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Moudafi's open question and simultaneous iterative algorithm for general split equality variational inclusion problems and general split equality optimization problems

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Abstract

The purpose of this paper is first to introduce and study *the general split equality variational inclusion problems* and *the general split equality optimization problems* in the setting of infinite-dimensional Hilbert spaces and then propose a new simultaneous iterative algorithm. Under suitable conditions, some strong convergence theorems for the sequences generated by the proposed algorithm converging strongly to a solution for these two kinds of problems are proved. As special cases, we shall utilize our results to study the split feasibility problems, the split equality equilibrium problems, and the split optimization problems. The results presented in the paper not only extend and improve the corresponding recent results announced by many authors, but they also provide an affirmative answer to an open question raised by Moudafi in his recent work.

Keywords: general split equality variational inclusion problem; general split equality optimization problem; split feasibility problem; split equality equilibrium problem; split optimization problem

1 Introduction

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (SFP) is formulated as

$$\text{to finding } x^* \in C \text{ and } Ax^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [1] first introduced the (SFP) in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the (SFP) can also be used in various disciplines such as image restoration, and computer tomograph and radiation therapy treatment planning [3–5]. The (SFP) in an infinite-dimensional real Hilbert space can be found in [2, 4, 6–10].

Assuming that the (SFP) is consistent, it is not hard to see that $x^* \in C$ solves (SFP) if and only if it solves the fixed-point equation

$$x^* = P_C\big(I - \gamma A^*(I - P_Q)A\big)x^*,$$

where P_C and P_Q are the metric projection from H_1 onto C and from H_2 onto Q, respectively, $\gamma > 0$ is a positive constant and A^* is the adjoint of A.

A popular algorithm to be used to solves the *SFP* (1.1) is due to Byrne's *CQ-algorithm* [2]:

$$x_{k+1} = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x_k, \quad k \geq 1, \tag{1.2}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A.

Recently, Moudafi [11, 12] introduced the following *split equality feasibility problem* (SEFP):

$$\text{to find } x \in C, y \in Q \text{ such that } Ax = By, \tag{1.3}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.3) reduces to (1.1). The kind of split equality problems (1.3) allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, such as decomposition methods for PDEs, and applications in game theory and intensity-modulated radiation therapy.

In order to solve the split equality feasibility problem (1.3), Moudafi [11] introduced the following simultaneous iterative method:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.4}$$

and under suitable conditions he proved the weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.3) in Hilbert spaces.

At the same time, he raised the following open question.

Moudafi's Open Question 1.1 Is there any strong convergence theorem of an alternating algorithm for the split equality feasibility problem (1.3) in real Hilbert spaces?

More recently, Eslamian and Latif [13], Chen *et al.* [14], Chuang [15] and Chang and Wang [16] introduced and studied some kinds of *general split feasibility problem*, *general split equality problem,* and *split variational inclusion problem* in real Hilbert spaces. Under suitable conditions some strong convergence theorems are proved. Also a comprehensive survey and update bibliography on split feasibility problems are given in Ansari and Rehan [17].

Motivated by the above works and related literature, in this paper, we continue to consider the problem (1.3). We obtain some strongly convergent theorems to a solution of the problem (1.3) which provide an affirmative answer to Moudafi's open question.

For the purpose we first introduce and consider the following more general problems.

(I) General split equality variational inclusion problem:

(GSEVIP) to find $x^* \in H_1$ and $y^* \in H_2$ such that

$$0 \in \bigcap_{i=1}^{\infty} U_i\bigl(x^*\bigr), \qquad 0 \in \bigcap_{i=1}^{\infty} K_i\bigl(y^*\bigr) \quad \text{and} \quad Ax^* = By^*, \tag{1.5}$$

where H_1, H_2 and H_3 are three real Hilbert spaces, $U_i : H_1 \to H_1$ and $K_i : H_2 \to H_2$, $i = 1, 2, \ldots$ are two families of set-valued maximal monotone mappings, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two linear and bounded operators.

(II) General split equality optimization problem:

$$\text{(GSEOP)} \quad \text{to find } x^* \in H_1 \text{ and } y^* \in H_2 \text{ such that for each } i \geq 1$$

$$h_i(x^*) = \min_{x \in H_1} h_i(x), \qquad g_i(y^*) = \min_{y \in H_2} g_i(y) \quad \text{and} \quad Ax^* = By^*, \tag{1.6}$$

where H_1, H_2, and H_3 are three real Hilbert spaces, $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two linear and bounded operators, $h_i : H_1 \to \mathbb{R}$ and $g_i : H_2 \to \mathbb{R}$ are two countable families of proper, convex, and lower semicontinuous functions.

The following problems are special cases of Problem I and II.

(III) Split equality feasibility problems.

Let $C \subset H_1$ and $Q \subset H_2$ be two nonempty closed convex subsets and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. As mentioned above the so-called 'split equality feasibility problem' (SEFP) is to find

$$x^* \in C, y^* \in Q \text{ such that } Ax^* = By^*. \tag*{(1.3)**}$$

Let i_C and i_Q be the indicator functions of C and Q, respectively, *i.e.*,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C; \end{cases} \qquad i_Q(y) = \begin{cases} 0, & \text{if } y \in C, \\ +\infty, & \text{if } y \notin Q. \end{cases} \tag{1.7}$$

Denote by $N_C(x)$ and $N_Q(y)$ the *normal cones of C and Q at x and y*, respectively:

$$N_C(x) = \{z \in H_1 : \langle z, v - x\rangle \leq 0, \forall v \in C\},$$
$$N_Q(y) = \{z \in H_2 : \langle z, v - y\rangle \leq 0, \forall v \in Q\}.$$

It is easy to know that i_C and i_Q both are proper convex and lower semicontinuous functions on H_1 and H_2, respectively, and the sub-differentials ∂i_C and ∂i_Q both are maximal monotone operators. We define the resolvent operator $J_\beta^{\partial i_C}$ of i_C by

$$J_\beta^{\partial i_C}(x) = (I + \beta \partial i_C)^{-1}(x), \quad \beta > 0, x \in H_1.$$

Here

$$\begin{aligned} \partial i_C(x) &= \{z \in H_1 : i_C(x) + \langle z, u - x\rangle \leq i_C(u), \forall u \in H_1\} \\ &= \{z \in H_1 : \langle z, u - x\rangle \leq 0, \forall u \in C\} = N_C(x), \quad x \in C. \end{aligned}$$

Hence we have

$$\begin{aligned} u = J_\beta^{\partial i_C}(x) \quad &\Leftrightarrow \quad x - u \in \beta N_C(u) \\ &\Leftrightarrow \quad \langle x - u, y - u\rangle \leq 0, \quad \forall y \in C \quad \Leftrightarrow \quad u = P_C(x). \end{aligned}$$

This implies that $J_\beta^{\partial i_C} = P_C$ for any $\beta > 0$. Similarly, we also have $\partial i_Q(y) = N_Q(y)$, and $J_\beta^{\partial i_Q} = P_Q$ for any $\beta > 0$. Therefore the (SEFP) (1.3) is equivalent to the following *split equality* *optimization problem*, *i.e.*, to find $x^* \in H_1$, and $y^* \in H_2$ such that

$$\begin{aligned} & i_C(x^*) = \min_{x \in H_1} i_C(x), \qquad i_Q(y^*) = \min_{y \in H_2} i_Q(y) \quad \text{and} \quad Ax^* = By^*; \\ & \quad \Leftrightarrow \quad 0 \in \partial i_C(x^*), \qquad 0 \in \partial i_Q(y^*) \quad \text{and} \quad Ax^* = By^*. \end{aligned}$$

(IV) *Split equality equilibrium problem.*

Let D be a nonempty closed and convex subset of a real Hilbert space H. A bifunction $g : D \times D \to (-\infty, +\infty)$ is said to be a *equilibrium function*, if it satisfies the following conditions:

(A1) $g(x,x) = 0$, for all $x \in D$;

(A2) g is monotone, *i.e.*, $g(x,y) + g(y,x) \leq 0$ for all $x, y \in D$;

(A3) $\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x,y)$ for all $x, y, z \in D$;

(A4) for each $x \in D$, $y \mapsto g(x,y)$ is convex and lower semicontinuous.

The so-called *equilibrium problem with respect to the equilibrium function g* is

$$\text{to find } x^* \in D \text{ such that } g(x^*, y) \geq 0, \quad \forall y \in D. \tag{1.8}$$

Its solution set is denoted by $EP(g)$.

For given $\lambda > 0$ and $x \in H$, the *resolvent of the equilibrium function g* is the operator $R_{\lambda,g} : H \to D$ defined by

$$R_{\lambda,g}(x) := \left\{ z \in D : g(z,y) + \frac{1}{\lambda} \langle y - z, z - x \rangle \geq 0, \forall y \in D \right\}. \tag{1.9}$$

Proposition 1.2 [18] *The resolvent operator $R_{\lambda,g}$ of the equilibrium function g has the following properties*:

(1) *$R_{\lambda,g}$ is single-valued*;

(2) *$F(R_{\lambda,g}) = EP(g)$ and $EP(g)$ is a nonempty closed and convex subset of D*;

(3) *$R_{\lambda,g}$ is a firmly nonexpansive mapping.*

Let $h, g : D \times D \to (-\infty, +\infty)$ be two equilibrium functions. For given $\lambda > 0$, let $R_{\lambda,h}$ and $R_{\lambda,g}$ be the resolvent of h and g (defined by (1.9)), respectively.

The so-called *split equality equilibrium problem with respective to h, g, and D* (*SEEP*(h, g, D)) is to find $x^* \in D$, $y^* \in D$ such that

$$h(x^*, u) \geq 0, \quad \forall u \in D, \qquad g(y^*, v) \geq 0, \quad \forall v \in D \quad \text{and} \quad Ax^* = By^*, \tag{1.10}$$

where $A, B : D \to D$ are two linear and bounded operators.

By Proposition 1.2, the (*SEEP*(h, g, D)) (1.10) is equivalent to find $x^* \in D$, $y^* \in D$ such that for each $\lambda > 0$

$$\begin{aligned} & x^* \in EP(h, D), \qquad y^* \in EP(g, D) \quad \text{and} \quad Ax^* = By^* \\ & \quad \Leftrightarrow \quad x^* \in F(R_{\lambda h}), \qquad y^* \in F(R_{\lambda g}) \quad \text{and} \quad Ax^* = By^*. \end{aligned}$$

Letting $C = F(R_{\lambda h})$, $Q = F(R_{\lambda g})$, by Proposition 1.2, C and Q both are nonempty closed and convex subset of D. Hence the problem (1.10) is equivalent to the following split equality feasibility problem:

$$\text{to find } x^* \in C, y^* \in Q \text{ such that } Ax^* = By^*. \tag{1.11}$$

(V) *Split optimization problem.*

Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operators, $h : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ be two proper convex and lower semicontinuous functions. The *split optimization problem* (SOP) is to find $x^* \in H_1$, $Ax^* \in H_2$ such that

$$h(x^*) = \min_{x \in H_1} h_i(x) \quad \text{and} \quad g(Ax^*) = \min_{z \in H_2} g(z). \tag{1.12}$$

Denote by $U = \partial h$ and $K = \partial g$, then the (SOP) (1.12) is equivalent to the following *split variational inclusion problem (SVIP)*: to find $x^* \in H_1$ such that

$$0 \in U(x^*), \qquad 0 \in K(Ax^*). \tag{1.13}$$

For solving (GSEVIP) (1.5) and (GSEOP) (1.6), in Sections 3 and 4, we propose a new simultaneous type iterative algorithm. Under suitable conditions some strong convergence theorems for the sequences generated by the algorithm are proved in the setting of infinite-dimensional Hilbert spaces. As special cases, we shall utilize our results to study the split feasibility problem, split equality equilibrium problem and the split optimization problem. By the way, we obtain a strongly convergent iterative sequence to a solution of the problem (1.3), which provides an affirmative answer to the open question raised by Moudafi [11]. The results presented in the paper extend and improve the corresponding results announced by Moudafi *et al.* [11, 12, 19], Eslamian and Latif [13], Chen *et al.* [14], Censor *et al.* [1, 3–5, 20], Chuang [15], Naraghirad [21], Chang and Wang [16], Ansari and Rehan [17], and some others.

2 Preliminaries

We first recall some definitions, notations, and conclusions.

Throughout this paper, we assume that H is a real Hilbert space and C is a nonempty closed convex subset of H. In the sequel, we denote by $F(T)$ the set of fixed points of a mapping T and by $x_n \to x^*$ and $x_n \rightharpoonup x^*$, the strong convergence, and weak convergence of a sequence $\{x_n\}$ to a point x^*, respectively.

Recall that a mapping $T : H \to H$ is said to be nonexpansive, if $\|Tx - Ty\| \le \|x - y\|$, $\forall x, y \in H$. A typical example of nonexpansive mapping is the metric projection P_C from H onto $C \subseteq H$ defined by $\|x - P_C x\| = \inf_{y \in C} \|x - y\|$. The metric projection P_C is *firmly nonexpansive*, if

$$\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle \quad \forall x, y \in H, \tag{2.1}$$

and it can be characterized by the fact that

$$P_C(x) \in C \quad \text{and} \quad \langle y - P_C(x), x - P_C(x)\rangle \le 0, \quad \forall x \in H, y \in C. \tag{2.2}$$

A mapping $T : H \to H$ is said to be *quasinonexpansive*, if $F(T) \ne \emptyset$, and

$$\|Tx - p\| \le \|x - p\|, \quad \text{for each } x \in H \text{ and } p \in F(T).$$

It is easy to see that if T is a quasi-nonexpansive mapping, then $F(T)$ is a closed and convex subset of C. Besides, T is said to be a *firmly nonexpansive*, if

$$\begin{aligned} &\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle \quad \forall x, y \in C \\ &\quad \Leftrightarrow \quad \|Tx - Ty\|^2 \leq \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2 \quad \forall x, y \in C. \end{aligned}$$

Lemma 2.1 [22] *Let H be a real Hilbert space, and $\{x_n\}$ be a sequence in H. Then, for any given sequence $\{\lambda_n\}$ of positive numbers with $\sum_{i=1}^{\infty} \lambda_n = 1$ for any positive integers i, j with $i < j$ the following holds*:

$$\left\|\sum_{i=1}^{\infty} \lambda_n x_n\right\|^2 \leq \sum_{i=1}^{\infty} \lambda_n \|x_n\|^2 - \lambda_i \lambda_j \|x_i - x_j\|^2.$$

Lemma 2.2 [23] *Let H be a real Hilbert space. For any $x, y \in H$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3 [24] *Let $\{t_n\}$ be a sequence of real numbers. If there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \geq 1$, then there exists a nondecreasing sequence $\{\tau(n)\}$ with $\tau(n) \to \infty$ such that for all* (*sufficiently large*) *positive integer number n, the following holds*:

$$t_{\tau(n)} \leq t_{\tau(n)+1}, \qquad t_n \leq t_{\tau(n)+1}.$$

In fact,

$$\tau(n) = \max\{k \leq n : t_k \leq t_{k+1}\}.$$

Definition 2.4 (Demiclosedness principle) Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a mapping with $F(T) \neq \emptyset$. Then $I - T$ is said to be *demiclosed at zero*, if for any sequence $\{x_n\} \subset C$ with $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, $x = Tx$.

Remark 2.5 [25] It is well known that if $T : C \to C$ is a nonexpansive mapping, then $I - T$ is demiclosed at zero.

Lemma 2.6 *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of positive real numbers satisfying $a_{n+1} \leq (1 - b_n)a_n + c_n$ for all $n \geq 1$. If the following conditions are satisfied*:

(1) $b_n \in (0, 1)$ *and* $\sum_{n=1}^{\infty} b_n = \infty$,

(2) $\sum_{n=1}^{\infty} c_n < \infty$, *or* $\limsup_{n\to\infty} \frac{c_n}{b_n} \leq 0$,

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 [15] *Let H be a real Hilbert space, $B : H \to 2^H$ be a set-valued maximal monotone mapping, $\beta > 0$, and let J_β^B be the resolvent mapping of B defined by $J_\beta^B := (I + \beta B)^{-1}$, then*

(i) *for each $\beta > 0$, J_β^B is a single-valued and firmly nonexpansive mapping*;

(ii) *$D(J_\beta^B) = H$ and $F(J_\beta^B) = B^{-1}(0)$;*
(iii) *$(I - J_\beta^B)$ is a firmly nonexpansive mapping for each $\beta > 0$;*
(iv) *suppose that $B^{-1}(0) \neq \emptyset$, then for each $x \in H$, each $x^* \in B^{-1}(0)$ and each $\beta > 0$*

$$\left\| x - J_\beta^B x \right\|^2 + \left\| J_\beta^B x - x^* \right\| \leq \left\| x - x^* \right\|^2;$$

(v) *suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_\beta^B x, J_\beta^B x - w\rangle \geq 0$ for each $x \in H$, each $w \in B^{-1}(0)$, and each $\beta > 0$.*

Lemma 2.8 *Let H_1, H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear bounded operator and A^* be the adjoint of A. Let $B : H_2 \to 2^{H_2}$ be a set-valued maximal monotone mapping, $\beta > 0$, and let J_β^B be the resolvent mapping of B, then*
(i) *$\|(I - J_\beta^B)Ax - (I - J_\beta^B)Ay\|^2 \leq \langle (I - J_\beta^B)Ax - (I - J_\beta^B)Ay, Ax - Ay\rangle$;*
(ii) *$\|A^*(I - J_\beta^B)Ax - A^*(I - J_\beta^B)Ay\|^2 \leq \|A\|^2 \langle (I - J_\beta^B)Ax - (I - J_\beta^B)Ay, Ax - Ay\rangle$;*
(iii) *if $\rho \in (0, \frac{2}{\|A\|^2})$, then $(I - \rho A^*(I - J_\beta^B)A)$ is a nonexpansive mapping.*

Proof By Lemma 2.7(iii), the mapping $(I - J_\beta^B)$ is firmly nonexpansive, hence the conclusions (i) and (ii) are obvious.

Now we prove the conclusion (iii).

In fact, for any $x, y \in H_1$, it follows from the conclusions (i) and (ii) that

$$\begin{aligned}
&\left\| \left(I - \rho A^*\left(I - J_\beta^B\right)A\right)x - \left(I - \rho A^*\left(I - J_\beta^B\right)A\right)y \right\|^2 \\
&\quad = \|x - y\|^2 - 2\rho\left\langle x - y, A^*\left(I - J_\beta^B\right)Ax - A^*\left(I - J_\beta^B\right)Ay\right\rangle \\
&\qquad + \rho^2 \left\| A^*\left(I - J_\beta^B\right)Ax - A^*\left(I - J_\beta^B\right)Ay \right\|^2 \\
&\quad \leq \|x - y\|^2 - 2\rho\left\langle Ax - Ay, \left(I - J_\beta^B\right)Ax - \left(I - J_\beta^B\right)Ay\right\rangle \\
&\qquad + \rho^2 \|A\|^2 \left\| \left(I - J_\beta^B\right)Ax - \left(I - J_\beta^B\right)Ay \right\|^2 \\
&\quad \leq \|x - y\|^2 - \rho\left(2 - \rho\|A\|^2\right) \left\| \left(I - J_\beta^B\right)Ax - \left(I - J_\beta^B\right)Ay \right\|^2 \\
&\quad \leq \|x - y\|^2 \quad \left(\text{since } \rho\left(2 - \rho\|A\|^2\right) \geq 0\right).
\end{aligned}$$

This completes the proof of Lemma 2.8. □

3 General split equality variational inclusion problem and strong convergence theorems

Throughout this section we assume that
(1) H_1, H_2, H_3 are three real Hilbert spaces;
(2) $\{U_i\}_{i=1}^\infty : H_1 \to 2^{H_1}$ and $\{K_i\}_{i=1}^\infty : H_2 \to 2^{H_2}$ are two families of set-valued maximal monotone mappings, $\beta > 0$ and $\gamma > 0$ are given positive numbers;
(3) $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators and A^*, B^* are the adjoint of A and B, respectively;
(4) $f = \left[\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right]$, where f_i, $i = 1, 2$ is a k-contractive mapping on H_i with $k \in (0, 1)$;
(5) the set of solutions of (GSEVIP) (1.5) $\Omega \neq \emptyset$,

$$J_{\mu_i}^{(U_i, K_i)} := \begin{bmatrix} J_{\mu_i}^{U_i} \\ J_{\mu_i}^{K_i} \end{bmatrix}, \qquad G = [A \ -B], \qquad G^* = \begin{bmatrix} A^* \\ -B^* \end{bmatrix}, \qquad G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix},$$

(6) for any given $w_0 \in H_1 \times H_2$, the iterative sequence $\{w_n\} \subset H_1 \times H_2$ is generated by

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \sum_{i=1}^{\infty} \gamma_{n,i}\big(J_{\mu_i}^{(U_i,K_i)}\big(I - \lambda_{n,i} G^* G\big) w_n\big), \quad n \geq 0, \tag{3.1}$$

or its equivalent form:

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n f_1(x_n) + \sum_{i=1}^{\infty} \gamma_{n,i}(J_{\mu_i}^{U_i}(x_n - \lambda_{n,i}(A^*(Ax_n - By_n)))), \\ y_{n+1} = \alpha_n y_n + \beta_n f_2(y_n) + \sum_{i=1}^{\infty} \gamma_{n,i}(J_{\mu_i}^{K_i}(y_n + \lambda_{n,i}(B^*(Ax_n - By_n)))), \end{cases} \tag{3.1$'$}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_{n,i}\}$ are the sequences of nonnegative numbers satisfying

$$\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1, \quad \text{for each } n \geq 0.$$

We are now in a position to give the following results.

Lemma 3.1 *Let H_1, H_2, H_3, A, B, A^*, B^*, $\{U_i\}$, $\{K_i\}$, $J_{\mu_i}^{(U_i,K_i)}$, G, G^* be the same as above. If $\Omega \neq \emptyset$ (the solution set of (GSEVIP) (1.5)), then $w^* := (x^*, y^*) \in H_1 \times H_2$ is a solution of (GSEVIP) (1.5) if and only if for each $i \geq 1$, and for any given $\gamma > 0$ and $\mu > 0$*

$$w^* = J_{\mu}^{(U_i,K_i)}\big(I - \gamma G^* G\big) w^*. \tag{3.2}$$

Proof Indeed, if $w^* = (x^*, y^*) \in H_1 \times H_2$ is a solution of (GSEVIP) (1.5), then by Lemma 2.7(ii), for each $i \geq 1$, and for any $\gamma > 0$ and $\mu > 0$ we have

$$\begin{aligned} & x^* \in U_i^{-1}(0) = F\big(J_{\mu}^{U_i}\big), \qquad y^* \in K_i^{-1}(0) = F\big(J_{\mu}^{K_i}\big) \quad \text{and} \quad Ax^* = By^* \\ & \quad \Leftrightarrow \quad x^* = J_{\mu}^{U_i} x^*, \qquad y^* = J_{\mu}^{K_i} y^* \quad \text{and} \quad Ax^* = By^*. \end{aligned}$$

Hence we have $G(w^*) = Ax^* - By^* = 0$, and so

$$J_{\mu_i}^{(U_i,K_i)}\big(I - \gamma G^* G\big)\big(w^*\big) = J_{\mu}^{(U_i,K_i)}\big(w^*\big) = \big(J_{\mu}^{U_i} x^*, J_{\mu}^{K_i} y^*\big) = w^*.$$

This implies that (3.2) is true.

Conversely, if $w^* = (x^*, y^*) \in H_1 \times H_2$ satisfies (3.2), then we have

$$\begin{cases} x^* = J_{\mu}^{U_i}[x^* - \gamma A^*(Ax^* - By^*)], \\ y^* = J_{\mu}^{K_i}[y^* + \gamma B^*(Ax^* - By^*)]. \end{cases} \tag{3.3}$$

We make the assumption that the solution set Ω of (GSEVIP) (1.5) is nonempty. Hence the sets $U_i^{-1}(0)$ and $K_i^{-1}(0)$ both are nonempty. By Lemma 2.7(v) and (3.3), we have

$$\big\langle x^* - \big(x^* - \gamma A^*\big(Ax^* - By^*\big)\big), x - x^*\big\rangle \geq 0, \quad \forall x \in U_i^{-1}(0),$$

and so

$$\big\langle Ax^* - By^*, Ax - Ax^*\big\rangle \geq 0, \quad \forall x \in U_i^{-1}(0). \tag{3.4}$$

Similarly, by Lemma 2.7(v) and (3.3) again, one gets

$$\langle Ax^* - By^*, By^* - By\rangle \geq 0, \quad \forall y \in K_i^{-1}(0). \tag{3.5}$$

Adding up (3.4) and (3.5), we have

$$\langle Ax^* - By^*, Ax - Ax^* + By^* - By\rangle \geq 0, \quad \forall x \in U_i^{-1}(0) \text{ and } y \in K_i^{-1}(0).$$

Simplifying it, we have

$$\|Ax^* - By^*\|^2 \leq \langle Ax^* - By^*, Ax - By\rangle, \quad \forall x \in U_i^{-1}(0) \text{ and } y \in K_i^{-1}(0). \tag{3.6}$$

Since $\Omega \neq \emptyset$, taking $\bar{w} = (\bar{x}, \bar{y}) \in \Omega$, for each $i \geq 1$, we have $\bar{x} \in U_i^{-1}(0)$ and $\bar{y} \in K_i^{-1}(0)$ and $A\bar{x} = B\bar{y}$. In (3.6), taking $x = \bar{x}$ and $y = \bar{y}$, we have

$$\|Ax^* - By^*\| = 0, \quad \textit{i.e.}, \ Ax^* = By^*. \tag{3.7}$$

Hence from (3.3) and (3.7)

$$\begin{cases} x^* = J_\mu^{U_i}(x^*), \\ y^* = J_\mu^{K_i}(y^*), \end{cases} \quad \Leftrightarrow \quad 0 \in U_i(x^*), \qquad 0 \in K_i(y^*), \quad \forall i \geq 1. \tag{3.8}$$

It follows from (3.7) and (3.8) that w^* is a solution of (GSEVIP) (1.5).

This completes the proof of Lemma 3.1. □

Lemma 3.2 *If $\lambda \in (0, \frac{2}{L})$, where $L = \|G\|^2$, then $(I - \lambda G^* G) : H_1 \times H_2 \to H_1 \times H_2$ is a non-expansive mapping.*

Proof In fact, for any $w, u \in H_1 \times H_2$, we have

$$\begin{aligned} &\|(I - \lambda G^*G)u - (I - \lambda G^*G)w\|^2 \\ &\quad = \|(u - w) - \lambda G^*G(u - w)\|^2 \\ &\quad = \|u - w\|^2 + \lambda^2\|G^*G(u - w)\|^2 - 2\lambda\langle u - w, G^*G(u - w)\rangle \\ &\quad \leq \|u - w\|^2 + \lambda^2 L\|G(u - w)\|^2 - 2\lambda\langle G(u - w), G(u - w)\rangle \\ &\quad = \|u - w\|^2 + \lambda^2 L\|G(u - w)\|^2 - 2\lambda\|G(u - w)\|^2 \\ &\quad = \|u - w\|^2 - \lambda(2 - \lambda L)\|G(u - w)\|^2 \\ &\quad \leq \|u - w\|^2. \end{aligned}$$

This completes the proof. □

Theorem 3.3 *Let $H_1, H_2, H_3, A, B, A^*, B^*, \{U_i\}, \{K_i\}, J_{\mu_i}^{(U_i, K_i)}, G, G^*, f$ be the same as above. Let $\{w_n\}$ be the sequence defined by* (3.1). *If the solution set Ω of* (*GSEVIP*) (1.5) *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$, *for each* $n \geq 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\liminf_{n\to\infty} \alpha_n \gamma_{n,i} > 0$ *for each* $i \geq 1$;
(iv) $\{\lambda_{n,i}\} \subset (0, \frac{2}{L})$ *for each* $i \geq 1$, *where* $L = \|G\|^2$,

then the sequence $\{w_n\}$ *converges strongly to* $w^* = P_\Omega f(w^*)$, *which is a solution of* (*GSEVIP*) (1.5).

Proof (I) First we prove that the sequence $\{w_n\}$ is bounded.

In fact, for any given $z \in \Omega$, it follows from Lemma 3.1, Lemma 3.2, and condition (iv) that

$$z = J_\mu^{(U_i,K_i)}\big(I - \lambda_{n,i} G^* G\big)z, \quad \text{for each } i \geq 1,$$

and $(I - \lambda_{n,i} G^* G) : H_1 \times H_2 \to H_1 \times H_2$ is a nonexpansive mapping. Also by Lemma 2.7(i), for each $i \geq 1$, $J_{\mu_i}^{(U_i,K_i)}$ is a firmly nonexpansive mapping. Hence we have

$$\begin{aligned}
\|w_{n+1} - z\| &= \left\| \left(\alpha_n w_n + \beta_n f(w_n) + \sum_{i=1}^{\infty} \gamma_{n,i} J_{\mu_i}^{(U_i,K_i)}\big(I - \lambda_{n,i} G^* G\big) w_n\right) - z \right\| \\
&\leq \alpha_n \|w_n - z\| + \beta_n \big\|f(w_n) - z\big\| + \sum_{i=1}^{\infty} \gamma_{n,i} \big\|J_{\mu_i}^{(U_i,K_i)}\big(I - \lambda_{n,i} G^* G\big) w_n - z\big\| \\
&\leq \alpha_n \|w_n - z\| + \beta_n \big\|f(w_n) - z\big\| + \sum_{i=1}^{\infty} \gamma_{n,i} \big\|\big(I - \lambda_{n,i} G^* G\big) w_n - z\big\| \\
&\leq \alpha_n \|w_n - z\| + \beta_n \big\|f(w_n) - z\big\| \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i} \big\|\big(I - \lambda_{n,i} G^* G\big) w_n - \big(I - \lambda_{n,i} G^* G\big) z\big\| \\
&\leq \alpha_n \|w_n - z\| + \beta_n \big\|f(w_n) - z\big\| + \sum_{i=1}^{\infty} \gamma_{n,i} \|w_n - z\| \\
&= (1 - \beta_n)\|w_n - z\| + \beta_n \big\|f(w_n) - z\big\| \\
&\leq (1 - \beta_n)\|w_n - z\| + \beta_n \big\|f(w_n) - f(z)\big\| + \beta_n \big\|f(z) - z\big\| \\
&\leq (1 - \beta_n)\|w_n - z\| + k\beta_n \|w_n - z\| + \beta_n \big\|f(z) - z\big\| \\
&= \big(1 - (1 - k)\beta_n\big)\|w_n - z\| + (1 - k)\beta_n \frac{1}{1 - k} \big\|f(z) - z\big\| \\
&\leq \max\left\{ \|w_n - z\|, \frac{1}{1 - k} \big\|f(z) - z\big\| \right\}.
\end{aligned}$$

By induction, we can prove that

$$\|w_n - z\| \leq \max\left\{ \|w_0 - z\|, \frac{1}{1 - k} \big\|f(z) - z\big\| \right\}, \quad \forall n \geq 0.$$

This shows that $\{w_n\}$ is bounded, and so is $\{f(w_n)\}$.

(II) Now we prove that the following inequality holds:

$$\begin{aligned}
&\alpha_n \gamma_{n,i} \big\|w_n - J_{\mu_i}^{(U_i,K_i)}\big(I - \lambda_{n,i} G^* G\big) w_n\big\|^2 \\
&\quad \leq \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + \beta_n \big\|f(w_n) - z\big\|^2 \quad \text{for each } i \geq 1.
\end{aligned} \tag{3.9}$$

Indeed, it follows from (3.1) and Lemma 2.1 that for each $i \geq 1$

$$\begin{aligned}
\|w_{n+1} - z\|^2 &= \left\| \alpha_n(w_n - z) + \beta_n\bigl(f(w_n) - z\bigr) + \sum_{j=1}^{\infty} \gamma_{n,j}\bigl(J_{\mu_j}^{(U_j,K_j)}\bigl(I - \lambda_{n,j}G^*G\bigr)w_n - z\bigr) \right\|^2 \\
&\leq \alpha_n\|w_n - z\|^2 + \beta_n\bigl\|f(w_n) - z\bigr\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j}\bigl\|J_{\mu_j}^{(U_j,K_j)}\bigl(I - \lambda_{n,j}G^*G\bigr)w_n - z\bigr\|^2 \\
&\quad - \alpha_n\gamma_{n,i}\bigl\|w_n - J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n\bigr\|^2 \\
&\leq \alpha_n\|w_n - z\|^2 + \beta_n\bigl\|f(w_n) - z\bigr\|^2 + \sum_{j=1}^{\infty} \gamma_{n,j}\|w_n - z\|^2 \\
&\quad - \alpha_n\gamma_{n,i}\bigl\|w_n - J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n\bigr\|^2 \\
&= (1 - \beta_n)\|w_n - z\|^2 + \beta_n\bigl\|f(w_n) - z\bigr\|^2 \\
&\quad - \alpha_n\gamma_{n,i}\bigl\|w_n - J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n\bigr\|^2.
\end{aligned}$$

This implies that for each $i \geq 1$

$$\alpha_n\gamma_{n,i}\bigl\|w_n - J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n\bigr\|^2 \leq \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + \beta_n\bigl\|f(w_n) - z\bigr\|^2.$$

Inequality (3.3) is proved.

It is easy to see that the solution set Ω of (GSEVIP) (1.5) is a closed and convex subset in $H_1 \times H_2$. By the assumption that Ω is nonempty, so it is a nonempty closed and convex subset in $H_1 \times H_2$. Hence the metric projection P_Ω is well defined. In addition, since $P_\Omega f : H_1 \times H_2 \to \Omega$ is a contractive mapping, there exists a unique $w^* \in \Omega$ such that

$$w^* = P_\Omega f\bigl(w^*\bigr). \tag{3.10}$$

(III) Now we prove that $\{w_n\}$ converges strongly to w^*.

For the purpose, we consider two cases.

Case I. Suppose that the sequence $\{\|w_n - w^*\|\}$ is monotone. Since $\{\|w_n - w^*\|\}$ is bounded, $\{\|w_n - w^*\|\}$ is convergent. Since $w^* \in \Omega$, in (3.9) taking $z = w^*$ and letting $n \to \infty$, in view of conditions (ii) and (iii), we have

$$\lim_{n\to\infty} \bigl\|w_n - J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n\bigr\| = 0, \quad \text{for each } i \geq 1. \tag{3.11}$$

On the other hand, by Lemma 2.2 and (3.1), we have

$$\begin{aligned}
\bigl\|w_{n+1} - w^*\bigr\|^2 &= \left\| \left(\alpha_n w_n + \beta_n f(w_n) + \sum_{i=1}^{\infty} \gamma_{n,i} J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n\right) - w^* \right\|^2 \\
&= \Bigl\| \alpha_n\bigl(w_n - w^*\bigr) + \beta_n\bigl(f(w_n) - w^*\bigr) \\
&\quad + \sum_{i=1}^{\infty} \gamma_{n,i}\bigl(J_{\mu_i}^{(U_i,K_i)}\bigl(I - \lambda_{n,i}G^*G\bigr)w_n - w^*\bigr) \Bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \left\| \alpha_n(w_n - w^*) + \sum_{i=1}^{\infty} \gamma_{n,i}\bigl(J_{\mu_i}^{(U_i,K_i)}(I - \lambda_{n,i}G^*G)w_n - w^*\bigr)\right\|^2 \\
&\quad + 2\beta_n\langle f(w_n) - w^*, w_{n+1} - w^*\rangle \quad \text{(by Lemma 2.2)} \\
&\le \left\{\alpha_n\|w_n - w^*\| + \sum_{i=1}^{\infty}\gamma_{n,i}\|w_n - w^*\|\right\}^2 \\
&\quad + 2\beta_n\langle f(w_n) - f(w^*), w_{n+1} - w^*\rangle + 2\beta_n\langle f(w^*) - w^*, w_{n+1} - w^*\rangle \\
&= (1-\beta_n)^2\|w_n - w^*\|^2 + 2\beta_n k\|w_n - w^*\|\|w_{n+1} - w^*\| \\
&\quad + 2\beta_n\langle f(w^*) - w^*, w_{n+1} - w^*\rangle \\
&\le (1-\beta_n)^2\|w_n - w^*\|^2 + \beta_n k\{\|w_n - w^*\|^2 + \|w_{n+1} - w^*\|^2\} \\
&\quad + 2\beta_n\langle f(w^*) - w^*, w_{n+1} - w^*\rangle.
\end{aligned}
$$

Simplifying it we have

$$
\begin{aligned}
\|w_{n+1} - w^*\|^2 &\le \frac{(1-\beta_n)^2 + \beta_n k}{1-\beta_n k}\|w_n - w^*\|^2 + \frac{2\beta_n}{1-\beta_n k}\langle f(w^*) - w^*, w_{n+1} - w^*\rangle \\
&= \frac{1 - 2\beta_n + \beta_n k}{1-\beta_n k}\|w_n - w^*\|^2 + \frac{\beta_n^2}{1-\beta_n k}\|w_n - w^*\|^2 \\
&\quad + \frac{2\beta_n}{1-\beta_n k}\langle f(w^*) - w^*, w_{n+1} - w^*\rangle \\
&= \left(1 - \frac{2(1-k)\beta_n}{1-\beta_n k}\right)\|w_n - w^*\|^2 \\
&\quad + \frac{2(1-k)\beta_n}{1-\beta_n k}\left\{\frac{\beta_n M}{2(1-k)} + \frac{1}{1-k}\langle f(w^*) - w^*, w_{n+1} - w^*\rangle\right\} \\
&= (1-\eta_n)\|w_n - w^*\|^2 + \eta_n\delta_n,
\end{aligned}
\tag{3.12}
$$

where

$$
\eta_n = \frac{2(1-k)\beta_n}{1-\beta_n k}, \qquad \delta_n = \frac{\beta_n M}{2(1-k)} + \frac{1}{1-k}\langle f(w^*) - w^*, w_{n+1} - w^*\rangle,
$$
$$
M = \sup_{n\ge 0}\|w_n - w^*\|^2.
$$

By condition (ii), $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$, and so is $\sum_{n=1}^{\infty}\eta_n = \infty$.

Next we prove that

$$
\limsup_{n\to\infty}\delta_n \le 0. \tag{3.13}
$$

In fact, since $\{w_n\}$ is bounded in $H_1 \times H_2$, there exists a subsequence $\{w_{n_k}\} \subset \{w_n\}$ with $w_{n_k} \rightharpoonup v^*$ (some point in $C \times Q$), and $\lambda_{n_k,i} \to \lambda_i \in (0, \frac{2}{L})$ such that

$$
\lim_{n\to\infty}\langle f(w^*) - w^*, w_{n_k} - w^*\rangle = \limsup_{n\to\infty}\langle f(w^*) - w^*, w_n - w^*\rangle.
$$

Since

$$\left\| w_{n_k} - J_{\mu_i}^{(U_i,K_i)}\left(I - \lambda_{n_k,i}G^*G\right)w_{n_k}\right\| \to 0, \quad \text{for each } i \geq 1$$

and $J_{\mu_i}^{(U_i,K_i)}(I - \lambda_{n_k,i}G^*G)$ is a nonexpansive mapping, by Remark 2.5, $I - J_{\mu_i}^{(B_i,K_i)}(I - \lambda_{n,i}G^*G)$ is demiclosed at zero, hence we have

$$v^* = J_{\mu_i}^{(U_i,K_i)}\left(I - \lambda_{n,i}G^*G\right)v^*, \quad \forall i \geq 1. \tag{3.14}$$

By Lemma 3.1, this implies that $v^* \in \Omega$. In addition, since $w^* = P_\Omega f(w^*)$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f\left(w^*\right) - w^*, w_n - w^*\rangle &= \lim_{n\to\infty}\langle f\left(w^*\right) - w^*, w_{n_k} - w^*\rangle \\ &= \langle f\left(w^*\right) - w^*, v^* - w^*\rangle \leq 0.\end{aligned}$$

This shows that (3.13) is true. Taking $a_n = \|w_n - w^*\|^2$, $b_n = \eta_n$, and $c_n = \delta_n\eta_n$ in Lemma 2.6, therefore all conditions in Lemma 2.6 are satisfied. We have $w_n \to w^*$.

Case II. If the sequence $\{\|w_n - w^*\|\}$ is not monotone, by Lemma 2.3, there exists a sequence of positive integers: $\{\tau(n)\}$, $n \geq n_0$ (where n_0 large enough) such that

$$\tau(n) = \max\left\{k \leq n : \left\|w_k - w^*\right\| \leq \left\|w_{k+1} - w^*\right\|\right\}. \tag{3.15}$$

Clearly $\{\tau(n)\}$ is a nondecreasing, $\tau(n) \to \infty$ as $n \to \infty$, and for all $n \geq n_0$

$$\left\|w_{\tau(n)} - w^*\right\| \leq \left\|w_{\tau(n)+1} - w^*\right\|; \qquad \left\|w_n - w^*\right\| \leq \left\|w_{\tau(n)+1} - w^*\right\|. \tag{3.16}$$

Therefore $\{\|w_{\tau(n)} - w^*\|\}$ is a nondecreasing sequence. According to Case I, $\lim_{n\to\infty}\|w_{\tau(n)} - w^*\| = 0$ and $\lim_{n\to\infty}\|w_{\tau(n)+1} - w^*\| = 0$. Hence we have

$$0 \leq \left\|w_n - w^*\right\| \leq \max\left\{\left\|w_n - w^*\right\|, \left\|w_{\tau(n)} - w^*\right\|\right\} \leq \left\|w_{\tau(n)+1} - w^*\right\| \to 0, \quad \text{as } n \to \infty.$$

This implies that $w_n \to w^*$ and $w^* = P_\Omega f(w^*)$ is a solution of (GSEVIP) (1.5).

This completes the proof of Theorem 3.3. □

Remark 3.4 Theorem 3.3 extends and improves the main results in Moudafi *et al.* [11, 12, 19], Eslamian and Latif [13], Chen *et al.* [14], Chuang [15], Naraghirad [21] and Ansari and Rehan [17].

4 General split equality optimization problem and strong convergence theorems

Let H_1, H_2, and H_3 be three real Hilbert spaces. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two linear and bounded operators. The so-called *general split equality optimization problem* (GSEOP) is to find $x^* \in H_1$, and $y^* \in H_2$ such that for each $i \geq 1$

$$h_i\left(x^*\right) = \min_{x\in H_1} h_i(x), \qquad g_i\left(y^*\right) = \min_{z\in H_2} g_i(z) \quad \text{and} \quad Ax^* = By^*, \tag{4.1}$$

where $h_i : H_1 \to \mathbb{R}$ and $g_i : H_2 \to \mathbb{R}$ are two families of proper, lower semicontinuous, and convex functions.

For each $i \geq 1$ denote by $\partial h_i = U_i$ and $\partial g_i = K_i$. Then the mappings $U_i : H_1 \to 2^{H_1}$ and $K_i : H_2 \to 2^{H_2}$, $i = 1, 2, \ldots$ both are set-valued maximal monotone mappings, and

$$h_i(x^*) = \min_{x \in H_1} h_i(x) \quad \Leftrightarrow \quad 0 \in \partial h_i(x^*) = U_i(x^*),$$
$$g_i(y^*) = \min_{z \in H_2} g_i(z) \quad \Leftrightarrow \quad 0 \in \partial g_i(y^*) = K_i(y^*).$$

Therefore (GSEOP) (4.1) is equivalent to the following general split equality variational inclusion problem (GSEVIP): to find $x^* \in H_1$ and $y^* \in H_2$ such that

$$0 \in \bigcap_{i=1}^{\infty} U_i(x^*), \qquad 0 \in \bigcap_{i=1}^{\infty} K_i(y^*) \quad \text{and} \quad Ax^* = By^*. \tag{4.2}$$

Therefore, the following theorem can be obtained from Theorem 3.3 immediately.

Theorem 4.1 *Let H_1, H_2, H_3, A, B, A^*, B^*, $\{U_i\}$, $\{K_i\}$ be the same as above. Let $J_{\mu_i}^{(U_i,K_i)}$, G, G^*, f be the same as in Theorem* 3.3. *Let $\{w_n\}$ be the sequence defined by* (3.1). *If the solution set Ω_1 of* (*GSEVIP*) (4.1) *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n + \beta_n + \sum_{i=1}^{\infty} \gamma_{n,i} = 1$, *for each* $n \geq 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(iii) $\liminf_{n\to\infty} \alpha_n \gamma_{n,i} > 0$ *for each* $i \geq 1$;

(iv) $\{\lambda_{n,i}\} \subset (0, \frac{2}{L})$ *for each* $i \geq 1$, *where* $L = \|G\|^2$,

then the sequence $\{w_n\}$ converges strongly to $w^* = P_{\Omega_1} f(w^*)$, which is a solution of (*GSEOP*) (4.1).

By using Theorem 3.3 and Theorem 4.1, now we give some corollaries for the split equality feasibility problem, the split equality equilibrium problem, and the split optimization problem.

Let H_1, H_2, H_3, C, Q, A, B be the same as in the split equality feasibility problem (1.3). Let i_C and i_Q be the indicator function of C and Q, respectively, defined by (1.7). In Theorem 4.1, take $\{U\} = \{\partial i_C\}$, $\{K\} = \{\partial i_Q\}$, and $J_{\mu}^{(U,K)} = P_{C \times Q} := \left[\begin{smallmatrix} P_C \\ P_Q \end{smallmatrix}\right]$, therefore we have the following.

Corollary 4.2 *Let H_1, H_2, H_3, A, B, A^*, B^*, $P_{C\times Q}$ be the same as above. Let G, G^*, f be the same as in Theorem* 4.1. *Let $\{w_n\}$ be the sequence generated by $w_0 \in H_1 \times H_2$*

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \gamma_n \big(P_{C\times Q}\big(I - \lambda_n G^* G\big) w_n\big), \quad n \geq 0, \tag{4.3}$$

or its equivalent form

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n f_1(x_n) + \gamma_n (P_C(x_n - \lambda_n (A^*(Ax_n - By_n)))), \\ y_{n+1} = \alpha_n y_n + \beta_n f_2(y_n) + \gamma_n (P_Q(y_n + \lambda_n (B^*(Ax_n - By_n)))). \end{cases} \tag{4.4}$$

If the solution set Γ_1 of (*SEFP*) (1.3) *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \geq 0$;

(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;
(iv) $\{\lambda_n\} \subset (0, \frac{2}{L})$ *for each* $i \geq 1$, *where* $L = \|G\|^2$,

then the sequence $\{w_n\}$ *converges strongly to* $w^* = P_{\Gamma_1} f(w^*)$, *which is a solution of* (*SEFP*) (1.3).

Remark 4.3 Since the simultaneous iterative sequence $\{(x_n, y_n)\}$ (4.4) converges strongly to a solution of (SEFP) (1.3). Therefore it provides an *affirmative answer to Moudafi's open question 1.1* [11].

Let $h, g : D \times D \to (-\infty, +\infty)$ be two equilibrium functions. For given $\lambda > 0$, let $R_{\lambda,h}$ and $R_{\lambda,g}$ be the resolvents of h and g (defined by (1.9)), respectively.

The so-called *split equality equilibrium problem with respective to h, g, and D* (*SEEP*(h, g, D)) is to find $x^* \in D$, $y^* \in D$ such that

$$h(x^*, u) \geq 0, \quad \forall u \in D, \qquad g(y^*, v) \geq 0, \quad \forall v \in D \quad \text{and} \quad Ax^* = By^*, \tag{4.5}$$

where $A, B : D \to D$ are two linear and bounded operators.

By Proposition 1.2, the (*SEEP*(h, g, D)) (4.5) is equivalent to find $x^* \in D$, $y^* \in D$ such that for each $\lambda > 0$

$$\begin{aligned} & x^* \in EP(h, D), \qquad y^* \in EP(g, D) \quad \text{and} \quad Ax^* = By^* \\ \Leftrightarrow \quad & x^* \in F(R_{\lambda h}), \qquad y^* \in F(R_{\lambda g}) \quad \text{and} \quad Ax^* = By^*. \end{aligned} \tag{4.6}$$

Letting $C = F(R_{\lambda h})$, $Q = F(R_{\lambda g})$, by Proposition 1.2, C and Q both are nonempty closed and convex subset of D. Hence the problem (4.5) (and so the problem (4.6)) is equivalent to the following split equality feasibility problem:

$$\text{to find } x^* \in C, y^* \in Q \text{ such that } Ax^* = By^*. \tag{4.7}$$

In Corollary 4.2 taking $H_1 = H_2 = H_3 = D$, from Corollary 4.2 we have the following.

Corollary 4.4 *Let* D, C, Q *be the same as above. Let* A, B, A^*, B^*, $P_{C\times Q}$, G, G^*, f *be the same as in Corollary* 4.2. *For any given* $w_0 \in D \times D$, *let* $\{w_n\}$ *be the sequence generated by*

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \gamma_n \big(P_{C\times Q}\big(I - \lambda_n G^* G\big) w_n\big), \quad n \geq 0. \tag{4.8}$$

If the solution set Γ_2 *of* (*SEEP*(h, g, D)) (4.5) *is nonempty and the following conditions are satisfied*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \geq 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;
(iv) $\{\lambda_n\} \subset (0, \frac{2}{L})$ *for each* $i \geq 1$, *where* $L = \|G\|^2$,

then the sequence $\{w_n\}$ *converges strongly to* $w^* = P_{\Gamma_2} f(w^*)$, *which is a solution of* (*SEEP*(h, g, D)) (4.5).

Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operators, $h : H_1 \to \mathbb{R}$ and $g : H_2 \to \mathbb{R}$ be two proper convex and lower semicontinuous functions. The *split optimization problem* (SOP) is to find $x^* \in H_1$, $Ax^* \in H_2$ such that

$$h(x^*) = \min_{x\in H_1} h_i(x) \quad \text{and} \quad g(Ax^*) = \min_{z\in H_2} g(z). \tag{4.9}$$

Denote $U = \partial h$ and $K = \partial g$, then the (SOP) (4.9) is equivalent to the following *split variational inclusion problem (SVIP)*: to find $x^* \in H_1$ such that

$$0 \in U(x^*), \qquad 0 \in K(Ax^*). \tag{4.10}$$

In Theorem 4.1 taking $H_3 = H_2$, $B = I$ (the identity mapping on H_2) and

$$\tilde{G} = [A \ -I], \qquad \tilde{G}^* = \begin{bmatrix} A^* \\ -I \end{bmatrix}, \qquad \tilde{G}^*\tilde{G} = \begin{bmatrix} A^*A & -A^* \\ -A & I \end{bmatrix},$$

then from Theorem 4.1 we have the following.

Corollary 4.5 *Let H_1, H_2, A, I, $\tilde{G}$, $\tilde{G}^*$, U, K, be the same as above. Let $J_\mu^{(U,K)}$, f be the same as in Theorem* 4.1. *For any given $w_0 = (x_0, y_0) \in H_1 \times H_2$, let $\{w_n = (x_n, y_n)\}$ be the sequence defined by*

$$\begin{cases} x_{n+1} = \alpha_n x_n + \beta_n f_1(x_n) + \gamma_n J_\mu^U(x_n - \lambda_n A^*(Ax_n - y_n)), \\ y_{n+1} = \alpha_n y_n + \beta_n f_2(y_n) + \gamma_n J_\mu^K(y_n + \lambda_n(Ax_n - y_n)), \end{cases} \tag{4.11}$$

or its equivalent form:

$$w_{n+1} = \alpha_n w_n + \beta_n f(w_n) + \gamma_n \big(J_\mu^{(U,K)}\big(I - \lambda_n \tilde{G}^*\tilde{G}\big) w_n\big), \quad n \geq 0, \tag{4.12}$$

If $\Gamma_3 := \{x^* \in U^{-1}(0) \cap A^{-1}K^{-1}(0)\}$, the solution set of (*SOP*) (4.9) *is nonempty, and the following conditions are satisfied*:

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for each* $n \geq 0$;
(ii) $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;
(iii) $\liminf_{n\to\infty} \alpha_n \gamma_n > 0$;
(iv) $\{\lambda_n\} \subset (0, \frac{2}{L})$, *where* $L = \|\tilde{G}\|^2$,

then the sequence $\{w_n\}$ converges strongly to $w^* = P_{\Gamma_3} f(w^*)$, which is a solution of (*SOP*) (4.9).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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