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Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem

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Abstract

The purpose of this paper is to study the split feasibility problem and fixed point problem involved in the pseudocontractive mappings. We construct an iterative algorithm and prove its strong convergence.

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Keywords: split feasibility problem; fixed point problem; pseudocontractive mapping

1 Introduction

Let H_1 and H_2 be two real Hilbert spaces and let $C \subset H_1$ and $Q \subset H_2$ be two nonempty, closed, and convex sets. Let $A: H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S: H_2 \to H_2$ and $T: H_1 \to H_1$ be two nonlinear mappings.

The purpose of this paper is to study the following split feasibility problem and fixed point problem:

$$\text{Find } x^* \in C \cap \text{Fix}(T) \text{ such that } Ax^* \in Q \cap \text{Fix}(S). \tag{1.1}$$

Special cases:
(i) Finding a point x^* which satisfies

$$x^* \in C \quad \text{and} \quad Ax^* \in Q. \tag{1.2}$$

This problem, referred to as the split feasibility problem, was introduced by Censor and Elfving [1], modeling phase retrieval and other image restoration problems, and further studied by many researchers; see, for instance, [2–7].
(ii) Find a point x^* with the property

$$x^* \in \text{Fix}(T) \quad \text{and} \quad Ax^* \in \text{Fix}(S). \tag{1.3}$$

This problem, referred to as the split common fixed point problem, was first introduced by Censor and Segal [8].

Next, we recall some existing algorithms for solving (1.1)-(1.3) in the literature.

In order to solve (1.2), Censor and Elfving [1] introduced the following algorithm:

$$x_{n+1} = A^{-1}P_Q\big(P_{A(C)}(Ax_n)\big), \quad n \in \mathbb{N}, \tag{1.4}$$

where C and Q are closed and convex sets in $\mathbb{R}^n$, A is a full rank $n \times n$ matrix and $A(C) = \{y \in \mathbb{R}^n \mid y = Ax, x \in C\}$.

Now (1.4) is not popular because it involves the computation of the inverse A^{-1}.

A more popular algorithm that solves (1.4) seems to be the *CQ* algorithm presented by Byrne [5, 7]:

$$x_{n+1} = P_C\big(x_n - \tau A^*(I - P_Q)Ax_n\big), \quad n \in \mathbb{N}, \tag{1.5}$$

where $\tau \in (0, \frac{2}{L})$, with L being the largest eigenvalue of the matrix A^*A.

Note that x^* solves (1.2) if and only if x^* solves the fixed point equation

$$x^* = P_C\big(I - \lambda A^*(I - P_Q)A\big)x^*. \tag{1.6}$$

The above equivalence relation (1.6) reminds us to use fixed point method to solve (1.2). Many authors have given a continuation of the study on the CQ algorithm and its variant form. For related work, please refer to [9–16]. Especially, the following regularized method was presented by Xu [6]:

$$x_{n+1} = P_C\big((1 - \alpha_n\gamma_n)x_n - \gamma_n A^*(I - P_Q)Ax_n\big), \quad n \in \mathbb{N}. \tag{1.7}$$

It should be pointed out that (1.7) can be used to find the minimum norm solution of (1.2).

For solving (1.3), Censor and Segal [8] invented an algorithm which generates a sequence $\{x_n\}$ according to the iterative procedure:

$$x_{n+1} = T\big(x_n - \gamma A^*(I - S)Ax_n\big), \quad n \in \mathbb{N}. \tag{1.8}$$

Note that (1.8) is more general than (1.5). Some further generations of this algorithm were studied by Moudafi [17] and Wang and Xu [18] and others; see, for example, [19–22].

Motivated by the results in this direction, the purpose of this paper is to study the split feasibility problem and the fixed point problem involved in the pseudocontractive mappings. We construct an iterative algorithm and prove its strong convergence.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty, closed, and convex subset of H.

Recall that a mapping $T: C \to C$ is called *pseudocontractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2$$

for all $x, y \in C$. It is well known that T is pseudocontractive if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \big\|(I - T)x - (I - T)y\big\|^2 \tag{2.1}$$

for all $x, y \in C$. A mapping $T: C \to C$ is called *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|$$

for all $x, y \in C$. If $L = 1$, we call T nonexpansive.

We will use Fix(T) to denote the set of fixed points of T, that is,

$$\text{Fix}(T) = \{x \in C : x = Tx\}.$$

We know that the metric projection $P_C: H \to C$ satisfies

$$\|x - P_C(x)\| = \inf\{\|x - y\| : y \in C\}.$$

It is well known that the metric projection $P_C: H \to C$ is firmly nonexpansive, that is,

$$\langle x - y, P_C(x) - P_C(y) \rangle \ge \|P_C(x) - P_C(y)\|^2$$
$$\Leftrightarrow \quad \|P_C(x) - P_C(y)\|^2 \le \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2 \tag{2.2}$$

for all $x, y \in H$.

For all $x, y \in H$, the following conclusions hold:

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \quad t \in [0, 1], \tag{2.3}$$
$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2, \tag{2.4}$$

and

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle. \tag{2.5}$$

Lemma 2.1 ([23]) *Let H be a real Hilbert space, C a closed convex subset of H. Let $T: C \to C$ be a continuous pseudocontractive mapping. Then*

  (i) Fix(T) *is a closed convex subset of C,*
  (ii) $(I - T)$ *is demiclosed at zero.*

Lemma 2.2 ([24]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n, \quad n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that
  (1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
  (2) $\limsup_{n \to \infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.3 ([25]) *Let $\{w_n\}$ be a sequence of real numbers. Assume $\{w_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \le w_{n_k+1}$ for all $k \ge 0$. For every $n \ge N_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \le n : w_{n_i} < w_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq N_0$

$$\max\{w_{\tau(n)}, w_n\} \leq w_{\tau(n)+1}.$$

3 Main results

Let H_1 and H_2 be two real Hilbert spaces and let $C \subset H_1$ and $Q \subset H_2$ be two nonempty closed convex sets. Let $A: H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $S: H_2 \to H_2$ nonexpansive mapping and let $T: H_1 \to H_1$ be an L-Lipschitzian pseudocontractive mapping with $L > 1$.

We use Γ to denote the set of solutions of (1.1), that is,

$$\Gamma = \{x^* \mid x^* \in C \cap \mathrm{Fix}(T), Ax^* \in Q \cap \mathrm{Fix}(S)\}.$$

In the sequel, we assume $\Gamma \neq \emptyset$.

Now, we present our algorithm for finding $x^* \in \Gamma$.

Algorithm 3.1 For fixed $u \in H_1$ and $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} u_n = P_C[\alpha_n u + (1 - \alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n)], \\ x_{n+1} = (1 - \beta_n)u_n + \beta_n T((1 - \gamma_n)u_n + \gamma_n Tu_n), \qquad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}_{n \in \mathbb{N}}$, $\{\beta_n\}_{n \in \mathbb{N}}$, and $\{\gamma_n\}_{n \in \mathbb{N}}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Theorem 3.2 *Assume the following conditions are satisfied*:
(C1) $\lim_{n \to \infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.
Then the sequence $\{x_n\}$ generated by algorithm (3.1) converges strongly to the point x^*, given by $x^* = P_\Gamma(u)$.

Proof Set $x^* = P_\Gamma(u)$. Then we have $x^* \in C \cap \mathrm{Fix}(T)$ and $Ax^* \in Q \cap \mathrm{Fix}(S)$. Set $z_n = P_Q Ax_n$ and $y_n = \alpha_n u + (1 - \alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n)$ for all $n \in \mathbb{N}$. Thus $u_n = P_C y_n$ for all $n \in \mathbb{N}$.

Since P_C and P_Q are nonexpansive, we have

$$\|z_n - Ax^*\| = \|P_Q Ax_n - P_Q Ax^*\| \leq \|Ax_n - Ax^*\| \tag{3.2}$$

and

$$\|u_n - x^*\| = \|P_C y_n - P_C x^*\| \leq \|y_n - x^*\|. \tag{3.3}$$

By (2.2), we get

$$\begin{aligned} \|Sz_n - Ax^*\|^2 &= \|SP_Q Ax_n - SP_Q Ax^*\|^2 \\ &\leq \|P_Q Ax_n - P_Q Ax^*\|^2 \\ &\leq \|Ax_n - Ax^*\|^2 - \|z_n - Ax_n\|^2. \end{aligned} \tag{3.4}$$

From (2.1), we have

$$\|Tu_n - x^*\|^2 \le \|u_n - x^*\|^2 + \|Tu_n - u_n\|^2 \tag{3.5}$$

and

$$
\begin{aligned}
\|T((1 - \gamma_n)u_n &+ \gamma_n Tu_n) - x^*\|^2 \\
&\le \|(1 - \gamma_n)(u_n - x^*) + \gamma_n(Tu_n - x^*)\|^2 \\
&\quad + \|(1 - \gamma_n)u_n + \gamma_n Tu_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2.
\end{aligned} \tag{3.6}
$$

Applying equality (2.3), we have

$$
\begin{aligned}
\|(1 - \gamma_n)u_n &+ \gamma_n Tu_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 \\
&= \|(1 - \gamma_n)(u_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)) + \gamma_n(Tu_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n))\|^2 \\
&= (1 - \gamma_n)\|u_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 + \gamma_n\|Tu_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 \\
&\quad - \gamma_n(1 - \gamma_n)\|u_n - Tu_n\|^2.
\end{aligned} \tag{3.7}
$$

Since T is L-Lipschitzian and $u_n - ((1 - \gamma_n)u_n + \gamma_n Tu_n) = \gamma_n(u_n - Tu_n)$, by (3.7), we get

$$
\begin{aligned}
\|(1 - \gamma_n)u_n &+ \gamma_n Tu_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 \\
&\le (1 - \gamma_n)\|u_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 + \gamma_n^3 L^2\|u_n - Tu_n\|^2 \\
&\quad - \gamma_n(1 - \gamma_n)\|u_n - Tu_n\|^2 \\
&= (1 - \gamma_n)\|u_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 + (\gamma_n^3 L^2 + \gamma_n^2 - \gamma_n)\|u_n - Tu_n\|^2.
\end{aligned} \tag{3.8}
$$

By (2.3) and (3.5), we have

$$
\begin{aligned}
\|(1 - \gamma_n)(u_n - x^*) &+ \gamma_n(Tu_n - x^*)\|^2 \\
&= (1 - \gamma_n)\|u_n - x^*\|^2 + \gamma_n\|Tu_n - x^*\|^2 - \gamma_n(1 - \gamma_n)\|u_n - Tu_n\|^2 \\
&\le (1 - \gamma_n)\|u_n - x^*\|^2 + \gamma_n(\|u_n - x^*\|^2 + \|u_n - Tu_n\|^2) \\
&\quad - \gamma_n(1 - \gamma_n)\|u_n - Tu_n\|^2 \\
&= \|u_n - x^*\|^2 + \gamma_n^2\|u_n - Tu_n\|^2.
\end{aligned} \tag{3.9}
$$

From (3.6), (3.8), and (3.9), we deduce

$$
\begin{aligned}
\|T((1 - \gamma_n)u_n &+ \gamma_n Tu_n) - x^*\|^2 \\
&\le \|u_n - x^*\|^2 + (1 - \gamma_n)\|u_n - T((1 - \gamma_n)u_n + \gamma_n Tu_n)\|^2 \\
&\quad - \gamma_n(1 - 2\gamma_n - \gamma_n^2 L^2)\|u_n - Tu_n\|^2.
\end{aligned} \tag{3.10}
$$

Since $\gamma_n < b < \frac{1}{\sqrt{1 + L^2} + 1}$, we derive that

$$1 - 2\gamma_n - \gamma_n^2 L^2 > 0, \quad \forall n \in \mathbb{N}.$$

This together with (3.10) implies that

$$
\begin{aligned}
&\left\| T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) - x^* \right\|^2 \\
&\quad \leq \left\| u_n - x^* \right\|^2 + (1 - \gamma_n)\left\| u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) \right\|^2.
\end{aligned}
\tag{3.11}
$$

By (2.3), (3.1), (3.11), and (C3), we have

$$
\begin{aligned}
\left\| x_{n+1} - x^* \right\|^2 &= \left\| (1 - \beta_n)u_n + \beta_n T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) - x^* \right\|^2 \\
&= (1 - \beta_n)\left\| u_n - x^* \right\|^2 + \beta_n \left\| T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) - x^* \right\|^2 \\
&\quad - \beta_n(1 - \beta_n)\left\| u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) \right\|^2 \\
&\leq \left\| u_n - x^* \right\|^2 - \beta_n(\gamma_n - \beta_n)\left\| T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) - x^* \right\|^2 \\
&\leq \left\| u_n - x^* \right\|^2.
\end{aligned}
\tag{3.12}
$$

By the convexity of the norm and by using (2.4), we get

$$
\begin{aligned}
\left\| y_n - x^* \right\|^2 &= \left\| \alpha_n\big(u - x^*\big) + (1 - \alpha_n)\big(x_n - x^* + \delta A^*(Sz_n - Ax_n)\big) \right\|^2 \\
&\leq (1 - \alpha_n)\left\| \big(x_n - x^* + \delta A^*(Sz_n - Ax_n)\big) \right\|^2 + \alpha_n \left\| u - x^* \right\|^2 \\
&= (1 - \alpha_n)\Big[\left\| x_n - x^* \right\| + \delta^2 \left\| A^*(Sz_n - Ax_n) \right\|^2 \\
&\quad + 2\delta\big\langle x_n - x^*, A^*(Sz_n - Ax_n)\big\rangle \Big] + \alpha_n \left\| u - x^* \right\|^2.
\end{aligned}
\tag{3.13}
$$

Since A is a linear operator with its adjoint A^*, we have

$$
\begin{aligned}
&\big\langle x_n - x^*, A^*(Sz_n - Ax_n)\big\rangle \\
&\quad = \big\langle A\big(x_n - x^*\big), Sz_n - Ax_n \big\rangle \\
&\quad = \big\langle Ax_n - Ax^* + Sz_n - Ax_n - (Sz_n - Ax_n), Sz_n - Ax_n \big\rangle \\
&\quad = \big\langle Sz_n - Ax^*, Sz_n - Ax_n \big\rangle - \| Sz_n - Ax_n \|^2.
\end{aligned}
\tag{3.14}
$$

Again using (2.4), we obtain

$$
\big\langle Sz_n - Ax^*, Sz_n - Ax_n \big\rangle = \frac{1}{2}\Big(\left\| Sz_n - Ax^* \right\|^2 + \| Sz_n - Ax_n \|^2 - \left\| Ax_n - Ax^* \right\|^2 \Big).
\tag{3.15}
$$

From (3.4), (3.14), and (3.15), we get

$$
\begin{aligned}
\big\langle x_n - x^*, A^*(Sz_n - Ax_n)\big\rangle &= \frac{1}{2}\Big(\left\| Sz_n - Ax^* \right\|^2 + \| Sz_n - Ax_n \|^2 - \left\| Ax_n - Ax^* \right\|^2 \Big) \\
&\quad - \| Sz_n - Ax_n \|^2 \\
&\leq \frac{1}{2}\Big(\left\| Ax_n - Ax^* \right\|^2 - \| z_n - Ax_n \|^2 + \| Sz_n - Ax_n \|^2 \\
&\quad - \left\| Ax_n - Ax^* \right\|^2 \Big) - \| Sz_n - Ax_n \|^2 \\
&= -\frac{1}{2}\| z_n - Ax_n \|^2 - \frac{1}{2}\| Sz_n - Ax_n \|^2.
\end{aligned}
\tag{3.16}
$$

Substituting (3.16) into (3.13) we deduce

$$\big\|y_n - x^*\big\|^2 \le (1 - \alpha_n)\bigg[\big\|x_n - x^*\big\|^2 + \delta^2\|A\|^2\|Sz_n - Ax_n\|^2$$
$$+ 2\delta\bigg(-\frac{1}{2}\|z_n - Ax_n\|^2 - \frac{1}{2}\|Sz_n - Ax_n\|^2\bigg)\bigg] + \alpha_n\big\|u - x^*\big\|^2$$
$$= (1 - \alpha_n)\big[\big\|x_n - x^*\big\|^2 + \big(\delta^2\|A\|^2 - \delta\big)\|Sz_n - Ax_n\|^2$$
$$- \delta\|z_n - Ax_n\|^2\big] + \alpha_n\big\|u - x^*\big\|^2$$
$$\le (1 - \alpha_n)\big\|x_n - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2. \tag{3.17}$$

From (3.3), (3.12), and (3.17), we get

$$\big\|x_{n+1} - x^*\big\|^2 \le \big\|y_n - x^*\big\|^2$$
$$\le (1 - \alpha_n)\big\|x_n - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2$$
$$\le \max\big\{\big\|x_n - x^*\big\|^2, \big\|u - x^*\big\|^2\big\}.$$

The boundedness of the sequence $\{x_n\}$ yields our result.

Using the firmly nonexpansivenessity of P_C (2.2), we have

$$\big\|u_n - x^*\big\|^2 = \big\|P_C y_n - x^*\big\|^2 \le \big\|y_n - x^*\big\|^2 - \|P_C y_n - y_n\|^2$$
$$= \big\|y_n - x^*\big\|^2 - \|u_n - y_n\|^2. \tag{3.18}$$

Thus

$$\big\|x_{n+1} - x^*\big\|^2 \le \big\|u_n - x^*\big\|^2$$
$$\le \big\|y_n - x^*\big\|^2 - \|u_n - y_n\|^2$$
$$\le (1 - \alpha_n)\big\|x_n - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2 - \|u_n - y_n\|^2.$$

It follows that

$$\|u_n - y_n\|^2 \le \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2. \tag{3.19}$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m > 0$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \ge m$. In this case, we know that $\lim_{n\to\infty}\|x_n - x^*\|$ exists. From (3.19), we deduce

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.20}$$

Returning to (3.17), we have

$$\big\|x_{n+1} - x^*\big\|^2 \le \big\|y_n - x^*\big\|^2$$
$$\le (1 - \alpha_n)\big\|x_n - x^*\big\|^2 + (1 - \alpha_n)\big(\delta^2\|A\|^2 - \delta\big)\|Sz_n - Ax_n\|^2$$
$$- (1 - \alpha_n)\delta\|z_n - Ax_n\|^2 + \alpha_n\big\|u - x^*\big\|^2.$$

Hence,

$$(1 - \alpha_n)\big(\delta - \delta^2 \|A\|^2\big)\|Sz_n - Ax_n\|^2 + (1 - \alpha_n)\delta\|z_n - Ax_n\|^2$$
$$\leq \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2,$$

which implies that

$$\lim_{n \to \infty} \|Sz_n - Ax_n\| = \lim_{n \to \infty} \|z_n - Ax_n\| = 0. \tag{3.21}$$

So,

$$\lim_{n \to \infty} \|Sz_n - z_n\| = 0. \tag{3.22}$$

Note that

$$\|y_n - x_n\| = \big\|\delta A^*(SP_Q - I)Ax_n + \alpha_n\big(x_n - \delta A^*(I - SP_Q)Ax_n - u\big)\big\|$$
$$\leq \delta\|A\|\|Sz_n - Ax_n\| + \alpha_n\big\|x_n - \delta A^*(I - SP_Q)Ax_n - u\big\|.$$

It follows from (3.21) that

$$\lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{3.23}$$

From (3.3), (3.12), and (3.17), we deduce

$$\big\|x_{n+1} - x^*\big\|^2 \leq \big\|u_n - x^*\big\|^2 - \beta_n(\gamma_n - \beta_n)\big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2$$
$$\leq \big\|x_n - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2$$
$$- \beta_n(\gamma_n - \beta_n)\big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2.$$

It follows that

$$\beta_n(\gamma_n - \beta_n)\big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\|^2$$
$$\leq \big\|x_n - x^*\big\|^2 - \big\|x_{n+1} - x^*\big\|^2 + \alpha_n\big\|u - x^*\big\|^2.$$

Therefore,

$$\lim_{n \to \infty} \big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\| = 0. \tag{3.24}$$

Observe that

$$\|u_n - Tu_n\| \leq \big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\| + \big\|T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big) - Tu_n\big\|$$
$$\leq \big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\| + L\gamma_n\|u_n - Tu_n\|.$$

Thus,

$$\|u_n - Tu_n\| \leq \frac{1}{1 - L\gamma_n}\big\|u_n - T\big((1 - \gamma_n)u_n + \gamma_n Tu_n\big)\big\|.$$

This together with (3.24) implies that

$$\lim_{n\to\infty} \|u_n - Tu_n\| = 0. \tag{3.25}$$

Now, we show that

$$\limsup_{n\to\infty} \langle u - x^*, y_n - x^* \rangle \le 0.$$

Choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty} \langle u - x^*, y_n - x^* \rangle = \lim_{i\to\infty} \langle u - x^*, y_{n_i} - x^* \rangle. \tag{3.26}$$

Since the sequence $\{y_{n_i}\}$ is bounded, we can choose a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ such that $y_{n_{i_j}} \rightharpoonup z$. For the sake of convenience, we assume (without loss of generality) that $y_{n_i} \rightharpoonup z$. Consequently, we derive from the above conclusions that

$$x_{n_i} \rightharpoonup z, \qquad u_{n_i} \rightharpoonup z, \qquad Ax_{n_i} \rightharpoonup Az \quad \text{and} \quad z_{n_i} \rightharpoonup Az. \tag{3.27}$$

Applying Lemma 2.1, we deduce

$$z \in \text{Fix}(T) \quad \text{and} \quad Az \in \text{Fix}(S).$$

Note that $u_{n_i} = P_C y_{n_i} \in C$ and $z_{n_i} = P_Q Ax_{n_i} \in Q$. From (3.27), we deduce

$$z \in C \quad \text{and} \quad Az \in Q.$$

To this end, we deduce

$$z \in C \cap \text{Fix}(T) \quad \text{and} \quad Az \in Q \cap \text{Fix}(S).$$

That is to say, $z \in \Gamma$.

Therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - x^*, y_n - x^* \rangle &= \lim_{i\to\infty} \langle u - x^*, y_{n_i} - x^* \rangle \\
&= \lim_{i\to\infty} \langle u - x^*, z - x^* \rangle \\
&\le 0.
\end{aligned} \tag{3.28}$$

Using (2.5), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|y_n - x^*\|^2 \\
&= \|(1 - \alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n - x^*) + \alpha_n(u - x^*)\|^2 \\
&\le (1 - \alpha_n)\|x_n - \delta A^*(I - SP_Q)Ax_n - x^*\|^2 + 2\alpha_n \langle u - x^*, y_n - x^* \rangle \\
&\le (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \langle u - x^*, y_n - x^* \rangle.
\end{aligned} \tag{3.29}$$

Applying Lemma 2.2 and (3.28) to (3.29), we deduce $x_n \to x^*$.

Case 2. Assume there exists an integer n_0 such that

$$\|x_{n_0} - x^*\| \le \|x_{n_0+1} - x^*\|.$$

Set $\omega_n = \{\|x_n - x^*\|\}$. Then we have

$$\omega_{n_0} \le \omega_{n_0+1}.$$

Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} \mid n_0 \le l \le n, \omega_l \le \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and

$$\omega_{\tau(n)} \le \omega_{\tau(n)+1},$$

for all $n \ge n_0$.

By a similar argument to that of Case 1, we can obtain

$$\lim_{n\to\infty} \|u_{\tau(n)} - y_{\tau(n)}\| = \lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|Sz_{\tau(n)} - Ax_{\tau(n)}\| = \lim_{n\to\infty} \|z_{\tau(n)} - Ax_{\tau(n)}\| = \lim_{n\to\infty} \|Sz_{\tau(n)} - z_{\tau(n)}\| = 0,$$

and

$$\lim_{n\to\infty} \|u_{\tau(n)} - Tu_{\tau(n)}\| = 0.$$

This implies that

$$\omega_w(y_{\tau(n)}) \subset \Gamma.$$

Thus, we obtain

$$\limsup_{n\to\infty} \langle u - x^*, y_{\tau(n)} - x^* \rangle \le 0. \tag{3.30}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from (3.29) that

$$\omega_{\tau(n)}^2 \le \omega_{\tau(n)+1}^2 \le (1 - \alpha_{\tau(n)})\omega_{\tau(n)}^2 + 2\alpha_{\tau(n)}\langle u - x^*, y_{\tau(n)} - x^* \rangle. \tag{3.31}$$

It follows that

$$\omega_{\tau(n)}^2 \le 2\langle u - x^*, y_{\tau(n)} - x^* \rangle. \tag{3.32}$$

Combining (3.30) and (3.32), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)} \le 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.33}$$

By (3.31), we obtain

$$\limsup_{n\to\infty} \omega^2_{\tau(n)+1} \le \limsup_{n\to\infty} \omega^2_{\tau(n)}.$$

This together with (3.33) implies that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.3 to get

$$0 \le \omega_n \le \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

Algorithm 3.3 For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} u_n = P_C[(1-\alpha_n)(x_n - \delta A^*(I - SP_Q)Ax_n)], \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n Tu_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.34}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, and $\{\gamma_n\}_{n\in\mathbb{N}}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.4 *Assume the following conditions are satisfied*:

  (C1) $\lim_{n\to\infty} \alpha_n = 0$;
  (C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
  (C3) $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.
  Then the sequence $\{x_n\}$ generated by algorithm (3.34) *converges strongly to* $x^* = P_\Gamma(0)$, *which is the minimum norm in* Γ.

Algorithm 3.5 For fixed $u \in H_1$ and $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = P_C[\alpha_n u + (1-\alpha_n)(x_n - \delta A^*(I - P_Q)Ax_n)], \quad n \in \mathbb{N}, \tag{3.35}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a real number sequence in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.6 *Suppose Γ_1, the set of the solutions of* (1.2), *is nonempty. Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by algorithm (3.35) *converges strongly to* $x^* = P_{\Gamma_1}(u)$.

Algorithm 3.7 For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = P_C\big[(1-\alpha_n)\big(x_n - \delta A^*(I - P_Q)Ax_n\big)\big], \quad n \in \mathbb{N}, \tag{3.36}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a real number sequence in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.8 *Suppose* Γ_1, *the set of the solutions of* (1.2), *is nonempty. Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequence $\{x_n\}$ generated by algorithm (3.36) *converges strongly to* $x^* = P_{\Gamma_1}(0)$, *which is the minimum norm in* Γ_1.

Algorithm 3.9 For fixed $u \in H_1$ and $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} u_n = \alpha_n u + (1-\alpha_n)(x_n - \delta A^*(I - S)Ax_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n Tu_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.37}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, and $\{\gamma_n\}_{n\in\mathbb{N}}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.10 *Suppose* Γ_2, *the set of the solutions of* (1.3), *is nonempty. Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ generated by algorithm (3.37) *converges strongly to* $x^* = P_{\Gamma_2}(u)$.

Algorithm 3.11 For and $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by

$$\begin{cases} u_n = (1-\alpha_n)(x_n - \delta A^*(I - S)Ax_n), \\ x_{n+1} = (1-\beta_n)u_n + \beta_n T((1-\gamma_n)u_n + \gamma_n Tu_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.38}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$, $\{\beta_n\}_{n\in\mathbb{N}}$, and $\{\gamma_n\}_{n\in\mathbb{N}}$ are three real number sequences in $(0,1)$ and δ is a constant in $(0, \frac{1}{\|A\|^2})$.

Corollary 3.12 *Suppose* Γ_2, *the set of the solutions of* (1.3), *is nonempty. Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $0 < a < \beta_n < c < \gamma_n < b < \frac{1}{\sqrt{1+L^2}+1}$.

Then the sequence $\{x_n\}$ generated by algorithm (3.38) *converges strongly to* $x^* = P_{\Gamma_2}(0)$ *which is the minimum norm in* Γ_2.

Example 3.13 Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x, y \rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C = [0, \infty)$ and $Q = \mathbb{R}$. Let $Sx = \frac{x}{2} - 1$ for all $x \in Q$ and let $Tx = x - 1 + \frac{4}{x+1}$ for all $x \in C$. Let $Ax = -\frac{2x}{3}$ for all $x \in \mathbb{R}$. Then A is a bounded linear operator with its adjoint $A^* = A$. Observe that $\mathrm{Fix}(T) = 3$ and $\mathrm{Fix}(S) = -2$. It is easy to see that

$$\langle Tx - Ty, x - y \rangle = \left\langle x - 1 + \frac{4}{x+1} - y + 1 - \frac{4}{y+1}, x - y \right\rangle$$

$$\leq \left[1 - \frac{4}{(x+1)(y+1)} \right] |x - y|^2$$

$$\leq |x - y|^2,$$

and

$$|Tx - Ty| \leq \left| x - 1 + \frac{4}{x+1} - y + 1 - \frac{4}{y+1} \right|$$

$$\leq \left| 1 - \frac{4}{(x+1)(y+1)} \right| |x - y|$$

$$\leq 5|x - y|,$$

for all $x, y \in C$.

But

$$\left| T\left(\frac{1}{4}\right) - T(0) \right| = \frac{11}{20} > \frac{1}{4}.$$

Hence, T is a Lipschitzian pseudocontractive mapping but not a nonexpansive one.

Note that $\|A\| = \|A^*\| = \frac{2}{3}$. Let $u = 3$ and $\delta = \frac{9}{8}$. Then we have

$$u_n = P_C\left[3\alpha_n + (1 - \alpha_n)\left(x_n - \frac{9}{8} \times \left(-\frac{2I}{3}\right) \times \left(\frac{I}{2} + 1\right) \times \left(\frac{-2x_n}{3}\right) \right) \right]$$

$$= P_C\left[3\alpha_n + (1 - \alpha_n)\left(\frac{3}{4}x_n + \frac{3}{4} \right) \right].$$

Let $\beta_n = \frac{1}{8}$ and $\gamma_n = \frac{1}{7}$ for all n. It is not hard to compute that

$$|x_{n+1} - 3| \leq |u_n - 3|$$

$$\leq (1 - \alpha_n)\left| \frac{3}{4}x_n + \frac{3}{4} - 3 \right|$$

$$\leq \frac{3}{4}|x_n - 3|$$

$$\leq \cdots$$

$$\leq \left(\frac{3}{4}\right)^n |x_1 - 3|,$$

which shows $x_n \to 3 \in \Gamma$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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