Fixed Point Theory and Applications
a SpringerOpen Journal

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Existence and uniqueness of fixed point for mixed monotone ternary operators with application

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Abstract

In this paper, partial order theory is used to study the fixed point of a mixed monotone ternary operator $A : P \times P \times P \to P$. The existence and uniqueness of a fixed point are obtained without assuming the operators to be compact or continuous. In the end, the application to an integral equation is presented. Our results unify, generalize, and complement various known comparable results from the current literature.

MSC: 47H05; 47H10

Keywords: fixed point; mixed monotone ternary operator; normal cone

1 Introduction

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach fixed point theorem. It is well known that mixed monotone operators were introduced by Guo and Lakshmikantham [1] in 1987. Later, Bhaskar and Lakshmikantham [2] introduced the notion of a coupled fixed point and proved some coupled fixed point results under certain conditions, in a complete metric space endowed with a partial order. Their study has not only important theoretical meaning but also wide applications in engineering, nuclear physics, biological chemistry technology, *etc.* (see [1–8] and the references therein).

Very recently, Harjani *et al.* [9] have established the existence results of coupled fixed point for mixed monotone operators, and further obtained their applications to integral equations. Berinde and Borcut [10] have introduced the concept of a triple fixed point and proved some related theorems for contractive type operators in partially ordered metric spaces. Zhai [11] has considered mixed monotone operators with convexity and get the existence and uniqueness of a fixed point ($A(u,u) = u$ type) without assuming the operator to be compact or continuous.

Motivated by the work reported in [9–11], the aim of this paper is to discuss the existence and uniqueness of a fixed point ($A(u,u,u) = u$ type) for mixed monotone ternary operators in the context of ordered metric spaces. Our results unify, generalize, and complement various known comparable results from the current literature.

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions and notations which will be used in the sequel. The existence and uniqueness of a fixed point for mixed monotone ternary operators (without assuming the operators to be compact or continuous) are obtained in Section 3. We also present an application in Section 4 to an integral equation to illustrate our results.

2 Preliminaries

In this section, we recall some standard definitions and notations needed in the following section. For the convenience of the reader, we suggest that one refers to [1, 2, 10–14] for details.

Throughout this paper, unless otherwise specified, suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \ge 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Further, P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \le x \le y$ implies $\|x\| \le N\|y\|$; in this case N is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \le x \le x_2\}$ is called the order interval between x_1 and x_2.

Definition 2.1 (see [10]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y, that is, for any $x, y \in P$,

$$\begin{aligned} &x_1, x_2 \in P, \quad x_1 \le x_2 \quad \Rightarrow \quad A(x_1, y) \le A(x_2, y), \\ &y_1, y_2 \in P, \quad y_1 \le y_2 \quad \Rightarrow \quad A(x, y_2) \le A(x, y_1). \end{aligned} \tag{2.1}$$

Definition 2.2 (see [11]) An element $x \in P$ is called a fixed point of $A : P \times P \to P$ if

$$A(x, x) = x.$$

Definition 2.3 (see [10]) $A : P \times P \times P \to P$ is said to be a mixed monotone operator if $A(x, y, z)$ is monotone non-decreasing in x, z and monotone non-increasing in y, that is, for any $x, y, z \in P$

$$\begin{aligned} &x_1, x_2 \in P, \quad x_1 \le x_2 \quad \Rightarrow \quad A(x_1, y, z) \le A(x_2, y, z), \\ &y_1, y_2 \in P, \quad y_1 \le y_2 \quad \Rightarrow \quad A(x, y_1, z) \ge A(x, y_2, z), \\ &z_1, z_2 \in P, \quad z_1 \le z_2 \quad \Rightarrow \quad A(x, y, z_1) \le A(x, y, z_2). \end{aligned} \tag{2.2}$$

Definition 2.4 An element $x \in P$ is called a fixed point of $A : P \times P \times P \to P$ if

$$A(x, x, x) = x.$$

3 Main results

In this section we consider the existence and uniqueness of a fixed point for mixed monotone ternary operators in ordered Banach spaces. Our first main result is the following.

Theorem 3.1 *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \times P \to P$ is a mixed monotone ternary operator which satisfies the following*:

(H_1) *for* $t \in (0,1)$, $x, y \in P$, *there exists* $\alpha(t,x,y) \in (1,+\infty)$, *such that*

$$A(tx,y,tx) \le t^{\alpha(t,x,y)} A(x,y,x); \tag{3.1}$$

(H_2) *there exist* $u_0, v_0, m_0 \in P$, $r \in (0,1)$, *such that*

$$\begin{aligned} &u_0 \le rv_0, \qquad m_0 \le rv_0, \\ &A(u_0,v_0,m_0) \ge u_0, \qquad A(v_0,u_0,v_0) \le v_0, \qquad A(m_0,v_0,u_0) \ge m_0, \\ &A(u_0,v_0,u_0) \ge u_0, \qquad A(m_0,v_0,m_0) \ge m_0. \end{aligned} \tag{3.2}$$

Then A has a unique fixed point u^* in $[u_0, rv_0] \cap [m_0, rv_0]$. Moreover, constructing successively the sequences

$$\begin{aligned} &x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}, y_{n-1}), \\ &z_n = A(z_{n-1}, y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \end{aligned}$$

for any initial values $x_0, y_0, z_0 \in [u_0, rv_0] \cap [m_0, rv_0]$, we have

$$\|x_n - u^*\| \to 0, \qquad \|y_n - u^*\| \to 0, \qquad \|z_n - u^*\| \to 0$$

as $n \to \infty$.

Proof Let $w_0 = rv_0$, $\varepsilon = r^{\alpha(r,v_0,u_0)-1}$. Then $w_0 \ge u_0$, $\varepsilon \in (0,1)$, and

$$\begin{aligned} A(w_0,u_0,w_0) = A(rv_0,u_0,rv_0) &\le r^{\alpha(r,v_0,u_0)} A(v_0,u_0,v_0) \\ &\le r^{\alpha(r,v_0,u_0)} v_0 = r^{\alpha(r,v_0,u_0)-1} \cdot rv_0 = \varepsilon w_0 \le w_0, \end{aligned} \tag{3.3}$$

$$A(u_0,w_0,m_0) = A(u_0,rv_0,m_0) \ge A(u_0,v_0,m_0) \ge u_0, \tag{3.4}$$

$$A(m_0,w_0,u_0) = A(m_0,rv_0,u_0) \ge A(m_0,v_0,u_0) \ge m_0. \tag{3.5}$$

Construct successively the sequences

$$\begin{aligned} &u_n = A(u_{n-1}, w_{n-1}, m_{n-1}), \qquad w_n = A(w_{n-1}, u_{n-1}, w_{n-1}), \qquad m_n = A(m_{n-1}, w_{n-1}, u_{n-1}), \\ &w'_n = \frac{1}{\varepsilon} A\bigl(w'_{n-1}, u_{n-1}, w'_{n-1}\bigr), \qquad w'_0 = w_0, \quad n = 1, 2, \ldots. \end{aligned}$$

From (3.3)-(3.5) and the mixed monotonicity of A, we have

$$u_0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0, \tag{3.6}$$

$$m_0 \le m_1 \le m_2 \le \cdots \le m_n \le \cdots \le w_n \le \cdots \le w_1 \le w_0. \tag{3.7}$$

Next we prove that

$$u_0 \le w'_n \le w_0. \tag{3.8}$$

From (3.2) and (3.3),

$$\begin{aligned}
w_1' &= \frac{1}{\varepsilon}A(w_0,u_0,w_0) \le \frac{1}{\varepsilon}\cdot\varepsilon w_0 = w_0,\\
w_1' &= \frac{1}{\varepsilon}A(w_0,u_0,w_0) \ge \frac{1}{\varepsilon}A(u_0,v_0,u_0) \ge \frac{1}{\varepsilon}u_0 \ge u_0,\\
w_2' &= \frac{1}{\varepsilon}A(w_1',u_1,w_1') \le \frac{1}{\varepsilon}A(w_0,u_0,w_0) \le \frac{1}{\varepsilon}\cdot\varepsilon w_0 = w_0,\\
w_2' &= \frac{1}{\varepsilon}A(w_1',u_1,w_1') \ge \frac{1}{\varepsilon}A(u_0,v_0,u_0) \ge \frac{1}{\varepsilon}u_0 \ge u_0.
\end{aligned}$$

Suppose that when $n = k$, we have

$$u_0 \le w_k' \le w_0,$$

then when $n = k + 1$, note that $u_k \le w_0 = rv_0 \le v_0$, we obtain

$$\begin{aligned}
w_{k+1}' &= \frac{1}{\varepsilon}A(w_k',u_k,w_k') \le \frac{1}{\varepsilon}A(w_0,u_0,w_0) \le \frac{1}{\varepsilon}\cdot\varepsilon w_0 = w_0,\\
w_{k+1}' &= \frac{1}{\varepsilon}A(w_k',u_k,w_k') \ge \frac{1}{\varepsilon}A(u_0,v_0,u_0) \ge \frac{1}{\varepsilon}u_0 \ge u_0.
\end{aligned}$$

By mathematical induction, we know that (3.8) holds. The same procedure may easily be adapted to obtain

$$m_0 \le w_n' \le w_0. \tag{3.9}$$

On the other hand, from (3.1),

$$\begin{aligned}
w_1 &= A(w_0,u_0,w_0) = \varepsilon\frac{1}{\varepsilon}A(w_0,u_0,w_0) = \varepsilon w_1',\\
w_2 &= A(w_1,u_1,w_1) = A(\varepsilon w_1',u_1,\varepsilon w_1') \le \varepsilon^{\alpha(\varepsilon,w_1',u_1)}A(w_1',u_1,w_1')\\
&= \varepsilon^{\alpha(\varepsilon,w_1',u_1)+1}\cdot\frac{1}{\varepsilon}A(w_1',u_1,w_1')\\
&\le \varepsilon^2 w_2'.
\end{aligned}$$

Suppose that when $n = k$, we have $w_k \le \varepsilon^k w_k'$. Then when $n = k + 1$, in view of (3.1), we obtain

$$\begin{aligned}
w_{k+1} &= A(w_k,u_k,w_k) \le A(\varepsilon^k w_k',u_k,\varepsilon^k w_k') \le (\varepsilon^k)^{\alpha(\varepsilon^k,w_k',u_k)}A(w_k',u_k,w_k')\\
&= \varepsilon^{k\alpha(\varepsilon^k,w_k',u_k)+1}\cdot\frac{1}{\varepsilon}A(w_k',u_k,w_k')\\
&\le \varepsilon^{k+1}w_{k+1}'.
\end{aligned}$$

By mathematical induction, we have

$$w_n \le \varepsilon^n w_n', \quad n = 1,2,\ldots. \tag{3.10}$$

By (3.6)-(3.10) we get

$$\begin{aligned}
&\theta \leq w_n - u_n \leq \varepsilon^n w_n' - u_n \leq \varepsilon^n w_n' - \varepsilon^n u_n = \varepsilon^n \left(w_n' - u_n\right) \leq \varepsilon^n (w_0 - u_0), \\
&\theta \leq u_{n+p} - u_n \leq w_n - u_n, \qquad \theta \leq w_n - w_{n+p} \leq w_n - u_n; \\
&\theta \leq w_n - m_n \leq \varepsilon^n w_n' - m_n \leq \varepsilon^n w_n' - \varepsilon^n m_n = \varepsilon^n \left(w_n' - m_n\right) \leq \varepsilon^n (w_0 - m_0), \\
&\theta \leq m_{n+p} - m_n \leq w_n - m_n.
\end{aligned}$$

Noting that P is normal and $\varepsilon \in (0,1)$, we have

$$\begin{aligned}
&\|w_n - u_n\| \leq N\varepsilon^n \|w_0 - u_0\| \to 0 \quad (\text{as } n \to \infty), \\
&\|w_n - m_n\| \leq N\varepsilon^n \|w_0 - m_0\| \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Further,

$$\begin{aligned}
&\|u_{n+p} - u_n\| \leq N\|w_n - u_n\| \to 0 \quad (\text{as } n \to \infty), \\
&\|w_n - w_{n+p}\| \leq N\|w_n - u_n\| \to 0 \quad (\text{as } n \to \infty), \\
&\|m_{n+p} - m_n\| \leq N\|w_n - m_n\| \to 0 \quad (\text{as } n \to \infty).
\end{aligned}$$

Here N is the normality constant.

So, we can claim that $\{u_n\}$, $\{w_n\}$, and $\{m_n\}$ are Cauchy sequences. Since E is complete, there exist $u^*, w^*, m^* \in P$ such that

$$u_n \to u^*, \qquad w_n \to w^*, \qquad m_n \to m^* \quad (\text{as } n \to \infty).$$

By (3.6), (3.7), respectively, we know that

$$\begin{aligned}
&u_0 \leq u_n \leq u^* \leq w^* \leq w_n \leq w_0, \\
&m_0 \leq m_n \leq m^* \leq w^* \leq w_n \leq w_0,
\end{aligned}$$

and then

$$\begin{aligned}
&\theta \leq w^* - u^* \leq w_n - u_n \leq \varepsilon^n (w_0 - u_0), \\
&\theta \leq w^* - m^* \leq w_n - m_n \leq \varepsilon^n (w_0 - m_0).
\end{aligned}$$

Further, $\|w^* - u^*\| \leq N\varepsilon^n \|w_0 - u_0\| \to 0$ (as $n \to \infty$), and thus $w^* = u^*$. Similarly, we get $\|w^* - m^*\| \leq N\varepsilon^n \|w_0 - m_0\| \to 0$ (as $n \to \infty$), and thus $w^* = m^*$. Consequently, $w^* = u^* = m^*$. Then we obtain

$$u_{n+1} = A(u_n, w_n, m_n) \leq A\left(u^*, u^*, u^*\right) \leq A(w_n, u_n, w_n) = w_{n+1}.$$

Letting $n \to \infty$, then we get

$$A\left(u^*, u^*, u^*\right) = u^*.$$

That is, u^* is a fixed point of A in $[u_0, rv_0] \cap [m_0, rv_0]$.

In the following, we prove that u^* is the unique fixed point of A in $[u_0, rv_0] \cap [m_0, rv_0]$. Suppose that there exists $x^* \in [u_0, rv_0] \cap [m_0, rv_0]$ such that $A(x^*, x^*, x^*) = x^*$. Then $u_0 \le x^* \le w_0$ and $m_0 \le x^* \le w_0$. By mathematical induction and the mixed monotonicity of A, we have

$$u_{n+1} = A(u_n, w_n, m_n) \le x^* = A\bigl(x^*, x^*, x^*\bigr) \le A(w_n, u_n, w_n) = w_{n+1}.$$

Then from the normality of P, we have $x^* = u^*$.

Moreover, constructing successively the sequences

$$\begin{aligned} &x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}, y_{n-1}), \\ &z_n = A(z_{n-1}, y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \end{aligned}$$

for any initial values $x_0, y_0, z_0 \in [u_0, rv_0] \cap [m_0, rv_0]$, we have $u_n \le x_n$, $w_n \ge y_n$, $m_n \le z_n$, $n = 1, 2, \ldots$. Letting $n \to \infty$ yields $x_n \to u^*$, $y_n \to u^*$, $z_n \to u^*$ as $n \to \infty$. □

Remark 3.1 It is evident from (3.1) that for $t \in (0,1)$, $x, y \in P$, there exists $\alpha(t, \frac{1}{t}x, y) \in (1, +\infty)$, such that

$$A\left(\frac{1}{t}x, y, \frac{1}{t}x\right) \ge \frac{1}{t^{\alpha(t, \frac{1}{t}x, y)}} A(x, y, x). \tag{3.11}$$

Remark 3.2 Let $\alpha(t, x, y)$ be a constant $\alpha \in (1, +\infty)$, then Theorem 3.1 also holds.

Corollary 3.2 *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \times P \to P$ is a mixed monotone ternary operator which satisfies* (H_2) *and, for $t \in (0,1)$, $x, y \in P$, there exists $\alpha \in (1, +\infty)$, such that $A(tx, y, tx) \le t^\alpha A(x, y, x)$. Then A has a unique fixed point u^* in $[u_0, rv_0] \cap [m_0, rv_0]$. Moreover, constructing successively the sequences*

$$\begin{aligned} &x_n = A(x_{n-1}, y_{n-1}, z_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}, y_{n-1}), \\ &z_n = A(z_{n-1}, y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \end{aligned}$$

for any initial values $x_0, y_0, z_0 \in [u_0, rv_0] \cap [m_0, rv_0]$, we have $u_n \le x_n$, $w_n \ge y_n$, $m_n \le z_n$, $n = 1, 2, \ldots$. Letting $n \to \infty$ yields $x_n \to u^*$, $y_n \to u^*$, $z_n \to u^*$ as $n \to \infty$.

Following the lines of the proof of Theorem 3.1, we obtain an immediate consequence.

Corollary 3.3 (see [11]) *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \to P$ is a mixed monotone operator which satisfies the following*:

(H_3) *for $t \in (0,1)$, $x, y \in P$, there exists $\alpha(t, x, y) \in (1, +\infty)$, such that*

$$A(tx, y) \le t^{\alpha(t,x,y)} A(x, y);$$

(H_4) *there exist $u_0, v_0 \in P$, $r \in (0,1)$, such that*

$$u_0 \le rv_0, \qquad A(u_0, v_0) \ge u_0, \qquad A(v_0, u_0) \le v_0.$$

Then A has a unique fixed point u^* in $[u_0, rv_0]$. Moreover, constructing successively the sequences

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

for any initial values $x_0, y_0 \in [u_0, rv_0]$, we have

$$\|x_n - u^*\| \to 0, \qquad \|y_n - u^*\| \to 0$$

as $n \to \infty$.

Theorem 3.4 *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \times P \to P$ is a mixed monotone ternary operator which satisfies* (3.1) *and*

(H_5) *for $R \in (1, +\infty)$, $x, y, z \in P$ there exist $\alpha(\frac{1}{R}, Rx, y, z), \alpha(\frac{1}{R}, x, y, Rz) \in (1, +\infty)$ such that*

$$A(Rx, y, z) \geq R^{\alpha(\frac{1}{R}, Rx, y, z)} A(x, y, z), \tag{3.12}$$

$$A(x, y, Rz) \geq R^{\alpha(\frac{1}{R}, x, y, Rz)} A(x, y, z); \tag{3.13}$$

(H_6) *there exist $u_0, v_0, m_0 \in P$, $R \in (1, +\infty)$, such that*

$$\begin{aligned} &v_0 \geq Ru_0, \qquad v_0 \geq Rm_0, \\ &A(u_0, v_0, m_0) \geq u_0, \qquad A(v_0, u_0, v_0) \leq v_0, \qquad A(m_0, v_0, u_0) \geq m_0, \\ &A(u_0, v_0, u_0) \geq u_0, \qquad A(m_0, v_0, m_0) \geq m_0. \end{aligned} \tag{3.14}$$

Then the operator equation $A(w, w, w) = bw$ has a unique solution w^* in $[Ru_0, v_0] \cap [Rm_0, v_0]$, where $b = \min\{R^{\alpha(\frac{1}{R}, Ru_0, v_0, m_0) - 1}, R^{\alpha(\frac{1}{R}, m_0, v_0, Ru_0) - 1}\}$. Moreover, constructing successively the sequences

$$\begin{aligned} &x_n = b^{-1} A(x_{n-1}, y_{n-1}, z_{n-1}), \qquad y_n = b^{-1} A(y_{n-1}, x_{n-1}, y_{n-1}), \\ &z_n = b^{-1} A(z_{n-1}, y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \end{aligned}$$

for any initial values $x_0, y_0, z_0 \in [Ru_0, v_0] \cap [Rm_0, v_0]$, we have

$$\|x_n - w^*\| \to 0, \qquad \|y_n - w^*\| \to 0, \qquad \|z_n - w^*\| \to 0$$

as $n \to \infty$.

Remark 3.3 Two comments with respect to conditions (3.12) and (3.13) are in order:

(a) A sufficient condition on A for (3.12) to be satisfied is that for $t \in (0, 1)$, $x, y, z \in P$, there exists $\alpha(t, x, y, z) \in (1, +\infty)$, such that

$$A(tx, y, z) \leq t^{\alpha(t, x, y, z)} A(x, y, z).$$

(b) A sufficient condition on A for (3.13) to be satisfied is that for $t \in (0, 1)$, $x, y, z \in P$, there exists $\alpha(t, x, y, z) \in (1, +\infty)$, such that

$$A(x, y, tz) \leq t^{\alpha(t, x, y, z)} A(x, y, z).$$

Proof of Theorem 3.4 Let $w_0 = Ru_0$. Then $v_0 \geq w_0$. Note that $b > 1$, from (3.12)-(3.14),

$$\begin{aligned} A(w_0, v_0, m_0) &= A(Ru_0, v_0, m_0) \geq R^{\alpha(\frac{1}{R}, Ru_0, v_0, m_0)} A(u_0, v_0, m_0) \\ &= R^{\alpha(\frac{1}{R}, Ru_0, v_0, m_0)-1} RA(u_0, v_0, m_0) \geq bRu_0 = bw_0 \geq w_0, \end{aligned} \tag{3.15}$$

$$A(v_0, w_0, v_0) = A(v_0, Ru_0, v_0) \leq A(v_0, u_0, v_0) \leq v_0, \tag{3.16}$$

$$\begin{aligned} A(m_0, v_0, w_0) &= A(m_0, v_0, Ru_0) \geq R^{\alpha(\frac{1}{R}, m_0, v_0, Ru_0)} A(m_0, v_0, u_0) \\ &= R^{\alpha(\frac{1}{R}, m_0, v_0, Ru_0)-1} RA(m_0, v_0, u_0) \geq bA(m_0, v_0, u_0) \geq bm_0 \geq m_0. \end{aligned} \tag{3.17}$$

Set $B(x, y, z) = b^{-1}A(x, y, z)$, $x, y, z \in P$. Then from the above inequalities, we have

$$\begin{aligned} &B(w_0, v_0, m_0) = b^{-1}A(w_0, v_0, m_0) \geq b^{-1}bw_0 = w_0, \\ &B(v_0, w_0, v_0) = b^{-1}A(v_0, w_0, v_0) \leq b^{-1}v_0 \leq v_0, \\ &B(m_0, v_0, w_0) = b^{-1}A(m_0, v_0, w_0) \geq b^{-1}bm_0 = m_0. \end{aligned} \tag{3.18}$$

Also, construct successively the sequences

$$\begin{aligned} &w_n = B(w_{n-1}, v_{n-1}, m_{n-1}), \qquad v_n = B(v_{n-1}, w_{n-1}, v_{n-1}), \qquad m_n = B(m_{n-1}, v_{n-1}, w_{n-1}), \\ &v'_n = bB\bigl(v'_{n-1}, w_{n-1}, v'_{n-1}\bigr), \qquad v'_0 = v_0, \quad n = 1, 2, \ldots. \end{aligned}$$

From (3.18) and the mixed monotonicity of A, we have

$$w_0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0, \tag{3.19}$$

$$m_0 \leq m_1 \leq m_2 \leq \cdots \leq m_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0. \tag{3.20}$$

Next we prove that

$$w_0 \leq v'_n \leq v_0, \quad n = 1, 2, \ldots. \tag{3.21}$$

By (3.11) and (3.14), we have

$$\begin{aligned} A(w_0, v_0, w_0) &= A(Ru_0, v_0, Ru_0) \geq R^{\alpha(\frac{1}{R}, Ru_0, v_0)} A(u_0, v_0, u_0) \\ &\geq RA(u_0, v_0, u_0) \\ &\geq Ru_0 \\ &= w_0. \end{aligned} \tag{3.22}$$

From (3.15)-(3.17) and (3.22),

$$\begin{aligned} &v'_1 = bB\bigl(v'_0, w_0, v'_0\bigr) = bB(v_0, w_0, v_0) = A(v_0, w_0, v_0) \leq v_0, \\ &v'_1 = bB\bigl(v'_0, w_0, v'_0\bigr) = bB(v_0, w_0, v_0) \geq bB(w_0, v_0, m_0) = A(w_0, v_0, m_0) \geq w_0, \\ &v'_2 = bB\bigl(v'_1, w_1, v'_1\bigr) \leq bB(v_0, w_0, v_0) = A(v_0, w_0, v_0) \leq v_0, \\ &v'_2 = bB\bigl(v'_1, w_1, v'_1\bigr) \geq bB(w_0, v_0, w_0) = A(w_0, v_0, w_0) = A(Ru_0, v_0, Ru_0) \geq w_0. \end{aligned}$$

Suppose that when $n = k$, we have

$$w_0 \le v'_k \le v_0,$$

then when $n = k + 1$, recalling (3.16) and (3.22), we obtain

$$\begin{aligned} v'_{k+1} &= bB(v'_k, w_k, v'_k) \le bB(v_0, w_0, v_0) = A(v_0, w_0, v_0) \le v_0, \\ v'_{k+1} &= bB(v'_k, w_k, v'_k) \ge bB(w_0, v_0, w_0) = A(w_0, v_0, w_0) = A(Ru_0, v_0, Ru_0) \ge w_0. \end{aligned}$$

By mathematical induction, we know that (3.21) holds. The same procedure may easily be adapted to obtain

$$m_0 \le v'_n \le v_0, \quad n = 1, 2, \ldots. \tag{3.23}$$

On the other hand, from (3.1),

$$\begin{aligned} v_1 &= B(v_0, w_0, v_0) = \frac{1}{b} bB(v_0, w_0, v_0) = \frac{1}{b} bB(v'_0, w_0, v'_0) = \frac{1}{b} v'_1, \\ v_2 &= B(v_1, w_1, v_1) = B\left(\frac{1}{b} v'_1, w_1, \frac{1}{b} v'_1\right) \le \left(\frac{1}{b}\right)^{\alpha(\frac{1}{b}, v'_1, w_1)} B(v'_1, w_1, v'_1) \\ &= \left(\frac{1}{b}\right)^{\alpha(\frac{1}{b}, v'_1, w_1)+1} bB(v'_1, w_1, v'_1) \le \left(\frac{1}{b}\right)^2 v'_2. \end{aligned}$$

Suppose that when $n = k$, we have $v_k \le (\frac{1}{b})^k v'_k$. Then when $n = k + 1$, in view of (3.1), we obtain

$$\begin{aligned} v_{k+1} &= B(v_k, w_k, v_k) \le B\left(\left(\frac{1}{b}\right)^k v'_k, w_k, \left(\frac{1}{b}\right)^k v'_k\right) \\ &\le \left(\left(\frac{1}{b}\right)^k\right)^{\alpha((\frac{1}{b})^k, v'_k, w_k)} B(v'_k, w_k, v'_k) \\ &\le \left(\frac{1}{b}\right)^{k\alpha((\frac{1}{b})^k, v'_k, w_k)+1} bB(v'_k, w_k, v'_k) \\ &\le \left(\frac{1}{b}\right)^{k+1} v'_{k+1}. \end{aligned}$$

By mathematical induction, we have

$$v_n \le \left(\frac{1}{b}\right)^n v'_n, \quad n = 1, 2, \ldots. \tag{3.24}$$

By (3.19)-(3.24) we get

$$\begin{aligned} \theta &\le v_n - w_n \le \left(\frac{1}{b}\right)^n v'_n - w_n \le \left(\frac{1}{b}\right)^n v'_n - \left(\frac{1}{b}\right)^n w_n \\ &= \left(\frac{1}{b}\right)^n (v'_n - w_n) \le \left(\frac{1}{b}\right)^n (v_0 - w_0), \end{aligned}$$

$$\theta \le w_{n+p} - w_n \le v_n - w_n, \qquad \theta \le v_n - v_{n+p} \le v_n - w_n;$$

$$\begin{aligned}\theta &\le v_n - m_n \le \left(\frac{1}{b}\right)^n v_n' - m_n \le \left(\frac{1}{b}\right)^n v_n' - \left(\frac{1}{b}\right)^n m_n \\ &= \left(\frac{1}{b}\right)^n \left(v_n' - m_n\right) \le \left(\frac{1}{b}\right)^n (v_0 - m_0),\end{aligned}$$

$$\theta \le m_{n+p} - m_n \le v_n - m_n, \qquad \theta \le v_n - v_{n+p} \le v_n - m_n.$$

Note that P is normal and $b > 1$, we have

$$\|v_n - w_n\| \le N\left(\frac{1}{b}\right)^n \|v_0 - w_0\| \to 0 \quad (\text{as } n \to \infty),$$

$$\|v_n - m_n\| \le N\left(\frac{1}{b}\right)^n \|v_0 - m_0\| \to 0 \quad (\text{as } n \to \infty).$$

Further,

$$\|w_{n+p} - w_n\| \le N\|v_n - w_n\| \to 0 \quad (\text{as } n \to \infty),$$

$$\|v_n - v_{n+p}\| \le N\|v_n - w_n\| \to 0 \quad (\text{as } n \to \infty),$$

$$\|m_{n+p} - m_n\| \le N\|v_n - m_n\| \to 0 \quad (\text{as } n \to \infty).$$

Here N is the normality constant.

So, we can claim that $\{w_n\}$, $\{v_n\}$, and $\{m_n\}$ are Cauchy sequences. Since E is complete, there exist $w^*, v^*, m^* \in P$ such that

$$w_n \to w^*, \qquad v_n \to v^*, \qquad m_n \to m^* \quad (\text{as } n \to \infty).$$

By (3.19), (3.20), respectively, we know that

$$w_0 \le w_n \le w^* \le v^* \le v_n \le v_0,$$

$$m_0 \le m_n \le m^* \le v^* \le v_n \le v_0,$$

and then

$$\theta \le v^* - w^* \le v_n - w_n \le \left(\frac{1}{b}\right)^n (v_0 - w_0),$$

$$\theta \le v^* - m^* \le v_n - m_n \le \left(\frac{1}{b}\right)^n (v_0 - m_0).$$

Further, $\|v^* - w^*\| \le N(\frac{1}{b})^n \|v_0 - w_0\| \to 0$ (as $n \to \infty$), and thus $v^* = w^*$. Similarly, we get $\|v^* - m^*\| \le N(\frac{1}{b})^n \|v_0 - m_0\| \to 0$ (as $n \to \infty$), and thus $v^* = m^*$. Consequently, $w^* = v^* = m^*$. Then we obtain

$$w_{n+1} = B(w_n, v_n, m_n) \le B\left(w^*, w^*, w^*\right) \le B(v_n, w_n, v_n) = v_{n+1}.$$

Letting $n \to \infty$, we get

$$B\left(w^*, w^*, w^*\right) = w^*.$$

That is, the operator equation $A(w,w,w) = bw$ has a unique solution w^* in $[Ru_0, v_0] \cap [Rm_0, v_0]$.

In the following, we prove that w^* is the unique solution of $A(w,w,w) = bw$ in $[Ru_0, v_0] \cap [Rm_0, v_0]$. Suppose that there exists $x^* \in [Ru_0, v_0] \cap [Rm_0, v_0]$ such that $A(x^*, x^*, x^*) = bx^*$. Then $w_0 \le x^* \le v_0$ and $m_0 \le x^* \le v_0$. By mathematical induction and the mixed monotonicity of A, we have

$$w_{n+1} = B(w_n, v_n, m_n) \le x^* = B\bigl(x^*, x^*, x^*\bigr) \le B(v_n, w_n, v_n) = v_{n+1}.$$

Then from the normality of P, we have $x^* = w^*$.

Moreover, constructing successively the sequences

$$\begin{aligned} &x_n = b^{-1}A(x_{n-1}, y_{n-1}, z_{n-1}), \qquad y_n = b^{-1}A(y_{n-1}, x_{n-1}, y_{n-1}), \\ &z_n = b^{-1}A(z_{n-1}, y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots, \end{aligned}$$

for any initial values $x_0, y_0, z_0 \in [Ru_0, v_0] \cap [Rm_0, v_0]$, we have $w_n \le x_n$, $v_n \ge y_n$, $m_n \le z_n$, $n = 1, 2, \ldots$. Letting $n \to \infty$ yields $x_n \to w^*$, $y_n \to w^*$, $z_n \to w^*$ as $n \to \infty$. □

From the proof of Theorem 3.4, we can easily obtain the following conclusion.

Corollary 3.5 (see [11]) *Let E be a real Banach space and let P be a normal cone in E. $A : P \times P \to P$ is a mixed monotone operator which satisfies* (H_3) *and*

(H_7) *there exist $u_0, v_0 \in P$, $R \in (1, +\infty)$ such that*

$$v_0 \ge Ru_0, \qquad A(u_0, v_0) \ge u_0, \qquad A(v_0, u_0) \le v_0.$$

Then the operator equation $A(w,w) = bw$ has a unique solution w^* in $[Ru_0, v_0]$, where $b = R^{\alpha(\frac{1}{R}, Ru_0, v_0)-1}$. Moreover, constructing successively the sequences

$$x_n = b^{-1}A(x_{n-1}, y_{n-1}), \qquad y_n = b^{-1}A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

for any initial values $x_0, y_0 \in [Ru_0, v_0]$, we have

$$\|x_n - w^*\| \to 0, \qquad \|y_n - w^*\| \to 0$$

as $n \to \infty$.

4 Application

As application of our results, we investigate the solvability of the following integral equation:

$$x(\tau) = \int_0^1 k(\tau, s)\left[\frac{x^{\alpha_1}(s)}{1 + x^{\alpha_2}(s)} + x^{\alpha_3}(s)\right] \mathrm{d}s \tag{4.1}$$

with $\alpha_1, \alpha_3 > 1$, $\alpha_2 > 0$.

Put $E = C[0,1]$ (the space of continuous functions defined on $[0,1]$ endowed with supremum norm). Let $P = \{x \in E \mid x(t) \ge 0, \forall t \in [0,1]\}$, then E is a Banach space and P is a normal cone. Suppose that $k(\tau, s) : [0,1] \times [0,1] \to R^{++}$ (R^{++} denotes the positive real numbers) is continuous and $0 < \int_0^1 k(\tau,s)\,\mathrm{d}s \le \frac{1}{2}$. In the following, we prove that (4.1) has a unique solution.

Consider the integral operator $A : P \times P \times P \to P$ defined by

$$A(u,v,m) = \int_0^1 k(\tau,s)\left[\frac{u^{\alpha_1}(s)}{1+v^{\alpha_2}(s)} + m^{\alpha_3}(s)\right]\mathrm{d}s$$

with $\alpha_1, \alpha_3 > 1$, $\alpha_2 > 0$.

It is clear that $A(u,v,m)$ is a mixed monotone ternary operator. We shall show that $A(u,v,m)$ satisfies (H_2) and for $t \in (0,1)$, $x,y \in P$, there exists $\alpha \in (1,+\infty)$, such that $A(tx,y,tx) \le t^\alpha A(x,y,x)$.

In fact, let $u_0(\tau) \equiv 0$, $m_0(\tau) \equiv 0$, $v_0(\tau) \equiv 1$, then

$$A(u_0,v_0,u_0) = 0 \ge u_0,$$

$$A(v_0,u_0,v_0) = 2\int_0^1 k(\tau,s)\,\mathrm{d}s \le 1.$$

On the other hand, noting that for any $t \in (0,1)$, letting

$$\alpha = \min\{\alpha_1,\alpha_3\} > 1,$$

we obtain

$$\begin{aligned} A(tx,y,tx) &= \int_0^1 k(\tau,s)\left[\frac{t^{\alpha_1}x^{\alpha_1}(s)}{1+y^{\alpha_2}(s)} + t^{\alpha_3}x^{\alpha_3}(s)\right]\mathrm{d}s \\ &\le \int_0^1 k(\tau,s)\left[\frac{t^{\alpha}x^{\alpha_1}(s)}{1+y^{\alpha_2}(s)} + t^{\alpha}x^{\alpha_3}(s)\right]\mathrm{d}s \\ &= t^\alpha \int_0^1 k(\tau,s)\left[\frac{x^{\alpha_1}(s)}{1+y^{\alpha_2}(s)} + x^{\alpha_3}(s)\right]\mathrm{d}s \\ &= t^\alpha A(x,y,x). \end{aligned}$$

Hence, all the hypotheses of Corollary 3.2 are satisfied. The operator

$$A(u,v,m) = \int_0^1 k(\tau,s)\left[\frac{u^{\alpha_1}(s)}{1+v^{\alpha_2}(s)} + m^{\alpha_3}(s)\right]\mathrm{d}s$$

has a unique fixed point in $[u_0,v_0]$, *i.e.*, the integral equation (4.1) has a unique solution in $[u_0,v_0]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
This research is supported by the Doctoral Fund of Education Ministry of China (20134219120003), the Natural Science Foundation of Hubei Province (2013CFA131), the National Natural Science Foundation of China (71231007) and Hubei Province Key Laboratory of Systems Science in Metallurgical Process (z201302).

Received: 19 June 2014 Accepted: 21 October 2014 Published: 31 Oct 2014

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10.1186/1687-1812-2014-223
Cite this article as: Bu et al.: Existence and uniqueness of fixed point for mixed monotone ternary operators with application. *Fixed Point Theory and Applications* 2014, 2014:223