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C^*-algebra-valued metric spaces and related fixed point theorems

Zhenhua Ma[1,2], Lining Jiang[1*] and Hongkai Sun[2]

*Correspondence:
jianglining@bit.edu.cn
[1]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, China
Full list of author information is available at the end of the article

Abstract
Based on the concept and properties of C^*-algebras, the paper introduces a concept of C^*-algebra-valued metric spaces and gives some fixed point theorems for self-maps with contractive or expansive conditions on such spaces. As applications, existence and uniqueness results for a type of integral equation and operator equation are given.

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Keywords: C^*-algebra; C^*-algebra-valued metric; contractive mapping; expansive mapping; fixed point theorem

1 Introduction

We begin with the concept of C^*-algebras.

Suppose that $\mathbb{A}$ is a unital algebra with the unit I. An involution on $\mathbb{A}$ is a conjugate-linear map $a \mapsto a^*$ on $\mathbb{A}$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in \mathbb{A}$. The pair $(\mathbb{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathbb{A}$ together with a complete sub-multiplicative norm such that $\|a^*\| = \|a\|$ ($\forall a \in \mathbb{A}$). A C^*-algebra is a Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ [1, 2].

Notice that the seeming mild requirement on a C^*-algebra above is in fact very strong. Moreover, the existence of the involution C^*-algebra theory can be thought of as infinite-dimensional real analysis. Clearly that under the norm topology, $L(H)$, the set of all bounded linear operators on a Hilbert space H, is a C^*-algebra.

As we have known, the Banach contraction principle is a very useful, simple and classical tool in modern analysis. Also it is an important tool for solving existence problems in many branches of mathematics and physics. In general, the theorem has been generalized in two directions. On the one side, the usual contractive (expansive) condition is replaced by weakly contractive (expansive) condition. On the other side, the action spaces are replaced by metric spaces endowed with an ordered or partially ordered structure. In recent years, O'Regan and Petrusel [3] started the investigations concerning a fixed point theory in ordered metric spaces. Later, many authors followed this research by introducing and investigating the different types of contractive mappings. For example in [4] Caballero *et al.* considered contractive like mapping in ordered metric spaces and applied their results in ordinary differential equations. In 2007, Huang and Zhang [5] introduced the concept of cone metric space, replacing the set of real numbers by an ordered Banach space. Later, many authors generalized their fixed point theorems on different type of metric spaces [6–13]. In [14], the authors studied the operator-valued metric spaces and gave some fixed point theorems on the spaces. In this paper, we introduce a new type of metric spaces which generalize the concepts of metric spaces and operator-valued metric spaces, and give some related fixed point theorems for self-maps with contractive or expansive conditions on such spaces.

The paper is organized as follows: Based on the concept and properties of C^*-algebras, we first introduce a concept of C^*-algebra-valued metric spaces. Moreover, some fixed point theorems for mappings satisfying the contractive or expansive conditions on C^*-algebra-valued metric spaces are established. Finally, as applications, existence and uniqueness results for a type of integral equation and operator equation are given.

2 Main results

To begin with, let us start from some basic definitions, which will be used later.

Throughout this paper, $\mathbb{A}$ will denote an unital C^*-algebra with a unit I. Set $\mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$. We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$, if $x \in \mathbb{A}_h$ and $\sigma(x) \subset \mathbb{R}_+ = [0, \infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows: $x \preceq y$ if and only if $y - x \succeq \theta$, where θ means the zero element in $\mathbb{A}$. From now on, by $\mathbb{A}_+$ we denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ and $|x| = (x^*x)^{\frac{1}{2}}$.

Remark 2.1 When $\mathbb{A}$ is a unital C^*-algebra, then for any $x \in \mathbb{A}_+$ we have $x \preceq I \Leftrightarrow \|x\| \leq 1$ [1, 2].

With the help of the positive element in $\mathbb{A}$, one can give the definition of a C^*-algebra-valued metric space.

Definition 2.1 Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{A}$ satisfies:

(1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta \Leftrightarrow x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

It is obvious that C^*-algebra-valued metric spaces generalize the concept of metric spaces, replacing the set of real numbers by $\mathbb{A}_+$.

Definition 2.2 Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$. If for any $\varepsilon > 0$ there is N such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$, then $\{x_n\}$ is said to be convergent with respect to $\mathbb{A}$ and $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote it by $\lim_{n\to\infty} x_n = x$.

If for any $\varepsilon > 0$ there is N such that for all $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$, then $\{x_n\}$ is called a Cauchy sequence with respect to $\mathbb{A}$.

We say $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.

It is obvious that if X is a Banach space, then $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space if we set

$$d(x,y) = \|x - y\|I.$$

The following are nontrivial examples of complete C^*-algebra-valued metric space.

Example 2.1 Let $X = L^\infty(E)$ and $H = L^2(E)$, where E is a Lebesgue measurable set. By $L(H)$ we denote the set of bounded linear operators on Hilbert space H. Clearly $L(H)$ is a C^*-algebra with the usual operator norm.

Define $d : X \times X \to L(H)$ by

$$d(f,g) = \pi_{|f-g|} \quad (\forall f,g \in X),$$

where $\pi_h : H \to H$ is the multiplication operator defined by

$$\pi_h(\varphi) = h \cdot \varphi,$$

for $\varphi \in H$. Then d is a C^*-algebra-valued metric and $(X, L(H), d)$ is a complete C^*-algebra-valued metric space.

Indeed, it suffices to verity the completeness. Let $\{f_n\}_{n=1}^\infty$ in X be a Cauchy sequence with respect to $L(H)$. Then for a given $\varepsilon > 0$, there is a natural number $N(\varepsilon)$ such that for all $n, m \geq N(\varepsilon)$,

$$\|d(f_n, f_m)\| = \|\pi_{|f_n - f_m|}\| = \|f_n - f_m\|_\infty \leq \varepsilon,$$

then $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in the space X. Thus, there is a function $f \in X$ and a natural number $N_1(\varepsilon)$ such that $\|f_n - f\|_\infty \leq \varepsilon$ if $n \geq N_1$.

It follows that

$$\|d(f_n, f)\| = \|\pi_{|f_n - f|}\| = \|f_n - f\|_\infty \leq \varepsilon.$$

Therefore, the sequence $\{f_n\}_{n=1}^\infty$ converges to the function f in X with respect to $L(H)$, that is, $(X, L(H), d)$ is complete with respect to $L(H)$.

Example 2.2 Let $X = \mathbb{R}$ and $\mathbb{A} = M_2(\mathbb{R})$. Define

$$d(x,y) = \operatorname{diag}\bigl(|x-y|, \alpha|x-y|\bigr),$$

where $x, y \in \mathbb{R}$ and $\alpha \geq 0$ is a constant. It is easy to verify d is a C^*-algebra-valued metric and $(X, M_2(\mathbb{R}), d)$ is a complete C^*-algebra-valued metric space by the completeness of $\mathbb{R}$.

Now we give the definition of a C^*-algebra-valued contractive mapping on X.

Definition 2.3 Suppose that $(X, \mathbb{A}, d)$ is a C^*-algebra-valued metric space. We call a mapping $T : X \to X$ is a C^*-algebra-valued contractive mapping on X, if there exists an $A \in \mathbb{A}$ with $\|A\| < 1$ such that

$$d(Tx, Ty) \preceq A^* d(x,y) A, \quad \forall x, y \in X.$$

Theorem 2.1 *If $(X, \mathbb{A}, d)$ is a complete C^*-algebra-valued metric space and T is a contractive mapping, there exists a unique fixed point in X.*

Proof It is clear that if $A = \theta$, T maps the X into a single point. Thus without loss of generality, one can suppose that $A \neq \theta$.

Choose $x_0 \in X$ and set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1, 2, \ldots$. For convenience, by B we denote the element $d(x_1, x_0)$ in $\mathbb{A}$.

Notice that in a C^*-algebra, if $a, b \in \mathbb{A}_+$ and $a \preceq b$, then for any $x \in \mathbb{A}$ both x^*ax and x^*bx are positive elements and $x^*ax \preceq x^*bx$ [1]. Thus

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \preceq A^* d(x_n, x_{n-1})A \\ &\preceq \left(A^*\right)^2 d(x_{n-1}, x_{n-2})A^2 \\ &\preceq \cdots \\ &\preceq \left(A^*\right)^n d(x_1, x_0)A^n \\ &= \left(A^*\right)^n BA^n. \end{aligned}$$

So for $n + 1 > m$,

$$\begin{aligned} d(x_{n+1}, x_m) &\preceq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \\ &\preceq \left(A^*\right)^n BA^n + \cdots + \left(A^*\right)^m BA^m \\ &= \sum_{k=m}^{n} \left(A^*\right)^k BA^k \\ &= \sum_{k=m}^{n} \left(A^*\right)^k B^{\frac{1}{2}} B^{\frac{1}{2}} A^k \\ &= \sum_{k=m}^{n} \left(B^{\frac{1}{2}} A^k\right)^* \left(B^{\frac{1}{2}} A^k\right) \\ &= \sum_{k=m}^{n} \left|B^{\frac{1}{2}} A^k\right|^2 \\ &\preceq \left\| \sum_{k=m}^{n} \left|B^{\frac{1}{2}} A^k\right|^2 \right\| I \\ &\preceq \sum_{k=m}^{n} \left\|B^{\frac{1}{2}}\right\|^2 \left\|A^k\right\|^2 I \\ &\preceq \left\|B^{\frac{1}{2}}\right\|^2 \sum_{k=m}^{n} \|A\|^{2k} I \\ &\preceq \left\|B^{\frac{1}{2}}\right\|^2 \frac{\|A\|^{2m}}{1 - \|A\|} I \to \theta \quad (m \to \infty). \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X, \mathbb{A}, d)$, there exists an $x \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = x$.

Since

$$\begin{aligned}\theta &\preceq d(Tx,x) \preceq d(Tx,Tx_n) + d(Tx_n,x)\\ &\preceq A^*d(x,x_n)A + d(x_{n+1},x) \to \theta \quad (n \to \infty),\end{aligned}$$

hence, $Tx = x$, *i.e.*, x is a fixed point of T.

Now suppose that y ($\neq x$) is another fixed point of T, since

$$\theta \preceq d(x,y) = d(Tx,Ty) \preceq A^*d(x,y)A,$$

we have

$$\begin{aligned}0 &\le \|d(x,y)\| = \|d(Tx,Ty)\|\\ &\le \|A^*d(x,y)A\|\\ &\le \|A^*\|\|d(x,y)\|\|A\|\\ &= \|A\|^2\|d(x,y)\|\\ &< \|d(x,y)\|.\end{aligned}$$

It is impossible. So $d(x,y) = \theta$ and $x = y$, which implies that the fixed point is unique. □

Similar to the concept of contractive mapping, we have the concept of an expansive mapping and furthermore have the related fixed point theorem.

Definition 2.4 Let X be a nonempty set. We call a mapping T is a C^*-algebra-valued expansion mapping on X, if $T : X \to X$ satisfies:

(1) $T(X) = X$;

(2) $d(Tx,Ty) \succeq A^*d(x,y)A$, $\forall x,y \in X$,

where $A \in \mathbb{A}$ is an invertible element and $\|A^{-1}\| < 1$.

Theorem 2.2 *Let $(X,\mathbb{A},d)$ be a complete C^*-algebra-valued metric space. Then for the expansion mapping T, there exists a unique fixed point in X.*

Proof Firstly, T is injective. Indeed, for any $x,y \in X$ with $x \neq y$, if $Tx = Ty$, we have

$$\theta = d(Tx,Ty) \succeq A^*d(x,y)A.$$

Since $A^*d(x,y)A \in \mathbb{A}_+$, $A^*d(x,y)A = \theta$. Also A is invertible, $d(x,y) = \theta$, which is impossible. Thus T is injective.

Next, we will show T has a unique fixed point in X. In fact, since T is invertible and for any $x,y \in X$,

$$d(Tx,Ty) \succeq A^*d(x,y)A.$$

In the above formula, substitute x, y with $T^{-1}x$, $T^{-1}y$, respectively, and we get

$$d(x,y) \succeq A^*d\bigl(T^{-1}x,T^{-1}y\bigr)A.$$

This means

$$\left(A^*\right)^{-1} d(x,y) A^{-1} \succeq d\left(T^{-1}x, T^{-1}y\right),$$

and thus

$$\left(A^{-1}\right)^* d(x,y) A^{-1} \succeq d\left(T^{-1}x, T^{-1}y\right).$$

Using Theorem 2.1, there exists a unique x such that $T^{-1}x = x$, which means there has a unique fixed point $x \in X$ such that $Tx = x$. □

Before introducing another fixed point theorem, we give a lemma first. Such a result can be found in [1, 15].

Lemma 2.1 *Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit I.*

(1) *If $a \in \mathbb{A}_+$ with $\|a\| < \frac{1}{2}$, then $I - a$ is invertible and $\|a(I-a)^{-1}\| < 1$;*
(2) *suppose that $a, b \in \mathbb{A}$ with $a, b \succeq \theta$ and $ab = ba$, then $ab \succeq \theta$;*
(3) *by $\mathbb{A}'$ we denote the set $\{a \in \mathbb{A} : ab = ba, \forall b \in \mathbb{A}\}$. Let $a \in \mathbb{A}'$, if $b, c \in \mathbb{A}$ with $b \succeq c \succeq \theta$ and $I - a \in \mathbb{A}'_+$ is a invertible operator, then*

$$(I-a)^{-1} b \succeq (I-a)^{-1} c.$$

Notice that in a C^*-algebra, if $\theta \preceq a, b$, one cannot conclude that $\theta \preceq ab$. Indeed, consider the C^*-algebra $\mathbb{M}_2(\mathbb{C})$ and set $a = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$, then $ab = \begin{pmatrix} -1 & 2 \\ -4 & 8 \end{pmatrix}$. Clearly $a, b \in \mathbb{M}_2(\mathbb{C})_+$, while ab is not.

Theorem 2.3 *Let $(X, \mathbb{A}, d)$ be a complete C^*-valued metric space. Suppose the mapping $T : X \to X$ satisfies for all $x, y \in X$*

$$d(Tx, Ty) \preceq A\left(d(Tx, y) + d(Ty, x)\right),$$

where $A \in \mathbb{A}'_+$ and $\|A\| < \frac{1}{2}$. Then there exists a unique fixed point in X.

Proof Without loss of generality, one can suppose that $A \neq \theta$. Notice that $A \in \mathbb{A}'_+$, $A(d(Tx,y) + d(Ty,x))$ is also a positive element.

Choose $x_0 \in X$, set $x_{n+1} = Tx_n = T^{n+1}x_0$, $n = 1, 2, \ldots$, by B we denote the element $d(x_1, x_0)$ in $\mathbb{A}$. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\preceq A\left(d(Tx_n, x_{n-1}) + d(Tx_{n-1}, x_n)\right) \\ &= A\left(d(Tx_n, Tx_{n-2}) + d(Tx_{n-1}, Tx_{n-1})\right) \\ &\preceq A\left(d(Tx_n, Tx_{n-1}) + d(Tx_{n-1}, Tx_{n-2})\right) \\ &= Ad(Tx_n, Tx_{n-1}) + Ad(Tx_{n-1}, Tx_{n-2}) \\ &= Ad(x_{n+1}, x_n) + Ad(x_n, x_{n-1}). \end{aligned}$$

Thus,

$$(I-A)d(x_{n+1},x_n) \preceq Ad(x_n,x_{n-1}).$$

Since $A \in \mathbb{A}'_+$ with $\|A\| < \frac{1}{2}$, one have $(I-A)^{-1} \in \mathbb{A}'_+$ and furthermore $A(I-A)^{-1} \in \mathbb{A}'_+$ with $\|A(I-A)^{-1}\| < 1$ by Lemma 2.1. Therefore,

$$d(x_{n+1},x_n) \preceq A(I-A)^{-1}d(x_n,x_{n-1}) = td(x_n,x_{n-1}),$$

where $t = A(I-A)^{-1}$.

For $n+1 > m$,

$$\begin{aligned}
d(x_{n+1},x_m) &\preceq d(x_{n+1},x_n) + d(x_n,x_{n-1}) + \cdots + d(x_{m+1},x_m) \\
&\preceq \left(t^n + t^{n-1} + \cdots + t^m\right)d(x_1,x_0) \\
&= \sum_{k=m}^{n} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} B^{\frac{1}{2}} \\
&= \sum_{k=m}^{n} B^{\frac{1}{2}} t^{\frac{k}{2}} t^{\frac{k}{2}} B^{\frac{1}{2}} \\
&= \sum_{k=m}^{n} \left(t^{\frac{k}{2}} B^{\frac{1}{2}}\right)^* \left(t^{\frac{k}{2}} B^{\frac{1}{2}}\right) \\
&= \sum_{k=m}^{n} \left|t^{\frac{k}{2}} B^{\frac{1}{2}}\right|^2 \\
&\preceq \left\| \sum_{k=m}^{n} \left|t^{\frac{k}{2}} B^{\frac{1}{2}}\right|^2 \right\| I \\
&\preceq \sum_{k=m}^{n} \left\|B^{\frac{1}{2}}\right\|^2 \left\|t^{\frac{k}{2}}\right\|^2 I \\
&\preceq \left\|B^{\frac{1}{2}}\right\|^2 \sum_{k=m}^{n} \|t\|^k I \\
&\preceq \left\|B^{\frac{1}{2}}\right\|^2 \frac{\|t\|^m}{1-\|t\|} I \to \theta \quad (m \to \infty).
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence with respect to $\mathbb{A}$. By the completeness of $(X,\mathbb{A},d)$, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, *i.e.* $\lim_{n\to\infty} Tx_{n-1} = x$. Since

$$\begin{aligned}
d(Tx,x) &\preceq d(Tx,Tx_n) + d(Tx_n,x) \\
&\preceq A\big(d(Tx,x_n) + d(Tx_n,x)\big) + d(x_{n+1},x) \\
&\preceq A\big(d(Tx,x) + d(x,x_n) + d(x_{n+1},x)\big) + d(x_{n+1},x).
\end{aligned}$$

This is equivalent to

$$(I-A)d(Tx,x) \preceq A\big(d(x,x_n) + d(x_{n+1},x)\big) + d(x_{n+1},x).$$

Then

$$\begin{aligned}\|d(Tx,x)\| &\le \|A(I-A)^{-1}\|\bigl(\|d(x,x_n)\| + \|d(x_{n+1},x)\|\bigr)\\ &\quad + \|(I-A)^{-1}\|\|d(x_{n+1},x)\|\\ &\to 0 \quad (n\to\infty).\end{aligned}$$

This implies that $Tx = x$ *i.e.*, x is a fixed point of T.

Now if y ($\neq x$) is another fixed point of T, then

$$\begin{aligned}\theta &\preceq d(x,y) = d(Tx,Ty)\\ &\preceq A\bigl(d(Tx,y)+d(Ty,x)\bigr)\\ &= A\bigl(d(x,y)+d(y,x)\bigr)\end{aligned}$$

i.e.,

$$d(x,y) \preceq A(I-A)^{-1}d(x,y).$$

Since $\|A(I-A)^{-1}\| < 1$,

$$\begin{aligned}0 &\le \|d(x,y)\| = \|d(Tx,Ty)\|\\ &\le \|A(I-A)^{-1}d(x,y)\|\\ &\le \|A(I-A)^{-1}\|\|d(x,y)\|\\ &< \|d(x,y)\|.\end{aligned}$$

This means that

$$d(x,y) = \theta \quad \Leftrightarrow \quad x = y.$$

Therefore the fixed point is unique and the proof is complete. □

3 Applications

As applications of contractive mapping theorem on complete C^*-algebra-valued metric spaces, existence and uniqueness results for a type of integral equation and operator equation are given.

Example 3.1 Consider the integral equation

$$x(t) = \int_E K\bigl(t,s,x(s)\bigr)\,\mathrm{d}s + g(t), \quad t \in E,$$

where E is a Lebesgue measurable set.

Suppose that

(1) $K : E\times E\times\mathbb{R} \to \mathbb{R}$ and $g \in L^\infty(E)$;

(2) there exists a continuous function $\varphi : E \times E \to \mathbb{R}$ and $k \in (0,1)$ such that

$$\bigl|K(t,s,u) - K(t,s,v)\bigr| \le k\bigl|\varphi(t,s)(u-v)\bigr|,$$

for $t,s \in E$ and $u,v \in \mathbb{R}$;

(3) $\sup_{t\in E} \int_E |\varphi(t,s)|\,\mathrm{d}s \le 1$.

Then the integral equation has a unique solution x^* in $L^\infty(E)$.

Proof Let $X = L^\infty(E)$ and $H = L^2(E)$. Set d as Example 2.1, then d is a C^*-algebra-valued metric and $(X, L(H), d)$ is a complete C^*-algebra-valued metric space with respect to $L(H)$.

Let $T : L^\infty(E) \to L^\infty(E)$ be

$$Tx(t) = \int_E K\bigl(t,s,x(s)\bigr)\,\mathrm{d}s + g(t), \quad t \in E.$$

Set $A = kI$, then $A \in L(H)_+$ and $\|A\| = k < 1$. For any $h \in H$,

$$\begin{aligned}
\bigl\|d(Tx,Ty)\bigr\| &= \sup_{\|h\|=1} (\pi_{|Tx-Ty|}h, h) \\
&= \sup_{\|h\|=1} \int_E \biggl[\biggl|\int_E \bigl(K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr)\mathrm{d}s\biggr|\biggr] h(t)\overline{h(t)}\,\mathrm{d}t \\
&\le \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr|\mathrm{d}s\biggr] \bigl|h(t)\bigr|^2\,\mathrm{d}t \\
&\le \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|k\varphi(t,s)\bigl(x(s)-y(s)\bigr)\bigr|\,\mathrm{d}s\biggr] \bigl|h(t)\bigr|^2\mathrm{d}t \\
&\le k \sup_{\|h\|=1} \int_E \biggl[\int_E \bigl|\varphi(t,s)\bigr|\mathrm{d}s\biggr] \bigl|h(t)\bigr|^2\,\mathrm{d}t \cdot \|x-y\|_\infty \\
&\le k \sup_{t\in E} \int_E \bigl|\varphi(t,s)\bigr|\,\mathrm{d}s \cdot \sup_{\|h\|=1} \int_E \bigl|h(t)\bigr|^2\,\mathrm{d}t \cdot \|x-y\|_\infty \\
&\le k\|x-y\|_\infty \\
&= \|A\|\bigl\|d(x,y)\bigr\|.
\end{aligned}$$

Since $\|A\| < 1$, the integral equation has a unique solution x^* in $L^\infty(E)$. □

Example 3.2 Suppose that H is a Hilbert space, $L(H)$ is the set of linear bounded operators on H. Let $A_1, A_2, \ldots, A_n \in L(H)$, which satisfy $\sum_{n=1}^\infty \|A_n\|^2 < 1$ and $X \in L(H)$, $Q \in L(H)_+$. Then the operator equation

$$X - \sum_{n=1}^\infty A_n^* X A_n = Q$$

has a unique solution in $L(H)$.

Proof Set $\alpha = \sum_{n=1}^\infty \|A_n\|^2$. Clear that if $\alpha = 0$, then the $A_n = \theta$ $(n \in \mathbb{N})$, and the equation has a unique solution in $L(H)$. Without loss of generality, one can suppose that $\alpha > 0$.

Choose a positive operator $T \in L(H)$. For $X, Y \in L(H)$, set

$$d(X,Y) = \|X - Y\| T.$$

It is easy to verify that $d(X,Y)$ is a C^*-algebra-valued metric and $(L(H), d)$ is complete since $L(H)$ is a Banach space.

Consider the map $F : L(H) \to L(H)$ defined by

$$F(X) = \sum_{n=1}^{\infty} A_n^* X A_n + Q.$$

Then

$$\begin{aligned} d\bigl(F(X), F(Y)\bigr) &= \bigl\|F(X) - F(Y)\bigr\| T \\ &= \left\| \sum_{n=1}^{\infty} A_n^*(X - Y)A_n \right\| T \\ &\preceq \sum_{n=1}^{\infty} \|A_n\|^2 \|X - Y\| T \\ &= \alpha d(X,Y) \\ &= \bigl(\alpha^{\frac{1}{2}} I\bigr)^* d(X,Y) \bigl(\alpha^{\frac{1}{2}} I\bigr). \end{aligned}$$

Using Theorem 2.1, there exists a unique fixed point X in $L(H)$. Furthermore, since $\sum_{n=1}^{\infty} A_n^* X A_n + Q$ is a positive operator, the solution is a Hermitian operator. □

As a special case of Example 3.2, one can consider the following matrix equation, which can also be found in [16]:

$$X - A_1^* X A_1 - \cdots - A_m^* X A_m = Q,$$

where Q is a positive definite matrix and $A_1, \ldots, A_m$ are arbitrary $n \times n$ matrices. Using Example 3.2, there exists a unique Hermitian matrix solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Statistics, Beijing Institute of Technology, Beijing, 100081, China. [2]Department of Mathematics and Physics, Hebei Institute of Architecture and Civil Engineering, Zhangjiakou, 075024, China.

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