Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Tripled common fixed point theorems under probabilistic φ-contractive conditions in generalized Menger probabilistic metric spaces

Ting Luo, Chuanxi Zhu* and Zhaoqi Wu

*Correspondence: chuanxizhu@126.com
Department of Mathematics, Nanchang University, Nanchang, 330031, P.R. China

Abstract

In this paper, the new concepts of generalized Menger probabilistic metric spaces and tripled common fixed point for a pair of mappings $T : X \times X \times X \to X$ and $A : X \to X$ are introduced. Utilizing the properties of the pseudo-metric and the triangular norm, some tripled common fixed point problems of hybrid probabilistic contractions with a gauge function φ are studied. The obtained results generalize some coupled common fixed point theorems in the corresponding literature. Finally, an example is given to illustrate our main results.

MSC: 47H10; 46S50

Keywords: generalized Menger probabilistic metric space; hybrid probabilistic contraction; gauge function; tripled common fixed point

1 Introduction

Coupled fixed points were considered by Bhaskar and Lakshmikantham [1]. Recently, some new results for the existence and uniqueness of coupled fixed points were presented for the cases of partially ordered metric spaces, cone metric spaces and fuzzy metric spaces (see [2–12]). The concept of probabilistic metric space was initiated and studied by Menger which is a generalization of the metric space notion [13]. Many results on the existence of fixed points or solutions of nonlinear equations under various types of conditions in Menger spaces have been extensively studied by many scholars (see [14–18]). In 2010, Jachymski established a fixed point theorem for probabilistic φ-contractions and give a characterization of function φ having the property that there exists a probabilistic φ-contraction, which is not a probabilistic k-contraction ($k \in [0,1)$) [19]. In 2011, Xiao *et al.* obtained some common coupled fixed point results for hybrid probabilistic contractions with a gauge function φ in Menger probabilistic metric spaces and in non-Archimedean Menger probabilistic metric spaces without assuming any continuity or monotonicity conditions for φ [20].

The purpose of this paper is to introduce the concept of generalized Menger probabilistic metric spaces and tripled common fixed point for a pair of mappings $T : X \times X \times X \to X$ and $A : X \to X$. Utilizing the properties of the pseudo-metric and the triangular norm, some tripled common fixed points problems for pairs of commutative mappings under hybrid probabilistic contractions with a gauge function φ are studied in generalized Menger PM-spaces and in generalized non-Archimedean Menger PM-space, respectively. The obtained results generalize some coupled common fixed point theorems in corresponding literatures. Finally, an example is given to illustrate our main results.

2 Preliminaries

Consistent with Menger [13] and Zhang [14], the following results will be needed in the sequel.

Denote by R the set of real numbers, R^+ the nonnegative real numbers, and Z^+ the set of all positive integers.

If $\varphi : R^+ \to R^+$ is a function such that $\varphi(0) = 0$, then φ is called a gauge function. If $t \in R^+$, then $\varphi^n(t)$ denotes the nth iteration of $\varphi(t)$ and $\varphi^{-1}(\{0\}) = \{t \in R^+ : \varphi(t) = 0\}$.

A mapping $f : R \to R^+$ is called a distribution function if it is nondecreasing and left-continuous with $\inf_{t\in R} f(t) = 0$, $\sup_{t\in R} f(t) = 1$.

We shall denote by $\mathscr{D}$ the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 2.1 ([14]) A function $\Delta : [0,1] \times [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a,b,c,d,e,f \in [0,1]$:

(Δ-1) $\Delta(a,1,1) = a$, $\Delta(0,0,0) = 0$;
(Δ-2) $\Delta(a,b,c) = \Delta(a,c,b) = \Delta(c,b,a)$;
(Δ-3) $a \ge d, b \ge e, c \ge f \Rightarrow \Delta(a,b,c) \ge \Delta(d,e,f)$;
(Δ-4) $\Delta(a,\Delta(b,c,d),e) = \Delta(\Delta(a,b,c),d,e) = \Delta(a,b,\Delta(c,d,e))$.

Two typical examples of t-norms are $\Delta_m(a,b,c) = \min\{a,b,c\}$ and $\Delta_p(a,b,c) = abc$ for all $a,b,c \in [0,1]$.

We now introduce the definition of generalized Menger probabilistic metric space.

Definition 2.2 A triplet $(X,\mathscr{F},\Delta)$ is called a generalized Menger probabilistic metric space (for short, a generalized Menger PM-space) if X is a non-empty set, Δ is a t-norm and $\mathscr{F}$ is a mapping from $X \times X$ into $\mathscr{D}$ (we shall denote the distribution function $\mathscr{F}(x,y)$ by $F_{x,y}$ and $F_{x,y}(t)$ will represent the value of $F_{x,y}$ at $t \in R$) satisfying the following conditions:

(GPM-1) $F_{x,y}(0) = 0$;
(GPM-2) $F_{x,y}(t) = H(t)$ for all $t \in R$ if and only if $x = y$;
(GPM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in X$ and $t \in R$;
(GPM-4) $F_{x,w}(t_1 + t_2 + t_3) \ge \Delta(F_{x,y}(t_1), F_{y,z}(t_2), F_{z,w}(t_3))$ for all $x,y,z,w \in X$ and $t_1,t_2,t_3 \in R^+$.

$(X,\mathscr{F},\Delta)$ is called a generalized non-Archimedean Menger PM-space if it is a generalized Menger PM-space satisfying the following condition:

(GPM-5) $F_{x,y}(\max\{t,s,r\}) \ge \Delta(F_{x,z}(t), F_{z,w}(s), F_{w,y}(r))$ for all $x,y,z,w \in X$ and $t,s,r \in R^+$.

Remark 2.1 In 1942, Menger [13] proposed a generalization of a metric space called a Menger probabilistic metric space (briefly a Menger PM-space). Our definition of a generalized Menger PM-space is different from the one of Menger, since the t-norm we used here is an associative function of three variables rather than a function of two variables. Note that Definition 2.1 is first used by Chang to define a probabilistic 2-metric space. Our definition is also different from the one of Chang since the distribution function of the latter is from $X \times X \times X$ to $\mathscr{D}$.

Example 2.1 Suppose that $X = [-1,1] \subset R$. Define $\mathscr{F} : X \times X \to \mathscr{D}$ by

$$\mathscr{F}(x,y)(t) = F_{x,y}(t) = \begin{cases} (\frac{t}{t+1})^{|x-y|}, & t > 0, \\ 0, & t \leq 0 \end{cases}$$

for $x, y \in X$. It is easy to verify that $(X, \mathscr{F}, \Delta_p)$ is a generalized Menger PM-space. Now, assume that $t, s, r > 0$ and $x, y, z, w \in X$. Then we have

$$\begin{aligned} \Delta_p\bigl(F_{x,z}(t), F_{z,w}(s), F_{w,y}(r)\bigr) &= \left(\frac{t}{t+1}\right)^{|x-z|} \left(\frac{s}{s+1}\right)^{|z-w|} \left(\frac{r}{r+1}\right)^{|w-y|} \\ &\leq \left(\frac{\max\{t,s,r\}}{\max\{t,s,r\}+1}\right)^{|x-z|+|z-w|+|w-y|} \\ &\leq \left(\frac{\max\{t,s,r\}}{\max\{t,s,r\}+1}\right)^{|x-y|} \\ &= F_{x,y}\bigl(\max\{t,s,r\}\bigr). \end{aligned}$$

Hence $(X, \mathscr{F}, \Delta_p)$ is a generalized non-Archimedean Menger PM-space.

Proposition 2.1 *Let $(X, \mathscr{F}, \Delta)$ be a generalized Menger PM-space and Δ be a continuous t-norm. Then $(X, \mathscr{F}, \Delta)$ is a Hausdorff topological space in the (ε, λ)-topology $\mathscr{T}$, i.e., the family of sets*

$$\bigl\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0,1], x \in X\bigr\}$$

is a base of neighborhoods of a point x for $\mathscr{T}$, where

$$U_x(\varepsilon, \lambda) = \bigl\{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\bigr\}.$$

Proof It suffices to prove that:

(i) For any $x \in X$, there exists an $U = U_x(\varepsilon, \lambda)$ such that $x \in U$.
(ii) For any given $U_x(\varepsilon_1, \lambda_1)$ and $U_x(\varepsilon_2, \lambda_2)$, there exist $\varepsilon > 0$ and $\lambda > 0$, such that $U_x(\varepsilon, \lambda) \subset U_x(\varepsilon_1, \lambda_1) \cap U_x(\varepsilon_2, \lambda_2)$.
(iii) For any $y \in U_x(\varepsilon, \lambda)$, there exist $\varepsilon' > 0$ and $\lambda' > 0$, such that $U_y(\varepsilon', \lambda') \subset U_x(\varepsilon, \lambda)$.
(iv) For any $x, y \in X$, $x \neq y$, there exist $U_x(\varepsilon_1, \lambda_1)$ and $U_y(\varepsilon_2, \lambda_2)$, such that $U_x(\varepsilon_1, \lambda_1) \cap U_y(\varepsilon_2, \lambda_2) = \emptyset$.

It is easy to check that (i)-(iii) are true. Now we prove that (iv) is also true. In fact, suppose that $x, y \in X$ and $x \neq y$. Then there exist $t_0 > 0$ and $0 \leq a < 1$, such that $F_{x,y}(t_0) = a$. Let

$$U_x = \left\{r : F_{x,r}\left(\frac{t_0}{3}\right) > b\right\}, \qquad U_y = \left\{r : F_{y,r}\left(\frac{t_0}{3}\right) > b\right\},$$

where $0 < b < 1$ and $\Delta(b,1,b) > a$ (since Δ is continuous and $\Delta(1,1,1) = 1$, such b exists). Now suppose that there exists a point $w \in U_x \cap U_y$, which implies that $F_{x,w}(\frac{t_0}{3}) > b$ and $F_{y,w}(\frac{t_0}{3}) > b$. Take $v = w$. Then we have

$$a = F_{x,y}(t_0) \geq \Delta\left(F_{x,w}\left(\frac{t_0}{3}\right), F_{w,v}\left(\frac{t_0}{3}\right), F_{v,y}\left(\frac{t_0}{3}\right)\right) \geq \Delta(b,1,b) > a,$$

which is a contradiction. Thus the conclusion (iv) is proved. This completes the proof. □

Definition 2.3 Let $(X,\mathscr{F},\Delta)$ be a generalized Menger PM-space, Δ be a continuous t-norm.

(i) A sequence $\{x_n\}$ in X is said to be $\mathscr{T}$-convergent to $x \in X$ if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$.
(ii) A sequence $\{x_n\}$ in X is said to be a $\mathscr{T}$-Cauchy sequence, if for any given $\varepsilon > 0$ and $\lambda \in (0,1]$, there exists a positive integer $N = N(\varepsilon,\lambda)$, such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ whenever $n,m \geq N$.
(iii) $(X,\mathscr{F},\Delta)$ is said to be $\mathscr{T}$-complete, if each $\mathscr{T}$-Cauchy sequence in X is $\mathscr{T}$-convergent to some point in X.

Definition 2.4 A t-norm Δ is said to be of H-type if the family of functions $\{\Delta^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t = 1$, where

$$\Delta^1(t) = t, \qquad \Delta^{n+1}(t) = \Delta\big(t,t,\Delta^n(t)\big), \quad n = 1,2,\ldots, t \in [0,1].$$

Definition 2.5 Let X be a non-empty set, $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings. A is said to be commutative with T, if $AT(x,y,z) = T(Ax,Ay,Az)$ for all $x,y,z \in X$. A point $u \in X$ is called a tripled common fixed point of T and A, if $u = Au = T(u,u,u)$.

Imitating the proof in [9], we can easily obtain the following lemma.

Lemma 2.1 *Let $(X,\mathscr{F},\Delta)$ be a generalized Menger PM-space. For each $\lambda \in (0,1]$, define a function $d_\lambda : X \times X \to R^+$ by*

$$d_\lambda(x,y) = \inf\big\{t > 0 : F_{x,y}(t) > 1-\lambda\big\}. \tag{2.1}$$

Then the following statements hold:
(1) $d_\lambda(x,y) < t$ *if and only if* $F_{x,y}(t) > 1-\lambda$;
(2) $d_\lambda(x,y) = d_\lambda(y,x)$ *for all* $x,y \in X$ *and* $\lambda \in (0,1]$;
(3) $d_\lambda(x,y) = 0$ *if and only if* $x = y$;
(4) $d_\lambda(x,z) \leq d_\mu(x,y) + d_\mu(y,z)$ *for all* $x,y,z \in X$ *and* $\mu \in (0,\lambda]$.

The following lemmas play an important role in proving our main results in Sections 3 and 4.

Lemma 2.2 ([17]) *Suppose that $F \in \mathscr{D}$. For any $n \in Z^+$, let $F_n : R \to [0,1]$ be nondecreasing, and $g_n : (0,+\infty) \to (0,+\infty)$ satisfy $\lim_{n\to\infty} g_n(t) = 0$ for all $t > 0$. If $F_n(g_n(t)) \geq F(t)$ for all $t > 0$, then $\lim_{n\to\infty} F_n(t) = 1$ for all $t > 0$.*

Lemma 2.3 *Let X be a nonempty set, $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings. If $T(X \times X \times X) \subset A(X)$, then there exist three sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ in X, such that $Ax_{n+1} = T(x_n, y_n, z_n)$, $Ay_{n+1} = T(y_n, x_n, z_n)$, and $Az_{n+1} = T(z_n, x_n, y_n)$.*

Proof Let x_0, y_0, z_0 be any given points in X. Since $T(X \times X \times X) \subset A(X)$, we can choose $x_1, y_1, z_1 \in X$, such that $Ax_1 = T(x_0, y_0, z_0)$, $Ay_1 = T(y_0, x_0, z_0)$, and $Az_1 = T(z_0, x_0, y_0)$. Continuing this process, we can construct three sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, and $\{z_n\}_{n=1}^{\infty}$ in X, such that $Ax_{n+1} = T(x_n, y_n, z_n)$, $Ay_{n+1} = T(y_n, x_n, z_n)$, and $Az_{n+1} = T(z_n, x_n, y_n)$. □

3 Tripled common fixed point results in generalized PM-spaces

Lemma 3.1 *Let $(X, \mathscr{F}, \Delta)$ be a generalized Menger PM-space, $\{d_\lambda\}_{\lambda \in (0,1]}$ be a family of pseudo-metrics on X defined by (2.1). If Δ is a t-norm of H-type, then for each $\lambda \in (0,1]$, there exists $\mu \in (0, \lambda]$ such that for all $m \in Z^+$ and $x_0, x_1, \ldots, x_m \in X$,*

$$d_\lambda(x_0, x_m) \le \sum_{i=0}^{m-1} d_\mu(x_i, x_{i+1}).$$

Proof Since Δ is a t-norm of H-type, $\{\Delta^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t = 1$, and so for each $\lambda \in (0,1]$, there exists $\mu \in (0, \lambda]$, such that

$$\Delta^n(1-\mu) > 1-\lambda, \quad \forall n \in Z^+. \tag{3.1}$$

For any given $m \in Z^+$ and $x_0, x_1, \ldots, x_m \in X$, we write $d_\mu(x_i, x_{i+1}) = t_i$ ($i = 0, 1, \ldots, m-1$). For any $\varepsilon > 0$, it is evident that $d_\mu(x_i, x_{i+1}) < t_i + \varepsilon$. By Lemma 2.1, we have

$$F_{x_i, x_{i+1}}(t_i + \varepsilon) > 1 - \mu, \quad i = 0, 1, \ldots, m-1. \tag{3.2}$$

It follows from (3.1)-(3.2), and (GPM-4) that

$$\begin{aligned}
&F_{x_0, x_m}\left(\sum_{i=0}^{m-1} t_i + m\varepsilon\right) \\
&\quad \ge \Delta\bigl(F_{x_0,x_1}(t_0+\varepsilon), F_{x_1,x_2}(t_1+\varepsilon), \Delta\bigl(F_{x_2,x_3}(t_2+\varepsilon), F_{x_3,x_4}(t_3+\varepsilon), \\
&\qquad \Delta\bigl(\ldots, \Delta\bigl(F_{x_{m-3},x_{m-2}}(t_{m-3}+\varepsilon), F_{x_{m-2},x_{m-1}}(t_{m-2}+\varepsilon), F_{x_{m-1},x_m}(t_{m-1}+\varepsilon)\bigr)\cdots\bigr)\bigr)\bigr) \\
&\quad \ge \Delta^m(1-\mu) > 1-\lambda.
\end{aligned}$$

Using Lemma 2.1 again, we have $d_\lambda(x_0, x_m) < \sum_{i=0}^{m-1} t_i + m\varepsilon$. By the arbitrariness of ε, we have

$$d_\lambda(x_0, x_m) \le \sum_{i=0}^{m-1} t_i = \sum_{i=0}^{m-1} d_\mu(x_i, x_{i+1}).$$

This completes the proof. □

Theorem 3.1 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized Menger PM-space with Δ a t-norm of H-type, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and*

$\lim_{n\to\infty}\varphi^n(t)=0$ *for any* $t>0$. *Let* $T:X\times X\times X\to X$ *and* $A:X\to X$ *be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big)\geq\big[F_{Ax,Ap}(t)F_{Ay,Aq}(t)F_{Az,Ar}(t)\big]^{\frac{1}{3}}\tag{3.3}$$

for all $x,y,z,p,q,r\in X$ *and* $t>0$, *where* $T(X\times X\times X)\subset A(X)$, A *is continuous and commutative with* T. *Then* T *and* A *have a unique tripled common fixed point in* X.

Proof By Lemma 2.3, we can construct three sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$, and $\{z_n\}_{n=0}^{\infty}$ in X, such that $Ax_{n+1}=T(x_n,y_n,z_n)$, $Ay_{n+1}=T(y_n,x_n,z_n)$, and $Az_{n+1}=T(z_n,x_n,y_n)$.

From (3.3), for all $t>0$ we have

$$\begin{aligned}F_{Ax_n,Ax_{n+1}}\big(\varphi(t)\big)&=F_{T(x_{n-1},y_{n-1},z_{n-1}),T(x_n,y_n,z_n)}\big(\varphi(t)\big)\\&\geq\big[F_{Ax_{n-1},Ax_n}(t)F_{Ay_{n-1},Ay_n}(t)F_{Az_{n-1},Az_n}(t)\big]^{\frac{1}{3}},\end{aligned}\tag{3.4}$$

$$\begin{aligned}F_{Ay_n,Ay_{n+1}}\big(\varphi(t)\big)&=F_{T(y_{n-1},x_{n-1},z_{n-1}),T(y_n,x_n,z_n)}\big(\varphi(t)\big)\\&\geq\big[F_{Ay_{n-1},Ay_n}(t)F_{Ax_{n-1},Ax_n}(t)F_{Az_{n-1},Az_n}(t)\big]^{\frac{1}{3}}\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}F_{Az_n,Az_{n+1}}\big(\varphi(t)\big)&=F_{T(z_{n-1},x_{n-1},y_{n-1}),T(z_n,x_n,y_n)}\big(\varphi(t)\big)\\&\geq\big[F_{Az_{n-1},Az_n}(t)F_{Ax_{n-1},Ax_n}(t)F_{Ay_{n-1},Ay_n}(t)\big]^{\frac{1}{3}}.\end{aligned}\tag{3.6}$$

Denote $P_n(t)=[F_{Ax_{n-1},Ax_n}(t)F_{Ay_{n-1},Ay_n}(t)F_{Az_{n-1},Az_n}(t)]^{\frac{1}{3}}$. From (3.4)-(3.6), we have

$$\begin{aligned}P_{n+1}\big(\varphi(t)\big)&=\big[F_{Ax_n,Ax_{n+1}}\big(\varphi(t)\big)F_{Ay_n,Ay_{n+1}}\big(\varphi(t)\big)F_{Az_n,Az_{n+1}}\big(\varphi(t)\big)\big]^{\frac{1}{3}}\\&\geq\big[P_n(t)P_n(t)P_n(t)\big]^{\frac{1}{3}}=P_n(t),\end{aligned}$$

which implies that

$$F_{Ax_n,Ax_{n+1}}\big(\varphi^n(t)\big)\geq P_n\big(\varphi^{n-1}(t)\big)\geq\cdots\geq P_1(t),\tag{3.7}$$

$$F_{Ay_n,Ay_{n+1}}\big(\varphi^n(t)\big)\geq P_n\big(\varphi^{n-1}(t)\big)\geq\cdots\geq P_1(t)\tag{3.8}$$

and

$$F_{Az_n,Az_{n+1}}\big(\varphi^n(t)\big)\geq P_n\big(\varphi^{n-1}(t)\big)\geq\cdots\geq P_1(t).\tag{3.9}$$

Since $P_1(t)=[F_{Ax_0,Ax_1}(t)F_{Ay_0,Ay_1}(t)F_{Az_0,Az_1}(t)]^{\frac{1}{3}}\in\mathscr{D}$ and $\lim_{n\to\infty}\varphi^n(t)=0$ for each $t>0$, using Lemma 2.2, we have

$$\lim_{n\to\infty}F_{Ax_n,Ax_{n+1}}(t)=1,\qquad\lim_{n\to\infty}F_{Ay_n,Ay_{n+1}}(t)=1,\qquad\lim_{n\to\infty}F_{Az_n,Az_{n+1}}(t)=1.\tag{3.10}$$

Thus

$$\lim_{n\to\infty}P_n(t)=1,\quad\forall t>0.\tag{3.11}$$

We claim that for any $k \in Z^+$ and $t > 0$,

$$F_{Ax_n,Ax_{n+k}}(t) \geq \Delta^k\left(P_n\left(\frac{t-\varphi(t)}{2}\right)\right), \tag{3.12}$$

$$F_{Ay_n,Ay_{n+k}}(t) \geq \Delta^k\left(P_n\left(\frac{t-\varphi(t)}{2}\right)\right) \tag{3.13}$$

and

$$F_{Az_n,Az_{n+k}}(t) \geq \Delta^k\left(P_n\left(\frac{t-\varphi(t)}{2}\right)\right). \tag{3.14}$$

In fact, by (3.7)-(3.9), it is easy to see that (3.12)-(3.14) hold for $k = 1$. Assume that (3.12)-(3.14) hold for some k. Since $\varphi(t) < t$, by (3.4) we have $F_{Ax_n,Ax_{n+1}}(t) \geq F_{Ax_n,Ax_{n+1}}(\varphi(t)) \geq P_n(t)$. By (3.3) and (3.12)-(3.14), we have

$$\begin{aligned} F_{Ax_{n+1},Ax_{n+k+1}}\bigl(\varphi(t)\bigr) &\geq \bigl[F_{Ax_n,Ax_{n+k}}(t)F_{Ay_n,Ay_{n+k}}(t)F_{Az_n,Az_{n+k}}(t)\bigr]^{\frac{1}{3}} \\ &\geq \Delta^k\left(P_n\left(\frac{t-\varphi(t)}{2}\right)\right). \end{aligned}$$

Hence, by the monotonicity of Δ, we have

$$\begin{aligned} F_{Ax_n,Ax_{n+k+1}}(t) &= F_{Ax_n,Ax_{n+k+1}}\bigl(t-\varphi(t)+\varphi(t)\bigr) \\ &\geq \Delta\left(F_{Ax_n,Ax_{n+1}}\left(\frac{t-\varphi(t)}{2}\right), F_{Ax_{n+1},Ax_{n+1}}\left(\frac{t-\varphi(t)}{2}\right),\right. \\ &\qquad \left. F_{Ax_{n+1},Ax_{n+k+1}}\bigl(\varphi(t)\bigr)\right) \\ &\geq \Delta\left(P_n\left(\frac{t-\varphi(t)}{2}\right), P_n\left(\frac{t-\varphi(t)}{2}\right), \Delta^k\left(P_n\left(\frac{t-\varphi(t)}{2}\right)\right)\right) \\ &= \Delta^{k+1}\left(P_n\left(\frac{t-\varphi(t)}{2}\right)\right). \end{aligned}$$

Similarly, we have $F_{Ay_n,Ay_{n+k+1}}(t) \geq \Delta^{k+1}(P_n(\frac{t-\varphi(t)}{2}))$ and $F_{Az_n,Az_{n+k+1}}(t) \geq \Delta^{k+1}(P_n(\frac{t-\varphi(t)}{2}))$. Therefore, by induction, (3.12)-(3.14) hold for all $k \in Z^+$ and $t > 0$.

Suppose that $\lambda \in (0,1]$ is given. Since Δ is a t-norm of H-type, there exists $\delta > 0$ such that

$$\Delta^k(s) > 1-\lambda, \quad s \in (1-\delta,1], k \in Z^+. \tag{3.15}$$

By (3.11), there exists $N \in Z^+$, such that $P_n(\frac{t-\varphi(t)}{2}) > 1-\delta$ for all $n \geq N$. Hence, from (3.12)-(3.15), we get $F_{Ax_n,Ax_{n+k}}(t) > 1-\lambda$, $F_{Ay_n,Ay_{n+k}}(t) > 1-\lambda$, $F_{Az_n,Az_{n+k}}(t) > 1-\lambda$ for all $n \geq N$, $k \in Z^+$. Therefore $\{Ax_n\}$, $\{Ay_n\}$, and $\{Az_n\}$ are Cauchy sequences.

Since $(X,\mathscr{F},\Delta)$ is complete, there exist $u,v,w \in X$, such that $\lim_{n\to\infty} Ax_n = u$, $\lim_{n\to\infty} Ay_n = v$ and $\lim_{n\to\infty} Az_n = w$. By the continuity of A, we have

$$\lim_{n\to\infty} AAx_n = Au, \qquad \lim_{n\to\infty} AAy_n = Av, \qquad \lim_{n\to\infty} AAz_n = Aw.$$

The commutativity of A with T implies that $AAx_{n+1} = AT(x_n, y_n, z_n) = T(Ax_n, Ay_n, Az_n)$. From (3.3) and $\varphi(t) < t$, we obtain

$$\begin{aligned} F_{AAx_{n+1},T(u,v,w)}(t) &\geq F_{AAx_{n+1},T(u,v,w)}\big(\varphi(t)\big) \\ &= F_{T(Ax_n,Ay_n,Az_n),T(u,v,w)}\big(\varphi(t)\big) \\ &\geq \big[F_{AAx_n,Au}(t)F_{AAy_n,Av}(t)F_{AAz_n,Aw}(t)\big]^{\frac{1}{3}}. \end{aligned} \tag{3.16}$$

Letting $n \to \infty$ in (3.16), we have $\lim_{n\to\infty} AAx_n = T(u, v, w)$. Hence, $T(u, v, w) = Au$. Similarly, we can show that $T(v, u, w) = Av$ and $T(w, u, v) = Aw$.

Next we show that $Au = v$, $Av = u$, and $Aw = w$. In fact, from (3.3), for all $t > 0$ we have

$$\begin{aligned} F_{Au,Ay_n}\big(\varphi(t)\big) &= F_{T(u,v,w),T(y_{n-1},x_{n-1},z_{n-1})}\big(\varphi(t)\big) \\ &\geq \big[F_{Au,Ay_{n-1}}(t)F_{Av,Ax_{n-1}}(t)F_{Aw,Az_{n-1}}(t)\big]^{\frac{1}{3}}, \end{aligned} \tag{3.17}$$

$$F_{Av,Ax_n}\big(\varphi(t)\big) \geq \big[F_{Av,Ax_{n-1}}(t)F_{Au,Ay_{n-1}}(t)F_{Aw,Az_{n-1}}(t)\big]^{\frac{1}{3}} \tag{3.18}$$

and

$$F_{Aw,Az_n}\big(\varphi(t)\big) \geq \big[F_{Aw,Az_{n-1}}(t)F_{Au,Ax_{n-1}}(t)F_{Av,Ay_{n-1}}(t)\big]^{\frac{1}{3}}. \tag{3.19}$$

Denote $Q_n(t) = F_{Au,Ay_n}(t)F_{Av,Ax_n}(t)F_{Aw,Az_n}(t)$. By (3.17)-(3.19), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$, and hence for all $t > 0$

$$Q_n\big(\varphi^n(t)\big) \geq Q_{n-1}\big(\varphi^{n-1}(t)\big) \geq \cdots \geq Q_0(t).$$

Thus, for all $t > 0$ we have

$$\begin{aligned} &F_{Au,Ay_n}\big(\varphi^n(t)\big) \geq \big[Q_0(t)\big]^{\frac{1}{3}}, \qquad F_{Av,Ax_n}\big(\varphi^n(t)\big) \geq \big[Q_0(t)\big]^{\frac{1}{3}}, \\ &F_{Aw,Az_n}\big(\varphi^n(t)\big) \geq \big[Q_0(t)\big]^{\frac{1}{3}}. \end{aligned}$$

Since $[Q_0(t)]^{\frac{1}{3}} \in \mathscr{D}$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, by Lemma 2.2, we conclude that

$$\lim_{n\to\infty} Ax_n = Av, \qquad \lim_{n\to\infty} Ay_n = Au, \qquad \lim_{n\to\infty} Az_n = Aw. \tag{3.20}$$

This shows that $Au = v$, $Av = u$, and $Aw = w$. Hence, $v = T(u, v, w)$, $u = T(v, u, w)$, and $w = T(w, u, v)$. Finally, we prove that $u = v$. By (3.3), for all $t > 0$ we have

$$\begin{aligned} F_{u,v}\big(\varphi(t)\big) &= F_{T(v,u,w),T(u,v,w)}\big(\varphi(t)\big) \\ &\geq \big[F_{Av,Au}(t)F_{Au,Av}(t)F_{Aw,Aw}(t)\big]^{\frac{1}{3}} = \big[F_{u,v}(t)\big]^{\frac{2}{3}}, \end{aligned} \tag{3.21}$$

which implies that $F_{u,v}(\varphi^n(t)) \geq [F_{u,v}(t)]^{(\frac{2}{3})^n}$. Using Lemma 2.2, we have $F_{u,v}(t) = 1$ for all $t > 0$, *i.e.*, $u = v$. Similarly, we can show that $u = w$. Hence, there exists $u \in X$, such that $u = Au = T(u, u, u)$.

Finally, we show the uniqueness of the tripled common fixed point of T and A. Suppose that $u' \in X$ is another tripled common fixed point of T and A, *i.e.*, $u' = Au' = T(u', u', u')$. By (3.3), for all $t > 0$ we have

$$\begin{aligned} F_{u,u'}\big(\varphi(t)\big) &= F_{T(u,u,u),T(u',u',u')}\big(\varphi(t)\big) \\ &\geq \big[F_{Au,Au'}(t)F_{Au,Au'}(t)F_{Au,Au'}(t)\big]^{\frac{1}{3}} \\ &\geq F_{Au,Au'}(t) = F_{u,u'}(t), \end{aligned} \tag{3.22}$$

which implies that $F_{u,u'}(\varphi^n(t)) \geq F_{u,u'}(t)$ for all $t > 0$. Using Lemma 2.2, we have $F_{u,u'}(t) = 1$ for all $t > 0$, *i.e.*, $u = u'$. This completes the proof. □

Corollary 3.1 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized Menger PM-space with Δ a t-norm of H-type and $\Delta \geq \Delta_p$, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, for each $t > 0$. Let $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big) \geq \big[\Delta\big(F_{Ax,Ap}(t), F_{Ay,Aq}(t), F_{Az,Ar}(t)\big)\big]^{\frac{1}{3}} \tag{3.23}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$, where $T(X \times X \times X) \subset A(X)$, A is continuous and commutative with T. Then T and A have a unique tripled common fixed point in X.

Letting $A = I$ (I is the identity mapping) in Corollary 3.1, we can obtain the following corollary.

Corollary 3.2 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized Menger PM-space with Δ a t-norm of H-type and $\Delta \geq \Delta_p$, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, for any $t > 0$. Let $T : X \times X \times X \to X$ be a mapping satisfying*

$$F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big) \geq \big[\Delta\big(F_{x,p}(t), F_{y,q}(t), F_{z,r}(t)\big)\big]^{\frac{1}{3}}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$. Then T has a unique fixed point in X.

Letting $\varphi(t) = \alpha t$ ($0 < \alpha < 1$) in Corollary 3.1, we can obtain the following corollary.

Corollary 3.3 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized Menger PM-space with Δ a t-norm of H-type and $\Delta \geq \Delta_p$, $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}(\alpha t) \geq \big[\Delta\big(F_{Ax,Ap}(t), F_{Ay,Aq}(t), F_{Az,Ar}(t)\big)\big]^{\frac{1}{3}}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$, where $0 < \alpha < 1$, $T(X \times X \times X) \subset A(X)$, A is continuous and commutative with T. Then T and A have a unique tripled common fixed point in X.

Letting $A = I$ (I is the identity mapping) in Theorem 3.1, we can obtain the following corollary.

Corollary 3.4 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized Menger PM-space with Δ a t-norm of H-type, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$, and*

$\lim_{n\to\infty} \varphi^n(t) = 0$ *for any* $t > 0$. *Let* $T : X \times X \times X \to X$ *be a mapping satisfying*

$$F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big) \geq \big[F_{x,p}(t)F_{y,q}(t)F_{z,r}(t)\big]^{\frac{1}{3}}$$

for all $x, y, z, p, q, r \in X$ *and* $t > 0$. *Then* T *has a unique fixed point in* X.

Letting $\varphi(t) = \alpha t$ ($0 < \alpha < 1$) in Theorem 3.1, we can obtain the following corollary.

Corollary 3.5 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete generalized Menger PM-space with* Δ *a t-norm of H-type,* $T : X \times X \times X \to X$ *and* $A : X \to X$ *be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}(\alpha t) \geq \big[F_{Ax,Ap}(t)F_{Ay,Aq}(t)F_{Az,Ar}(t)\big]^{\frac{1}{3}}$$

for all $x, y, z, p, q, r \in X$ *and* $t > 0$, *where* $0 < \alpha < 1$, $T(X \times X \times X) \subset A(X)$, A *is continuous and commutative with* T. *Then* T *and* A *have a unique tripled common fixed point in* X.

From the proof of Theorem 3.1, we can similarly prove the following result.

Theorem 3.2 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete generalized Menger PM-space with* Δ *a t-norm of H-type,* $\varphi : R^+ \to R^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, *and* $\lim_{n\to\infty} \varphi^n(t) = +\infty$ *for any* $t > 0$. *Let* $T : X \times X \times X \to X$ *and* $A : X \to X$ *be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}(t) \geq \min\big\{F_{Ax,Ap}\big(\varphi(t)\big), F_{Ay,Aq}\big(\varphi(t)\big), F_{Az,Ar}\big(\varphi(t)\big)\big\} \tag{3.24}$$

for all $x, y, z, p, q, r \in X$ *and* $t > 0$, *where* $T(X \times X \times X) \subset A(X)$, A *is continuous and commutative with* T. *Then* T *and* A *have a unique tripled common fixed point in* X.

Letting $A = I$ (I is the identity mapping) in Theorem 3.2, we can obtain the following corollary.

Corollary 3.6 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete generalized Menger PM-space with* Δ *a t-norm of H-type,* $\varphi : R^+ \to R^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) > t$, *and* $\lim_{n\to\infty} \varphi^n(t) = +\infty$ *for any* $t > 0$. *Let* $T : X \times X \times X \to X$ *be a mapping satisfying*

$$F_{T(x,y,z),T(p,q,r)}(t) \geq \min\big\{F_{x,p}\big(\varphi(t)\big), F_{y,q}\big(\varphi(t)\big), F_{z,r}\big(\varphi(t)\big)\big\}$$

for all $x, y, z, p, q, r \in X$ *and* $t > 0$. *Then* T *has a unique fixed point in* X.

4 Tripled common fixed point results in generalized non-Archimedean PM-spaces

In this section, we will use the results in Section 3 to get some corresponding results in generalized non-Archimedean Menger spaces.

Lemma 4.1 *Let* $(X, \mathscr{F}, \Delta)$ *be a complete generalized non-Archimedean Menger PM-space,* $\{d_\lambda\}_{\lambda\in(0,1]}$ *be a family of pseudo-metrics on* X *defined by* (2.1). *If* Δ *is a t-norm of H-type,* *then for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$, such that for all $m \in Z^+$ and $x_0, x_1, \ldots, x_m \in X$,*

$$d_\lambda(x_0, x_m) \le \max_{0 \le i \le m-1} d_\mu(x_i, x_{i+1}).$$

Proof Since Δ is a t-norm of H-type, $\{\Delta^n(t)\}_{n=1}^\infty$ is equicontinuous at $t = 1$, and so for each $\lambda \in (0,1]$, there exists $\mu \in (0,\lambda]$ such that

$$\Delta^n(1-\mu) > 1-\lambda, \quad \forall n \in Z^+. \tag{4.1}$$

For any given $m \in Z^+$, and $x_0, x_1, \ldots, x_m \in X$, write $d_\mu(x_i, x_{i+1}) = t_i$ ($i = 0, 1, \ldots, m-1$). For any $\varepsilon > 0$, we have $F_{x_i, x_{i+1}}(t_i + \varepsilon) > 1 - \mu$. It follows from (4.1) and (GPM-5) that

$$\begin{aligned}
&F_{x_0,x_m}\Big(\max_{0\le i\le m-1} t_i + \varepsilon\Big)\\
&\quad \ge \Delta\big(F_{x_0,x_1}(t_0+\varepsilon), F_{x_1,x_2}(t_1+\varepsilon), \Delta\big(F_{x_2,x_3}(t_2+\varepsilon), F_{x_3,x_4}(t_3+\varepsilon),\\
&\qquad \Delta\big(\ldots, \Delta\big(F_{x_{m-3},x_{m-2}}(t_{m-3}+\varepsilon), F_{x_{m-2},x_{m-1}}(t_{m-2}+\varepsilon), F_{x_{m-1},x_m}(t_{m-1}+\varepsilon)\big)\cdots\big)\big)\big)\\
&\quad \ge \Delta^m(1-\mu) > 1-\lambda.
\end{aligned}$$

By Lemma 2.1, we have $d_\lambda(x_0, x_m) < \max_{0\le i\le m-1} t_i + \varepsilon$. By the arbitrariness of ε, we have

$$d_\lambda(x_0, x_m) \le \max_{0 \le i \le m-1} t_i = \max_{0 \le i \le m-1} d_\mu(x_i, x_{i+1}).$$

This completes the proof. □

Theorem 4.1 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized non-Archimedean Menger PM-space such that $\sup_{0<t<1} \Delta(t,t,t) = 1$ and $\Delta \ge \Delta_p$, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}(t) \ge \big[\Delta\big(F_{Ax,Ap}\big(\varphi(t)\big), F_{Ay,Aq}\big(\varphi(t)\big), F_{Az,Ar}\big(\varphi(t)\big)\big)\big]^{\frac{1}{3}} \tag{4.2}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$, where $T(X \times X \times X) \subset A(X)$, A is continuous and commutative with T. Suppose that there exist $b, c, d \in X$, such that for any $t > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \prod_{i=n}^\infty F_{Ab,T(b,c,d)}\big(\varphi^i(t)\big) = 1, \qquad \lim_{n\to\infty} \prod_{i=n}^\infty F_{Ac,T(c,d,b)}\big(\varphi^i(t)\big) = 1,\\
&\lim_{n\to\infty} \prod_{i=n}^\infty F_{Ad,T(d,b,c)}\big(\varphi^i(t)\big) = 1.
\end{aligned} \tag{4.3}$$

Then T and A have a unique tripled common fixed point in X.

Proof Take $x_0 = b$, $y_0 = c$, and $z_0 = d$. By Lemma 2.3, we can construct three sequences $\{x_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, and $\{z_n\}_{n=0}^\infty$ in X, such that $Ax_{n+1} = T(x_n, y_n, z_n)$, $Ay_{n+1} = T(y_n, x_n, z_n)$, and $Az_{n+1} = T(z_n, x_n, y_n)$.

From (4.2), for all $t > 0$, we have

$$\begin{aligned} F_{Ax_n,Ax_{n+1}}(t) &= F_{T(x_{n-1},y_{n-1},z_{n-1}),T(x_n,y_n,z_n)}(t) \\ &\geq \bigl[\Delta\bigl(F_{Ax_{n-1},Ax_n,}\bigl(\varphi(t)\bigr),F_{Ay_{n-1},Ay_n}\bigl(\varphi(t)\bigr),F_{Az_{n-1},Az_n}\bigl(\varphi(t)\bigr)\bigr)\bigr]^{\frac{1}{3}}, \qquad (4.4) \\ F_{Ay_n,Ay_{n+1}}(t) &= F_{T(y_{n-1},x_{n-1},z_{n-1}),T(y_n,x_n,z_n)}(t) \\ &\geq \bigl[\Delta\bigl(F_{Ay_{n-1},Ay_n}\bigl(\varphi(t)\bigr),F_{Ax_{n-1},Ax_n}\bigl(\varphi(t)\bigr),F_{Az_{n-1},Az_n}\bigl(\varphi(t)\bigr)\bigr)\bigr]^{\frac{1}{3}} \qquad (4.5) \end{aligned}$$

and

$$\begin{aligned} F_{Az_n,Az_{n+1}}(t) &= F_{T(z_{n-1},x_{n-1},y_{n-1}),T(z_n,x_n,y_n)}(t) \\ &\geq \bigl[\Delta\bigl(F_{Az_{n-1},Az_n}\bigl(\varphi(t)\bigr),F_{Ax_{n-1},Ax_n}\bigl(\varphi(t)\bigr),F_{Ay_{n-1},Ay_n}\bigl(\varphi(t)\bigr)\bigr)\bigr]^{\frac{1}{3}}. \qquad (4.6) \end{aligned}$$

Denote $G_n(t) = [\Delta(F_{Ax_{n-1},Ax_n}(t),F_{Ay_{n-1},Ay_n}(t),F_{Az_{n-1},Az_n}(t))]^{\frac{1}{3}}$. From (4.4)-(4.6), and $\Delta \geq \Delta_p$, we obtain

$$\begin{aligned} G_{n+1}(t) &\geq \bigl[\Delta\bigl(G_n\bigl(\varphi(t)\bigr),G_n\bigl(\varphi(t)\bigr),G_n\bigl(\varphi(t)\bigr)\bigr)\bigr]^{\frac{1}{3}} \\ &\geq \bigl[G_n\bigl(\varphi(t)\bigr)G_n\bigl(\varphi(t)\bigr)G_n\bigl(\varphi(t)\bigr)\bigr]^{\frac{1}{3}} = G_n\bigl(\varphi(t)\bigr), \end{aligned}$$

which implies that

$$G_{n+1}(t) \geq G_n\bigl(\varphi(t)\bigr) \geq G_{n-1}\bigl(\varphi^2(t)\bigr) \geq \cdots \geq G_1\bigl(\varphi^n(t)\bigr). \qquad (4.7)$$

Thus, by (4.4)-(4.7), we have

$$\begin{aligned} &F_{Ax_n,Ax_{n+1}}(t) \geq G_1\bigl(\varphi^n(t)\bigr), \qquad F_{Ay_n,Ay_{n+1}}(t) \geq G_1\bigl(\varphi^n(t)\bigr), \\ &F_{Az_n,Az_{n+1}}(t) \geq G_1\bigl(\varphi^n(t)\bigr). \end{aligned} \qquad (4.8)$$

Suppose that $\varepsilon > 0$ and $\lambda \in (0,1]$. By (4.3), there exists $N \in Z^+$, such that

$$\prod_{i=n}^{n+k-1} F_{Ax_0,Ax_1}\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right) > 1 - \lambda, \qquad \prod_{i=n}^{n+k-1} F_{Ay_0,Ay_1}\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right) > 1 - \lambda$$

and

$$\prod_{i=n}^{n+k-1} F_{Az_0,Az_1}\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right) > 1 - \lambda$$

for all $n \geq N$ and $k \in Z^+$.

Hence, it follows from (4.8) and (GPM-4) that

$$\begin{aligned} &F_{Ax_n,Ax_{n+k}}(\varepsilon) \\ &\quad \geq \Delta\left(F_{Ax_n,Ax_{n+1}}\left(\frac{\varepsilon}{k}\right),F_{Ax_{n+1},Ax_{n+2}}\left(\frac{\varepsilon}{k}\right),\Delta\left(F_{Ax_{n+2},Ax_{n+3}}\left(\frac{\varepsilon}{k}\right),F_{Ax_{n+3},Ax_{n+4}}\left(\frac{\varepsilon}{k}\right),\right.\right. \end{aligned}$$

$$
\begin{aligned}
&\Delta\left(\ldots,\Delta\left(F_{Ax_{n+k-3},Ax_{n+k-2}}\left(\frac{\varepsilon}{k}\right),F_{Ax_{n+k-2},Ax_{n+k-1}}\left(\frac{\varepsilon}{k}\right),F_{Ax_{n+k-1},Ax_{n+k}}\left(\frac{\varepsilon}{k}\right)\right)\cdots\right)\right)\\
&\geq \Delta\left(G_1\left(\varphi^n\left(\frac{\varepsilon}{k}\right)\right),G_1\left(\varphi^{n+1}\left(\frac{\varepsilon}{k}\right)\right),\Delta\left(G_1\left(\varphi^{n+2}\left(\frac{\varepsilon}{k}\right)\right),G_1\left(\varphi^{n+3}\left(\frac{\varepsilon}{k}\right)\right),\right.\right.\\
&\quad\Delta\left(\ldots,\Delta\left(G_1\left(\varphi^{n+k-3}\left(\frac{\varepsilon}{k}\right)\right),G_1\left(\varphi^{n+k-2}\left(\frac{\varepsilon}{k}\right)\right),G_1\left(\varphi^{n+k-1}\left(\frac{\varepsilon}{k}\right)\right)\right)\cdots\right)\right)\\
&\geq \prod_{i=n}^{n+k-1} G_1\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right)\\
&\geq \prod_{i=n}^{n+k-1}\left[F_{Ax_0,Ax_1}\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right)F_{Ay_0,Ay_1}\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right)F_{Az_0,Az_1}\left(\varphi^i\left(\frac{\varepsilon}{k}\right)\right)\right]^{\frac{1}{3}}\\
&> 1-\lambda.
\end{aligned}
\tag{4.9}
$$

This shows that $\{Ax_n\}$ is a Cauchy sequence. Similarly, we can show that $\{Ay_n\}$ and $\{Az_n\}$ are Cauchy sequences.

Since $(X,\mathscr{F},\Delta)$ is complete, there exist $u,v,w\in X$, such that $\lim_{n\to\infty}Ax_n=u$, $\lim_{n\to\infty}Ay_n=v$, and $\lim_{n\to\infty}Az_n=w$. By the continuity of A, we have

$$
\lim_{n\to\infty}AAx_n=Au,\qquad \lim_{n\to\infty}AAy_n=Av,\qquad \lim_{n\to\infty}AAz_n=Aw. \tag{4.10}
$$

From (4.2) and the commutativity of A with T, we have

$$
\begin{aligned}
F_{AAx_{n+1},T(u,v,w)}(t) &= F_{AT(x_n,y_n,z_n),T(u,v,w)}(t)=F_{T(Ax_n,Ay_n,Az_n),T(u,v,w)}(t)\\
&\geq \left[\Delta\left(F_{AAx_n,Au}\left(\varphi(t)\right),F_{AAy_n,Av}\left(\varphi(t)\right),F_{AAz_n,Aw}\left(\varphi(t)\right)\right)\right]^{\frac{1}{3}}\\
&\geq \left[F_{AAx_n,Au}\left(\varphi(t)\right)F_{AAy_n,Av}\left(\varphi(t)\right)F_{AAz_n,Aw}\left(\varphi(t)\right)\right]^{\frac{1}{3}}.
\end{aligned}
\tag{4.11}
$$

Letting $n\to\infty$ in (4.11), we have $\lim_{n\to\infty}AAx_n=T(u,v,w)$. Hence, $T(u,v,w)=Au$. Similarly, we have $T(v,u,w)=Av$ and $T(w,u,v)=Aw$.

Next we claim that $Au=v$, $Av=u$, and $Aw=w$. In fact, by (4.2), we have

$$
\begin{aligned}
F_{Au,Ay_n}(t) &= F_{T(u,v,w),T(y_{n-1},x_{n-1},z_{n-1})}(t)\\
&\geq \left[\Delta\left(F_{Au,Ay_{n-1}}\left(\varphi(t)\right),F_{Av,Ax_{n-1}}\left(\varphi(t)\right),F_{Aw,Az_{n-1}}\left(\varphi(t)\right)\right)\right]^{\frac{1}{3}}\\
&\geq \left[F_{Au,Ay_{n-1}}\left(\varphi(t)\right)F_{Av,Ax_{n-1}}\left(\varphi(t)\right)F_{Aw,Az_{n-1}}\left(\varphi(t)\right)\right]^{\frac{1}{3}},
\end{aligned}
\tag{4.12}
$$

$$
F_{Av,Ax_n}(t)\geq\left[F_{Av,Ax_{n-1}}\left(\varphi(t)\right)F_{Au,Ay_{n-1}}\left(\varphi(t)\right)F_{Aw,Az_{n-1}}\left(\varphi(t)\right)\right]^{\frac{1}{3}} \tag{4.13}
$$

and

$$
F_{Aw,Az_n}(t)\geq\left[F_{Aw,Az_{n-1}}\left(\varphi(t)\right)F_{Au,Ax_{n-1}}\left(\varphi(t)\right)F_{Av,Ay_{n-1}}\left(\varphi(t)\right)\right]^{\frac{1}{3}}. \tag{4.14}
$$

Denote $Q_n(t)=F_{Au,Ay_n}(t)F_{Av,Ax_n}(t)F_{Aw,Az_n}(t)$. It follows from (4.12)-(4.14) that

$$
Q_n(t)\geq Q_{n-1}\left(\varphi(t)\right)\geq\cdots\geq Q_0\left(\varphi^n(t)\right),
$$

and thus

$$\begin{aligned}&F_{Au,Ay_n}(t) \geq \left[Q_0\left(\varphi^n(t)\right)\right]^{\frac{1}{3}}, \qquad F_{Av,Ax_n}(t) \geq \left[Q_0\left(\varphi^n(t)\right)\right]^{\frac{1}{3}},\\ &F_{Aw,Az_n}(t) \geq \left[Q_0\left(\varphi^n(t)\right)\right]^{\frac{1}{3}}.\end{aligned} \tag{4.15}$$

Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$, we have

$$\left[Q_0\left(\varphi^n(t)\right)\right]^{\frac{1}{3}} = \left[F_{Au,Ay_0}\left(\varphi^n(t)\right)F_{Av,Ax_0}\left(\varphi^n(t)\right)F_{Aw,Az_0}\left(\varphi^n(t)\right)\right]^{\frac{1}{3}} \to 1,$$

as $n \to \infty$. From (4.15), we have

$$\lim_{n\to\infty} Ax_n = Av, \qquad \lim_{n\to\infty} Ay_n = Au, \qquad \lim_{n\to\infty} Az_n = Aw. \tag{4.16}$$

Hence, $Au = v$, $Av = u$, and $Aw = w$, *i.e.*, $v = T(u,v,w)$, $u = T(v,u,w)$, and $w = T(w,u,v)$. Now we prove that $u = v$. In fact, by (4.2), we have

$$\begin{aligned}F_{u,v}(t) &= F_{T(v,u,w),T(u,v,w)}(t)\\ &\geq \left[\Delta\left(F_{Av,Au}\left(\varphi(t)\right), F_{Au,Av}\left(\varphi(t)\right), F_{Aw,Aw}\left(\varphi(t)\right)\right)\right]^{\frac{1}{3}}\\ &\geq \left[F_{u,v}\left(\varphi(t)\right)\right]^{\frac{2}{3}},\end{aligned} \tag{4.17}$$

which implies that $F_{u,v}(t) \geq [F_{u,v}(\varphi^n(t))]^{(\frac{2}{3})^n}$. Letting $n \to \infty$, we have $F_{u,v}(t) = 1$ for all $t > 0$, *i.e.*, $u = v$. Similarly, we can show that $v = w$. Hence, there exists $u \in X$, such that $u = Au = T(u,u,u)$.

Finally, we show the uniqueness of the tripled common fixed point of T and A. Suppose that $u' \in X$ is another tripled common fixed point of T and A, *i.e.*, $u' = Au' = T(u',u',u')$. By (4.2), for all $t > 0$, we have

$$\begin{aligned}F_{u,u'}(t) &= F_{T(u,u,u),T(u',u',u')}(t)\\ &\geq \left[\Delta\left(F_{Au,Au'}\left(\varphi(t)\right), F_{Au,Au'}\left(\varphi(t)\right), F_{Au,Au'}\left(\varphi(t)\right)\right)\right]^{\frac{1}{3}}\\ &\geq F_{Au,Au'}\left(\varphi(t)\right) = F_{u,u'}\left(\varphi(t)\right),\end{aligned} \tag{4.18}$$

which implies that $F_{u,u'}(t) \geq F_{u,u'}(\varphi^n(t))$ for all $t > 0$. Letting $n \to \infty$, we have $F_{u,u'}(t) = 1$ for all $t > 0$, *i.e.*, $u = u'$. This completes the proof. □

Letting $A = I$ (I is the identity mapping) in Theorem 4.1, we can obtain the following corollary.

Corollary 4.1 *Let* $(X,\mathscr{F},\Delta)$ *be a complete generalized non-Archimedean Menger PM-space such that* $\sup_{0<t<1} \Delta(t,t,t) = 1$ *and* $\Delta \geq \Delta_p$, $\varphi : R^+ \to R^+$ *be a gauge function such that* $\varphi^{-1}(\{0\}) = \{0\}$ *and* $\lim_{n\to\infty} \varphi^n(t) = +\infty$ *for any* $t > 0$. *Let* $T : X \times X \times X \to X$ *be a mapping satisfying*

$$F_{T(x,y,z),T(p,q,r)}(t) \geq \left[\Delta\left(F_{x,p}\left(\varphi(t)\right), F_{y,q}\left(\varphi(t)\right), F_{z,r}\left(\varphi(t)\right)\right)\right]^{\frac{1}{3}}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$. Suppose that there exist $b, c, d \in X$, such that for any $t > 0$,

$$\begin{aligned} &\lim_{n\to\infty}\prod_{i=n}^{\infty} F_{b,T(b,c,d)}\big(\varphi^i(t)\big) = 1, \qquad \lim_{n\to\infty}\prod_{i=n}^{\infty} F_{c,T(c,d,b)}\big(\varphi^i(t)\big) = 1, \\ &\lim_{n\to\infty}\prod_{i=n}^{\infty} F_{d,T(d,b,c)}\big(\varphi^i(t)\big) = 1. \end{aligned} \tag{4.19}$$

Then T has a unique fixed point in X.

In a similar way, we can obtain the following result.

Theorem 4.2 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized non-Archimedean Menger PM-space such that Δ is a t-norm of H-type, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big) \ge \min\big\{F_{Ax,Ap}(t), F_{Ay,Aq}(t), F_{Az,Ar}(t)\big\} \tag{4.20}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$, where $T(X \times X \times X) \subset A(X)$, A is continuous and commutative with T. Then T and A have a unique tripled common fixed point in X.

Letting $A = I$ (I is the identity mapping) in Theorem 4.2, we can obtain the following corollary.

Corollary 4.2 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized non-Archimedean Menger PM-space such that Δ is a t-norm of H-type, $\varphi : R^+ \to R^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $T : X \times X \times X \to X$ be a mapping satisfying*

$$F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big) \ge \min\big\{F_{x,p}(t), F_{y,q}(t), F_{z,r}(t)\big\} \tag{4.21}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$. Then T has a unique fixed point in X.

Letting $\varphi(t) = \alpha t$ ($0 < \alpha < 1$) in Theorem 4.2, we can obtain the following corollary.

Corollary 4.3 *Let $(X, \mathscr{F}, \Delta)$ be a complete generalized non-Archimedean Menger PM-space such that Δ is a t-norm of H-type. Let $T : X \times X \times X \to X$ and $A : X \to X$ be two mappings satisfying*

$$F_{T(x,y,z),T(p,q,r)}(\alpha t) \ge \min\big\{F_{Ax,Ap}(t), F_{Ay,Aq}(t), F_{Az,Ar}(t)\big\}$$

for all $x, y, z, p, q, r \in X$ and $t > 0$, where $0 < \alpha < 1$, $T(X \times X \times X) \subset A(X)$, A is continuous and commutative with T. Then T and A have a unique tripled common fixed point in X.

Remark 4.1 If $(X, \mathscr{F}, \Delta)$ is a generalized non-Archimedean Menger PM-space, then the hypotheses concerning gauge functions can be weakened. Let us note that in Theorem 4.2 the gauge function only satisfies $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$, and it does not necessarily satisfy $\varphi(t) < t$ for all $t > 0$.

5 An application

In this section, we shall provide an example to show the validity of the main results of this paper.

Example 5.1 Suppose that $X = [-1,1] \subset R$, $\Delta = \Delta_m$. Then Δ_m is a t-norm of H-type and $\Delta_m \geq \Delta_p$. Define $\mathscr{F} : X \times X \to \mathscr{D}$ by

$$\mathscr{F}(x,y)(t) = F_{x,y}(t) = \begin{cases} e^{-\frac{|x-y|}{t}}, & t > 0, x, y \in X, \\ 0, & t \leq 0, x, y \in X. \end{cases}$$

We claim that $(X, \mathscr{F}, \Delta_m)$ is a generalized Menger PM-space. In fact, it is easy to verify (GPM-1), (GPM-2), and (GPM-3). Assume that for any $s, t, r > 0$ and $x, y, z, w \in X$,

$$\Delta_m\big(F_{x,z}(t), F_{z,w}(s), F_{w,y}(r)\big) = \min\big\{e^{-\frac{|x-z|}{t}}, e^{-\frac{|z-w|}{s}}, e^{-\frac{|w-y|}{r}}\big\} = e^{-\frac{|x-z|}{t}}.$$

Then we have $t|z-w| \leq s|x-z|$, $t|w-y| \leq r|x-z|$, and so $\frac{t+s+r}{t}|x-z| = |x-z| + \frac{s}{t}|x-z| + \frac{r}{t}|x-z| \geq |x-z| + |z-w| + |w-y| \geq |x-y|$. It follows that

$$F_{x,y}(t+s+r) = e^{-\frac{|x-y|}{t+s+r}} \geq e^{-\frac{|x-z|}{t}} = \Delta_m\big(F_{x,z}(t), F_{z,w}(s), F_{w,y}(r)\big).$$

Hence (GPM-4) holds. It is obvious that $(X, \mathscr{F}, \Delta_m)$ is complete. Suppose that $\varphi(t) = \frac{t}{3}$. For $x, y, z \in X$, define $T : X \times X \times X \to X$ as follows:

$$T(x,y,z) = \frac{1}{81} - \frac{x^2}{81} - \frac{y^2}{81} - \frac{|z|}{27}.$$

Then for each $t > 0$ and $x, y, z, p, q, r \in X$, we have

$$\begin{aligned} \big|p^2 - x^2 + q^2 - y^2 + 3\big(|r| - |z|\big)\big| &\leq |p-x|\big(|p| + |x|\big) + |q-y|\big(|q| + |y|\big) + 3|r-z| \\ &\leq 9\max\big\{|x-p|, |y-q|, |z-r|\big\}, \end{aligned}$$

and so

$$\begin{aligned} F_{T(x,y,z),T(p,q,r)}\big(\varphi(t)\big) &= F_{T(x,y,z),T(p,q,r)}\left(\frac{t}{3}\right) \\ &= e^{-\frac{|p^2-x^2+q^2-y^2+3(|r|-|z|)|}{27t}} \\ &\geq \min\big\{e^{-\frac{|x-p|}{3t}}, e^{-\frac{|y-q|}{3t}}, e^{-\frac{|z-r|}{3t}}\big\} \\ &= \big(\min\big\{e^{-\frac{|x-p|}{t}}, e^{-\frac{|y-q|}{t}}, e^{-\frac{|z-r|}{t}}\big\}\big)^{\frac{1}{3}} \\ &= \big[\Delta_m\big(F_{x,p}(t), F_{y,q}(t), F_{z,r}(t)\big)\big]^{\frac{1}{3}}. \end{aligned}$$

Thus, all the conditions of Corollary 3.2 are satisfied. Therefore, T has a unique fixed point in X.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the editor and the referees for their constructive comments and suggestions. The research was supported by the Natural Science Foundation of China (11361042, 11071108, 11326099), the Natural Science Foundation of Jiangxi Province of China (20132BAB201001, 20142BAB211016, 2010GZS0147), and the Youth Foundation of the Education Department of Jiangxi (GJJ13012).

Received: 10 December 2013 Accepted: 25 June 2014 Published: 22 Jul 2014

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10.1186/1687-1812-2014-158
Cite this article as: Luo et al.: **Tripled common fixed point theorems under probabilistic φ-contractive conditions in generalized Menger probabilistic metric spaces.** *Fixed Point Theory and Applications* 2014, 2014:158