{3}(\frac{1}{3^{n+1}} + \frac{1}{3^n}), \\ \frac{1}{3^n} - x & \text{if } \frac{1}{3}(\frac{1}{3^{n+1}} + \frac{1}{3^n}) \leq x < \frac{1}{3^n} \end{cases}$$

for all $n \geq 0$. Then $F(T) = \{0\}$ and for any $x \in K$, there exists $j(x-0) \in J(x-0)$ satisfying

$$\langle T^n x - T^n 0, j(x-0) \rangle = T^n x \cdot x \leq \frac{1}{3} \|x\|^2 < \|x\|^2$$

for all $n \geq 1$. That is, T is an asymptotically pseudo-contractive mapping with sequence $\{k_n\} = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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