Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some convergence results for modified SP-iteration scheme in hyperbolic spaces

Aynur Şahin* and Metin Başarır

*Correspondence:
ayuce@sakarya.edu.tr
Department of Mathematics, Faculty of Sciences and Arts, Sakarya University, Sakarya, 54187, Turkey

Abstract

In this paper, we prove some strong and Δ-convergence theorems for a modified SP-iteration scheme for total asymptotically nonexpansive mappings in hyperbolic spaces by employing recent technical results of Khan *et al.* (Fixed Point Theory Appl. 2012:54, 2012). The results presented here extend and improve some well-known results in the current literature.

MSC: 47H09; 47H10

Keywords: hyperbolic space; fixed point; total asymptotically nonexpansive mapping; modified SP-iteration scheme; strong convergence; Δ-convergence

1 Introduction and preliminaries

Iterative schemes play a key role in approximating fixed points of nonlinear mappings. Structural properties of the underlying space, such as strict convexity and uniform convexity, are very much needed for the development of iterative fixed point theory in it. Hyperbolic spaces are general in nature and inherit rich geometrical structure suitable to obtain new results in topology, graph theory, multi-valued analysis and metric fixed point theory.

Kohlenbach [1] introduced the hyperbolic spaces, defined below, which play a significant role in many branches of mathematics.

A hyperbolic space (X,d,W) is a metric space (X,d) together with a mapping $W : X \times X \times [0,1] \to X$ satisfying

(W1) $d(z,W(x,y,\lambda)) \le (1-\lambda)d(z,x) + \lambda d(z,y)$,

(W2) $d(W(x,y,\lambda_1),W(x,y,\lambda_2)) = |\lambda_1-\lambda_2|d(x,y)$,

(W3) $W(x,y,\lambda) = W(y,x,(1-\lambda))$,

(W4) $d(W(x,z,\lambda),W(y,w,\lambda)) \le (1-\lambda)d(x,y) + \lambda d(z,w)$

for all $x,y,z,w \in X$ and $\lambda,\lambda_1,\lambda_2 \in [0,1]$.

A subset K of a hyperbolic space X is convex if $W(x,y,\lambda) \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$. If a space satisfies only (W1), it coincides with the convex metric space introduced by Takahashi [2]. The concept of hyperbolic spaces in [1] is more restrictive than the hyperbolic type introduced by Goebel *et al.* [3] since (W1)-(W3) together are equivalent to (X,d,W) being a space of hyperbolic type in [3]. Also it is slightly more general than the hyperbolic space defined by Reich *et al.* [4]. The class of hyperbolic spaces in [1] contains all normed linear spaces and convex subsets thereof, $\mathbb{R}$-trees, the Hilbert ball with the hyperbolic metric (see [5]), Cartesian products of Hilbert balls, Hadamard manifolds and CAT(0) spaces (see [6]) as special cases. Recently, the concept of p-uniformly convexity has been defined by Naor *et al.* [7] and its nonlinear version for $p = 2$ in hyperbolic spaces was studied by Khan [8]. Any CAT(0) space is 2-uniformly convex (see [9]).

The following example accentuates the importance of hyperbolic spaces.

Let B_H be an open unit ball in a complex Hilbert space $(H, \langle\cdot\rangle)$ w.r.t. the metric (also known as the Kobayashi distance)

$$k_{B_H}(x,y) = \arg\tanh\bigl(1 - \sigma(x,y)\bigr)^{\frac{1}{2}},$$

where

$$\sigma(x,y) = \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x,y\rangle|^2} \quad \text{for all } x,y \in B_H.$$

Then (B_H, k_{B_H}, W) is a hyperbolic space where $W(x,y,\lambda)$ defines a unique point z in a unique geodesic segment $[x,y]$ for all $x,y \in B_H$. For more on hyperbolic spaces and a detailed treatment of examples, we refer the readers to [1].

A hyperbolic space (X,d,W) is said to be

(i) *strictly convex* [2] if for any $x,y \in X$ and $\lambda \in [0,1]$, there exists a unique element $z \in X$ such that $d(z,x) = \lambda d(x,y)$ and $d(z,y) = (1-\lambda)d(x,y)$;

(ii) *uniformly convex* [10] if for all $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r$ whenever $d(x,u) \leq r$, $d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing such $\delta = \eta(r,\varepsilon)$ for given $r > 0$ and $\varepsilon \in (0,2]$ is called *modulus of uniform convexity*. We call η *monotone* if it decreases with r (for a fixed ε). A uniformly convex hyperbolic space is strictly convex (see [11]).

Let K be a nonempty subset of a metric space (X,d) and T be a self-mapping on K. Denote by $F(T) = \{x \in K : T(x) = x\}$ the set of fixed points of T and $d(x,F(T)) = \inf\{d(x,p) : p \in F(T)\}$. A self-mapping T is said to be

(1) nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in K$;

(2) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ such that $d(T^n x, T^n y) \leq k_n d(x,y)$ for all $x,y \in K$ and $n \geq 1$;

(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq L d(x,y)$ for all $x,y \in K$ and $n \geq 1$;

(4) total asymptotically nonexpansive if there exist non-negative real sequences $\{\mu_n\}$, $\{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that

$$d\bigl(T^n x, T^n y\bigr) \leq d(x,y) + \nu_n \zeta\bigl(d(x,y)\bigr) + \mu_n$$

for all $x,y \in K$ and $n \geq 1$ (see [12, Definition 2.1]).

It follows from the above definitions that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1$, $\forall n \geq 1$ and that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\nu_n = k_n - 1$, $\mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$. Moreover, each asymptotically nonexpansive mapping is a uniformly L-Lipschitzian mapping with $L = \sup_{n \in \mathbb{N}}\{k_n\}$. However, the converse of these statements is not true, in general.

It has been shown that every total asymptotically nonexpansive mapping defined on a nonempty bounded closed convex subset of a complete uniformly convex hyperbolic space always has a fixed point (see [13, Theorem 3.1]).

The following iteration process is a translation of the SP-iteration scheme introduced in [14] from Banach spaces to hyperbolic spaces. The SP-iteration is equivalent to Mann, Ishikawa, Noor iterations and converges faster than the others for the class of continuous and non-decreasing functions (see [14]).

$$\begin{cases} x_1 \in K, \\ z_n = W(x_n, T^n x_n, \gamma_n), \\ y_n = W(z_n, T^n z_n, \beta_n), \\ x_{n+1} = W(y_n, T^n y_n, \alpha_n), \quad n \geq 1, \end{cases} \tag{1.1}$$

where K is a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity and $T : K \to K$ is a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping.

Inspired and motivated by Khan *et al.* [15], Khan [16], Fukhar-ud-din and Khan [17], Wan [18], Zhao *et al.* [19] and Zhao *et al.* [20], we prove some strong and $\triangle$-convergence theorems of the modified SP-iteration process for approximating a fixed point of total asymptotically nonexpansive mappings in hyperbolic spaces. The results presented in the paper extend and improve some recent results given in [14, 21].

The concept of $\triangle$-convergence in a metric space was introduced by Lim [22] and its analogue in CAT(0) spaces was investigated by Dhompongsa and Panyanak [23]. In order to define the concept of $\triangle$-convergence in the general setup of hyperbolic spaces, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The *asymptotic radius* $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by

$$\rho = \inf\{r(x, \{x_n\}) : x \in X\}.$$

The *asymptotic center* of $\{x_n\}$ with respect to a subset K of X is defined as follows:

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \leq r(y, \{x_n\}), \forall y \in K\}.$$

This is the set of minimizer of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$.

Recall that a sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that 'bounded sequences have unique asymptotic centers with respect to closed convex subsets.' The following lemma is due to Leustean [24] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1 [24, Proposition 3.3] *Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

In the sequel, we shall need the following results.

Lemma 2 [15, Lemma 2.5] *Let (X, d, W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n, x) \le r, \qquad \limsup_{n\to\infty} d(y_n, x) \le r, \qquad \lim_{n\to\infty} d\big(W(x_n, y_n, \alpha_n), x\big) = r$$

for some $r \ge 0$, then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 3 [15, Lemma 2.6] *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

Lemma 4 [25, Lemma 2] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of non-negative real numbers such that*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

2 Main results

We begin with Δ-convergence of the modified SP-iterative sequence $\{x_n\}$ defined by (1.1) for total asymptotically nonexpansive mappings in hyperbolic spaces.

Theorem 1 *Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : K \to K$ be a uniformly L-Lipschitzian and total asymptotically nonexpansive mapping with $F(T) \neq \emptyset$. If the following conditions are satisfied*:

(i) $\sum_{n=1}^{\infty} \nu_n < \infty$ *and* $\sum_{n=1}^{\infty} \mu_n < \infty$;

(ii) *there exist constants $a, b \in (0, 1)$ such that $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, b]$*;

(iii) *there exists a constant $M > 0$ such that $\zeta(r) \le Mr$, $\forall r \ge 0$*,

then the sequence $\{x_n\}$ defined by (1.1), *Δ-converges to a fixed point of T.*

Proof We divide our proof into three steps.

Step 1. First we prove that the following limits exist:

$$\lim_{n\to\infty} d(x_n,p) \quad \text{for each } p \in F(T) \quad \text{and} \quad \lim_{n\to\infty} d\big(x_n, F(T)\big). \tag{2.1}$$

Since T is a total asymptotically nonexpansive mapping, by condition (iii), we get

$$\begin{aligned} d(z_n,p) &= d\big(W\big(x_n, T^n x_n, \gamma_n\big), p\big) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n d\big(T^n x_n, p\big) \\ &\leq (1-\gamma_n)d(x_n,p) + \gamma_n\big\{d(x_n,p) + \nu_n\zeta\big(d(x_n,p)\big) + \mu_n\big\} \\ &= d(x_n,p) + \gamma_n\nu_n\zeta\big(d(x_n,p)\big) + \gamma_n\mu_n \\ &\leq (1+\gamma_n\nu_n M)d(x_n,p) + \gamma_n\mu_n \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(y_n,p) &= d\big(W\big(z_n, T^n z_n, \beta_n\big), p\big) \\ &\leq (1-\beta_n)d(z_n,p) + \beta_n d\big(T^n z_n, p\big) \\ &\leq (1-\beta_n)d(z_n,p) + \beta_n\big\{d(z_n,p) + \nu_n\zeta\big(d(z_n,p)\big) + \mu_n\big\} \\ &\leq (1+\beta_n\nu_n M)d(z_n,p) + \beta_n\mu_n. \end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.3) and simplifying it, we have

$$\begin{aligned} d(y_n,p) &\leq (1+\beta_n\nu_n M)\big\{(1+\gamma_n\nu_n M)d(x_n,p) + \gamma_n\mu_n\big\} + \beta_n\mu_n \\ &\leq \big(1+\nu_n M(\beta_n+\gamma_n+\beta_n\gamma_n\nu_n M)\big)d(x_n,p) \\ &\quad + \mu_n(\beta_n+\gamma_n+\beta_n\gamma_n\nu_n M). \end{aligned} \tag{2.4}$$

Similarly, we obtain

$$d(x_{n+1},p) \leq (1+\alpha_n\nu_n M)d(y_n,p) + \alpha_n\mu_n. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$d(x_{n+1},p) \leq (1+\sigma_n)d(x_n,p) + \xi_n, \quad \forall n \geq 1 \text{ and } p \in F(T), \tag{2.6}$$

and so

$$d\big(x_{n+1}, F(T)\big) \leq (1+\sigma_n)d\big(x_n, F(T)\big) + \xi_n, \quad \forall n \geq 1,$$

where $\sigma_n = \nu_n M(\alpha_n + \beta_n + \gamma_n + \nu_n M(\alpha_n\beta_n + \beta_n\gamma_n + \alpha_n\gamma_n + \alpha_n\beta_n\gamma_n\nu_n M))$ and $\xi_n = \alpha_n + \beta_n + \gamma_n + \nu_n M(\alpha_n\beta_n + \beta_n\gamma_n + \alpha_n\gamma_n + \alpha_n\beta_n\gamma_n\nu_n M)$. By virtue of condition (i),

$$\sum_{n=1}^{\infty}\sigma_n < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}\xi_n < \infty.$$

By Lemma 4, $\lim_{n\to\infty} d(x_n, F(T))$ and $\lim_{n\to\infty} d(x_n,p)$ exist for each $p \in F(T)$.

Step 2. Next we prove that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

In fact, it follows from (2.1) that $\lim_{n\to\infty} d(x_n, p)$ exists for each given $p \in F(T)$. We may assume that $\lim_{n\to\infty} d(x_n, p) = r$. The case $r = 0$ is trivial. Next, we deal with the case $r > 0$. Taking lim sup on both sides in inequality (2.4), we have

$$\limsup_{n\to\infty} d(y_n, p) \leq r. \tag{2.7}$$

Since

$$\begin{aligned} d(T^n y_n, p) &\leq d(y_n, p) + \nu_n \zeta\big(d(y_n, p)\big) + \mu_n \\ &\leq (1 + \nu_n M) d(y_n, p) + \mu_n, \quad \forall n \geq 1, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(T^n y_n, p) \leq r. \tag{2.8}$$

In addition,

$$\lim_{n\to\infty} d(x_{n+1}, p) = \lim_{n\to\infty} d\big(W(y_n, T^n y_n, \alpha_n), p\big) = r. \tag{2.9}$$

With the help of (2.7)-(2.9) and Lemma 2, we have

$$\lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{2.10}$$

On the other hand, since

$$\begin{aligned} d(x_{n+1}, p) &\leq d(x_{n+1}, T^n y_n) + d(T^n y_n, p) \\ &\leq (1 - \alpha_n) d(y_n, T^n y_n) + (1 + \nu_n M) d(y_n, p) + \mu_n, \end{aligned}$$

we have $\liminf_{n\to\infty} d(y_n, p) \geq r$. Combined with (2.7), it yields that $\lim_{n\to\infty} d(y_n, p) = r$. This implies that

$$\lim_{n\to\infty} d\big(W(z_n, T^n z_n, \beta_n), p\big) = r. \tag{2.11}$$

Taking lim sup on both sides in inequality (2.2), we have

$$\limsup_{n\to\infty} d(z_n, p) \leq r. \tag{2.12}$$

Since

$$\begin{aligned} d(T^n z_n, p) &\leq d(z_n, p) + \nu_n \zeta\big(d(z_n, p)\big) + \mu_n \\ &\leq (1 + \nu_n M) d(z_n, p) + \mu_n, \quad \forall n \geq 1, \end{aligned}$$

we have

$$\limsup_{n\to\infty} d(T^n z_n, p) \leq r. \tag{2.13}$$

With the help of (2.11)-(2.13) and Lemma 2, we have

$$\lim_{n\to\infty} d\big(z_n, T^n z_n\big) = 0. \tag{2.14}$$

By the same method, we can also prove that

$$\lim_{n\to\infty} d\big(x_n, T^n x_n\big) = 0. \tag{2.15}$$

By (2.10), we get

$$d(x_{n+1}, y_n) \le d\big(W\big(y_n, T^n y_n, \alpha_n\big), y_n\big) \le \alpha_n d\big(y_n, T^n y_n\big) \to 0 \quad \text{as } n \to \infty.$$

In a similar way, we have

$$d(y_n, z_n) \le \beta_n d\big(z_n, T^n z_n\big) \to 0 \quad \text{as } n \to \infty$$

and

$$d(z_n, x_n) \le \alpha_n d\big(x_n, T^n x_n\big) \to 0 \quad \text{as } n \to \infty.$$

It follows that

$$d(x_{n+1}, x_n) \le d(x_{n+1}, y_n) + d(y_n, z_n) + d(z_n, x_n) \to 0 \quad \text{as } n \to \infty. \tag{2.16}$$

Since T is uniformly L-Lipschitzian, therefore we obtain

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d\big(x_{n+1}, T^{n+1} x_{n+1}\big) + d\big(T^{n+1} x_{n+1}, T^{n+1} x_n\big) + d\big(T^{n+1} x_n, Tx_n\big) \\ &\le (1 + L) d(x_{n+1}, x_n) + d\big(x_{n+1}, T^{n+1} x_{n+1}\big) + L d\big(T^n x_n, x_n\big). \end{aligned}$$

Hence, (2.15) and (2.16) imply that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{2.17}$$

Step 3. Now we prove that the sequence $\{x_n\}$ $\triangle$-converges to a fixed point of T.

In fact, for each $p \in F(T)$, $\lim_{n\to\infty} d(x_n, p)$ exists. This implies that the sequence $\{x_n\}$ is bounded. Hence by virtue of Lemma 1, $\{x_n\}$ has a unique asymptotic center $A_K(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A_K(\{u_n\}) = \{u\}$. Then, by (2.17), we have

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0. \tag{2.18}$$

We claim that $u \in F(T)$. In fact, we define a sequence $\{z_m\}$ in K by $z_m = T^m u$. So, we calculate

$$\begin{aligned} d(z_m, u_n) &\le d\big(T^m u, T^m u_n\big) + d\big(T^m u_n, T^{m-1} u_n\big) + \cdots + d(Tu_n, u_n) \\ &\le d(u, u_n) + \nu_n \zeta\big(d(u, u_n)\big) + \mu_n + \sum_{i=1}^{m} d\big(T^i u_n, T^{i-1} u_n\big) \\ &\le (1 + \nu_n M) d(u, u_n) + \mu_n + \sum_{i=1}^{m} d\big(T^i u_n, T^{i-1} u_n\big). \end{aligned} \tag{2.19}$$

Since T is uniformly L-Lipschitzian, from (2.19), we have

$$d(z_m, u_n) \leq (1 + \nu_n M) d(u, u_n) + \mu_n + mLd(Tu_n, u_n).$$

Taking lim sup on both sides of the above estimate and using (2.18), we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 3 that $\lim_{m\to\infty} T^m u = u$. Utilizing the uniform continuity of T, we have that

$$Tu = T\Big(\lim_{m\to\infty} T^m u\Big) = \lim_{m\to\infty} T^{m+1} u = u.$$

Hence $u \in F(T)$. Moreover, $\lim_{n\to\infty} d(x_n, u)$ exists by (2.1). Suppose that $x \neq u$. By the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\
&\leq \limsup_{n\to\infty} d(x_n, x) \\
&< \limsup_{n\to\infty} d(x_n, u) \\
&= \limsup_{n\to\infty} d(u_n, u)
\end{aligned}$$

a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$, that is, $\{x_n\}$ $\triangle$-converges to $x \in F(T)$. The proof is completed. □

We now discuss the strong convergence of the modified SP-iteration for total asymptotically nonexpansive mappings in hyperbolic spaces.

Theorem 2 *Let K, X, T and $\{x_n\}$ be the same as in Theorem* 1. *Suppose that conditions* (i)-(iii) *in Theorem* 1 *are satisfied. Then $\{x_n\}$ converges strongly to some $p \in F(T)$ if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof If $\{x_n\}$ converges to $p \in F(T)$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F(T)) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from (2.1) that $\lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence. In fact, it follows from (2.6) that for any $p \in F(T)$,

$$d(x_{n+1}, p) \leq (1 + \sigma_n) d(x_n, p) + \xi_n, \quad \forall n \geq 1,$$

where $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. Hence for any positive integers n, m, we have

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p) + d(p, x_n) \leq (1 + \sigma_{n+m-1}) d(x_{n+m-1}, p) + \xi_{n+m-1} + d(x_n, p).$$

Since for each $x \geq 0$, $1 + x \leq e^x$, we have

$$\begin{aligned}
d(x_{n+m}, x_n) &\leq e^{\sigma_{n+m-1}} d(x_{n+m-1}, p) + \xi_{n+m-1} + d(x_n, p) \\
&\leq e^{\sigma_{n+m-1}+\sigma_{n+m-2}} d(x_{n+m-2}, p) + e^{\sigma_{n+m-1}} \xi_{n+m-2} + \xi_{n+m-1} + d(x_n, p) \\
&\leq \cdots \\
&\leq e^{\sum_{i=n}^{n+m-1} \sigma_i} d(x_n, p) + e^{\sum_{i=n+1}^{n+m-1} \sigma_i} \xi_n + e^{\sum_{i=n+2}^{n+m-2} \sigma_i} \xi_{n+1} + \cdots \\
&\quad + e^{\sigma_{n+m-1}} \xi_{n+m-2} + \xi_{n+m-1} + d(x_n, p) \\
&\leq (1 + N) d(x_n, p) + N \sum_{i=n}^{n+m-1} \xi_i,
\end{aligned}$$

where $N = e^{\sum_{i=1}^{\infty} \sigma_i} < \infty$. Therefore we have

$$d(x_{n+m}, x_n) \leq (1 + N) d\bigl(x_n, F(T)\bigr) + N \sum_{i=n}^{n+m-1} \xi_i \to 0 \quad \text{as } n, m \to \infty.$$

This shows that $\{x_n\}$ is a Cauchy sequence in K. Since K is a closed subset in a complete hyperbolic space X, it is complete. We can assume that $\{x_n\}$ converges strongly to some point $p^\star \in K$. It is easy to prove that $F(T)$ is a closed subset in K, so is $F(T)$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, we obtain $p^\star \in F(T)$. This completes the proof. □

Remark 1 In Theorem 2, the condition $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F(T)) = 0$.

Example 1 Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K = [-1, 1]$. Define two mappings $T_1, T_2 : K \to K$ by

$$T_1(x) = \begin{cases} -2\sin\frac{x}{2} & \text{if } x \in [0, 1], \\ 2\sin\frac{x}{2} & \text{if } x \in [-1, 0), \end{cases}$$

and

$$T_2(x) = \begin{cases} x & \text{if } x \in [0, 1], \\ -x & \text{if } x \in [-1, 0). \end{cases}$$

It is proved in [26, Example 3.1] that both T_1 and T_2 are asymptotically nonexpansive mappings with $k_n = 1$, $\forall n \geq 1$. Therefore they are total asymptotically nonexpansive mappings with $\nu_n = \mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$. Additionally, they are uniformly L-Lipschitzian mappings with $L = 1$. Clearly, $F(T_1) = \{0\}$ and $F(T_2) = \{x \in K; 0 \leq x \leq 1\}$. Set

$$\alpha_n = \frac{n}{2n+1}, \qquad \beta_n = \frac{n}{3n+1} \quad \text{and} \quad \gamma_n = \frac{n}{4n+1} \quad \text{for all } n \geq 1. \tag{2.20}$$

Thus, the conditions of Theorem 1 are fulfilled. Therefore the results of Theorem 1 and Theorem 2 can be easily seen.

Example 2 Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $K = [0,\infty)$. Define two mappings $S_1, S_2 : K \to K$ by $S_1(x) = \sin x$ and $S_2(x) = x$. It is proved in [27, Example 1] that both S_1 and S_2 are total asymptotically nonexpansive mappings with $\nu_n = \frac{1}{n^2}$, $\mu_n = \frac{1}{n^3}$, $\forall n \geq 1$. Additionally, they are uniformly L-Lipschitzian mappings with $L = 1$. Clearly, $F(S_1) = \{0\}$ and $F(S_2) = \{x \in K; 0 \leq x < \infty\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the same as in (2.20). Similarly, the conditions of Theorem 1 are satisfied. So, the results of Theorem 1 and Theorem 2 also can be received.

Recall that a mapping T from a subset K of a metric space (X,d) into itself is semi-compact if every bounded sequence $\{x_n\} \subset K$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

Senter and Dotson [28, p.375] introduced the concept of condition (I) as follows.

A mapping $T : K \to K$ with $F(T) \neq \emptyset$ is said to satisfy condition (I) if there exists a non-decreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$d(x, Tx) \geq f\bigl(d\bigl(x, F(T)\bigr)\bigr) \quad \text{for all } x \in K.$$

By using the above definitions, we obtain the following strong convergence theorems.

Theorem 3 *Under the assumptions of Theorem* 1, *if* T *is semi-compact, then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof It follows from (2.1) that $\{x_n\}$ is a bounded sequence. Also, by (2.17), we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, by the semi-compactness of T, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $p \in K$. Moreover, by the uniform continuity of T, we have

$$d(p, Tp) = \lim_{k\to\infty} d(x_{n_k}, Tx_{n_k}) = 0.$$

This implies that $p \in F(T)$. Again, by (2.1), $\lim_{n\to\infty} d(x_n, p)$ exists. Hence p is the strong limit of the sequence $\{x_n\}$. As a result, $\{x_n\}$ converges strongly to a fixed point p of T. □

Theorem 4 *Under the assumptions of Theorem* 1, *if* T *satisfies condition* (I), *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof By virtue of (2.1), $\lim_{n\to\infty} d(x_n, F(T))$ exists. Further, by condition (I) and (2.17), we have

$$\lim_{n\to\infty} f\bigl(d\bigl(x_n, F(T)\bigr)\bigr) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now Theorem 2 implies that $\{x_n\}$ converges strongly to a point p in $F(T)$. □

Remark 2 Theorems 1-4 contain the corresponding theorems proved for asymptotically nonexpansive mappings when $\nu_n = k_n - 1$, $\mu_n = 0$, $\forall n \geq 1$, $\zeta(t) = t$, $\forall t \geq 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the editor and referees for their careful reading and valuable comments and suggestions which led to the present form of the paper. This paper was supported by Sakarya University Scientific Research Foundation (Project number: 2013-02-00-003).

Received: 14 January 2014 Accepted: 16 May 2014 Published: 02 Jun 2014

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Cite this article as: Şahin and Başarır: **Some convergence results for modified SP-iteration scheme in hyperbolic spaces.** *Fixed Point Theory and Applications* 2014, 2014:133