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Coupled coincidence point theorems for a ϕ-contractive mapping in partially ordered G-metric spaces without mixed g-monotone property

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Abstract

In this work, we show the existence of a coupled coincidence point and a coupled common fixed point for a ϕ-contractive mapping in G-metric spaces without the mixed g-monotone property, using the concept of a (F^*, g)-invariant set. We also show the uniqueness of a coupled coincidence point and give some examples, which are not applied to the existence of a coupled coincidence point by using the mixed g-monotone property. Further, we apply our results to the existence and uniqueness of a coupled coincidence point of the given mapping in partially ordered G-metric spaces.

Keywords: coupled fixed point; coupled coincidence point; compatible; mixed g-monotone; partially ordered set; G-metric spaces

1 Introduction

In 2004, the existence and uniqueness of a fixed point for contraction type of mappings in partially ordered complete metric spaces has been first considered by Ran and Reurings [1]. Following this initial work, Nieto and Lopez [2, 3] extended the results in [1] for a non-decreasing mapping. Later, Agarwal *et al.* [4] presented some new results for contractions in partially ordered metric spaces.

One of the interesting concepts, a coupled fixed point theorem, was introduced by Guo and Lakshmikantham [5]. Afterwards, Bhaskar and Lakshmikantham [6] introduced the concept of the mixed monotone property and also proved some coupled fixed point theorems for mappings satisfying the mixed monotone property in partially ordered metric spaces. Lakshimikantham and Ćirić [7] extended the results in [6] by defining the mixed g-monotone property and proved the existence and uniqueness of a coupled coincidence point for such mapping satisfying the mixed g-monotone property in partially ordered metric spaces. As a continuation of this work, several results of a coupled fixed point and a coupled coincidence point have been discussed in the recent literature (see, *e.g.*, [7–25]).

Recently, Sintunavarat *et al.* [23] proved the existence and uniqueness of a coupled fixed point for nonlinear contractions in partially ordered metric spaces without mixed monotone property and extended some coupled fixed point theorems of Bhaskar and Lakshmikantham [6]. Later, Charoensawan and Klanarong [17] proved the existence and uniqueness of a coupled coincidence point in partially ordered metric space without the mixed g-monotone property which extended some coupled fixed point theorems of Sintunavarat *et al.* [23].

The concept of a new class of generalized metric spaces, called G-metric space, was introduced by Mustafa and Sims [26]. Choudhury and Maily [27] proved the existence of a coupled fixed point of nonlinear contraction mappings with mixed monotone property in partially ordered G-metric spaces. Later, Abbas *et al.* [28] extended the results of a coupled fixed point for a mixed monotone mapping obtained in [27].

In the case of the coupled coincidence point theory in partially ordered G-metric space, Aydi *et al.* [29] established some coupled coincidence and coupled common fixed point theory for a mixed g-monotone mapping satisfying nonlinear contractions in partially ordered G-metric spaces. They extended the results obtained in [27]. Later, Karapınar *et al.* [30] extended the results of coupled coincidence and coupled common fixed point theorems for a mixed g-monotone mapping obtained in [29]. Some examples dealing with G-metric spaces are discussed in [21, 30–44].

In this work, we generalize the results of Aydi *et al.* [29] by extending the coupled coincidence point theorem of nonlinear contraction mappings in partially ordered G-metric spaces without the mixed g-monotone property using the concept of a (F^*, g)-invariant set in partially ordered G-metric spaces.

2 Preliminaries

In this section, we give some definitions, propositions, examples, and remarks which are used in our main results. Throughout this paper, $(X, \preceq)$ denotes a partially ordered set with the partial order $\preceq$. By $x \preceq y$, we mean $y \succeq x$. A mapping $f : X \to X$ is said to be non-decreasing (resp., non-increasing) if for all $x, y \in X$, $x \preceq y$ implies $f(x) \preceq f(y)$ (resp. $f(y) \succeq f(x)$).

Definition 2.1 [26] Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$.
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables).
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically, a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.2 Let (X, d) be a metric space. The function $G : X \times X \times X \to [0, +\infty)$, defined by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, for all $x, y, z \in X$, is a G-metric on X.

Definition 2.3 [26] Let (X, G) be a G-metric space, and let (x_n) be a sequence of point of X. We say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$, for all $n, m \geq N$. We call x the limit of the sequence (x_n) and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 2.4 [26] *Let* (X, G) *be a* G*-metric space, the following are equivalent*:

(1) (x_n) *is* G*-convergent to* x.
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$.
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$.
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 2.5 [26] Let (X, G) be a G-metric space. A sequence (x_n) is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$. That is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.

Proposition 2.6 [26] *Let* (X, G) *be a* G*-metric space, the following are equivalent*:

(1) *the sequence* (x_n) *is* G*-Cauchy*;
(2) *for any* $\varepsilon > 0$, *there exists* $N \in \mathbb{N}$ *such that* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $n, m \geq N$.

Proposition 2.7 [26] *Let* (X, G) *be a* G*-metric space. A mapping* $f : X \to X$ *is* G*-continuous at* $x \in X$ *if and only if it is* G*-sequentially continuous at* x, *that is, whenever* (x_n) *is* G*-convergent to* x, $(f(x_n))$ *is* G*-convergent to* $f(x)$.

Definition 2.8 [26] A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).

Definition 2.9 [27] Let (X, G) be a G-metric space. A mapping $F : X \times X \to X$ is said to be continuous if for any two G-convergent sequences (x_n) and (y_n) converging to x and y, respectively, $(F(x_n, y_n))$ is G-convergent to $F(x, y)$.

Bhaskar and Lakshmikantham in [6] introduced the following condition.

Definition 2.10 [6] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. We say F has the mixed monotone property if for any $x, y \in X$

$$x_1, x_2 \in X, \quad x_1 \preceq x_2 \quad \text{implies} \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preceq y_2 \quad \text{implies} \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 2.11 [6] An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Lakshmikantham and Ćirić [7] introduced the concept of a mixed g-monotone mapping and a coupled coincidence point as follows.

Definition 2.12 [7] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if for any $x, y \in X$

$$x_1, x_2 \in X, \quad g(x_1) \preceq g(x_2) \quad \text{implies} \quad F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \preceq g(y_2) \quad \text{implies} \quad F(x, y_1) \succeq F(x, y_2).$$

Definition 2.13 [7] An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 2.14 [7] Let X be a nonempty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$.

The following class of functions was considered by Lakshmikantham and Ćirić in [7]. Let Φ denote the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

1. $\phi^{-1}(\{0\}) = \{0\}$,
2. $\phi(t) < t$ for all $t > 0$,
3. $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$.

Lemma 2.15 [7] *Let $\phi \in \Phi$. For all $t > 0$, we have* $\lim_{n \to \infty} \phi^n(t) = 0$.

Aydi *et al.* [29] proved the following theorem.

Theorem 2.16 [29] *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$, and $g : X \to X$ such that*

$$G\big(F(x, u), F(y, v), F(z, w)\big) \leq \phi\left(\frac{G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))}{2}\right)$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \succeq g(y) \succeq g(z)$ and $g(u) \preceq g(v) \preceq g(w)$.

Suppose also that F is continuous and has the mixed g-monotone property, $F(X \times X) \subseteq G(X)$, and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \quad \textit{and} \quad g(y_0) \succeq F(y_0, x_0),$$

then there exists $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Definition 2.17 [29] Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X. We say that $(X, G, \preceq)$ is regular if the following conditions hold:

1. if a non-decreasing sequence $(x_n) \to x$, then $x_n \preceq x$ for all n,
2. if a non-increasing sequence $(y_n) \to y$, then $y \preceq y_n$ for all n.

Theorem 2.18 [29] *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that $(X, G, \preceq)$ is regular. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$, and $g : X \to X$ such that*

$$G\big(F(x, u), F(y, v), F(z, w)\big) \leq \phi\left(\frac{G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))}{2}\right)$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \succeq g(y) \succeq g(z)$ and $g(u) \preceq g(v) \preceq g(w)$.

Suppose also that $(g(X), G)$ is complete, F has the mixed g-monotone property, $F(X \times X) \subseteq G(X)$, and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0, x_0),$$

then there exists $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Batra and Vashistha [45] introduced an (F, g)-invariant set which is a generalization of the F-invariant set introduced by Samet and Vetro [46].

Definition 2.19 [45] Let (X, d) be a metric space and $F : X \times X \to X$, $g : X \to X$ be given mappings. Let M be a nonempty subset of X^4. We say that M is an (F, g)-invariant subset of X^4 if and only if, for all $x, y, z, w \in X$,

(i) $(x, y, z, w) \in M \Leftrightarrow (w, z, y, x) \in M$;
(ii) $(g(x), g(y), g(z), g(w)) \in M \Rightarrow (F(x, y), F(y, x), F(z, w), F(w, z)) \in M$.

Now, we give the notion of an F^*-invariant set and an (F^*, g)-invariant set, which is useful for our main results.

Definition 2.20 Let (X, G) be a G-metric space and $F : X \times X \to X$ be given mapping. Let M be a nonempty subset of X^6. We say that M is an F^*-invariant subset of X^6 if and only if, for all $x, y, z, u, v, w \in X$,

1. $(x, u, y, v, z, w) \in M \Leftrightarrow (w, z, v, y, u, x) \in M$;
2. $(x, u, y, v, z, w) \in M \Rightarrow (F(x, u), F(u, x), F(y, v), F(v, y), F(z, w), F(w, z)) \in M$.

Definition 2.21 Let (X, G) be a G-metric space and $F : X \times X \to X$ and $g : X \to X$ are given mapping. Let M be a nonempty subset of X^6. We say that M is an (F^*, g)-invariant subset of X^6 if and only if, for all $x, y, z, u, v, w \in X$,

1. $(x, u, y, v, z, w) \in M \Leftrightarrow (w, z, v, y, u, x) \in M$;
2. $(g(x), g(u), g(y), g(v), g(z), g(w)) \in M \Rightarrow$
   $(F(x, u), F(u, x), F(y, v), F(v, y), F(z, w), F(w, z)) \in M$.

Definition 2.22 Let (X, G) be a G-metric space and M be a subset of X^6. We say that satisfies *the transitive property* if and only if, for all $x, y, w, z, a, b, c, d, e, f \in X$,

$$(x, y, w, z, a, b) \in M \quad \text{and} \quad (a, b, c, d, e, f) \in M \to (x, y, w, z, e, f) \in M.$$

Remarks

1. The set $M = X^6$ is trivially (F^*, g)-invariant, which satisfies the transitive property.
2. Every F^*-invariant set is (F^*, I_X)-invariant when I_X denote identity map on X.

Example 2.23 Let $(X, \preceq)$ be a partially ordered set and suppose there is a G-metric d on X such that (X, G) is a complete G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be a mapping satisfying the mixed g-monotone property. Define a subset $M \subseteq X^6$ by $M = \{(a, b, c, d, e, f) \in X^6, a \succeq c \succeq e, b \preceq d \preceq f\}$. Then M is an (F^*, g)-invariant subset of X^6, which satisfies the transitive property.

Example 2.24 Let $X = R$ and $F : X \times X \to X$ be defined by $F(x,y) = 1 - x^2$. Let $g : X \to X$ be given by $g(x) = x - 1$. Then it is easy to show that $M = \{(x,0,0,0,0,w) \in X^6 : x = w\}$ is (F^*,g)-invariant subset of X^6 but not F^*-invariant subset of X^6 as $(1,0,0,0,0,1) \in M$ but $(F(1,0),F(0,1),F(0,0),F(0,0),F(0,1),F(1,0)) = (0,1,1,1,1,0) \notin M$.

3 Main results

Theorem 3.1 *Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space and M be a nonempty subset of X^6. Assume that there exists $\phi \in \Phi$ and also suppose that $F : X \times X \to X$ and $g : X \to X$ such that*

$$G\bigl(F(x,u),F(y,v),F(z,w)\bigr) \le \phi\left(\frac{G(g(x),g(y),g(z)) + G(g(u),g(v),g(w))}{2}\right) \tag{1}$$

for all $(g(x),g(u),g(y),g(v),g(z),g(w)) \in M$.

Suppose also that F is continuous, $F(X \times X) \subseteq G(X)$, and g is continuous and commutes with F. If there exist $x_0,y_0 \in X \times X$ such that

$$\bigl(F(x_0,y_0),F(y_0,x_0),F(x_0,y_0),F(y_0,x_0),g(x_0),g(y_0)\bigr) \in M$$

and M is an (F^*,g)-invariant set which satisfies the transitive property, then there exist $x,y \in X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$, that is, F has a coupled coincident point.

Proof Let $(x_0,y_0) \in X \times X$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1,y_1 \in X$ such that

$$g(x_1) = F(x_0,y_0) \quad \text{and} \quad g(y_1) = F(y_0,x_0).$$

Similarly, we can choose $x_2,y_2 \in X$ such that

$$g(x_2) = F(x_1,y_1) \quad \text{and} \quad g(y_2) = F(y_1,x_1).$$

Continuing this process we can construct sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that

$$g(x_n) = F(x_{n-1},y_{n-1}) \quad \text{and} \quad g(y_n) = F(y_{n-1},x_{n-1}) \quad \text{for all } n \ge 1. \tag{2}$$

If there exists $k \in N$ such that $(g(x_{k+1}),g(y_{k+1})) = (g(x_k),g(y_k))$ then $g(x_k) = g(x_{k+1}) = F(x_k,y_k)$ and $g(y_k) = g(y_{k+1}) = F(y_k,x_k)$. Thus, (x_k,y_k) is a coupled coincidence point of F. The proof is completed.

Now we assume that $(g(x_{k+1}),g(y_{k+1})) \neq (g(x_k),g(y_k))$ for all $n \ge 0$. Thus, we have either $g(x_{n+1}) = F(x_n,y_n) \neq g(x_n)$ or $g(y_{n+1}) = F(y_n,x_n) \neq g(y)$ for all $n \ge 0$. Since

$$\begin{aligned}&\bigl(F(x_0,y_0),F(y_0,x_0),F(x_0,y_0),F(y_0,x_0),g(x_0),g(y_0)\bigr)\\&\quad = \bigl(g(x_1),g(y_1),g(x_1),g(y_1),g(x_0),g(y_0)\bigr) \in M\end{aligned}$$

and M is an (F^*,g)-invariant set, we have

$$\begin{aligned}&\bigl(F(x_1,y_1),F(y_1,x_1),F(x_1,y_1),F(y_1,x_1),F(x_0,y_0),F(y_0,x_0)\bigr)\\&\quad = \bigl(g(x_2),g(y_2),g(x_2),g(y_2),g(x_1),g(y_1)\bigr) \in M.\end{aligned}$$

By repeating this argument, we get

$$\begin{aligned}&\big(F(x_2,y_2),F(y_2,x_2),F(x_2,y_2),F(y_2,x_2),F(x_1,y_1),F(y_1,x_1)\big)\\&\quad=\big(g(x_3),g(y_3),g(x_3),g(y_3),g(x_2),g(y_2)\big)\in M\end{aligned}$$

and

$$\begin{aligned}&\big(F(x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1}),F(x_{n-1},y_{n-1}),F(y_{n-1},x_{n-1}),F(x_{n-2},y_{n-2}),F(y_{n-2},x_{n-2})\big)\\&\quad=\big(g(x_n),g(y_n),g(x_n),g(y_n),g(x_{n-1}),g(y_{n-1})\big)\in M.\end{aligned}\tag{3}$$

From (1), (2) and (3), we have

$$\begin{aligned}G\big(g(x_{n+1}),g(x_{n+1}),g(x_n)\big)&=G\big(F(x_n,y_n),F(x_n,y_n),F(x_{n-1},y_{n-1})\big)\\&\leq\phi\left(\frac{G(g(x_n),g(x_n),g(x_{n-1}))+G(g(y_n),g(y_n),g(y_{n-1}))}{2}\right).\end{aligned}\tag{4}$$

From (3) and using the fact that M is an (F^*,g)-invariant set and (1), we have

$$\big(g(y_{n-1}),g(x_{n-1}),g(y_n),g(x_n),g(y_n),g(x_n)\big)\in M,$$

and

$$\begin{aligned}G\big(g(y_{n+1}),g(y_{n+1}),g(y_n)\big)&=G\big(F(y_n,x_n),F(y_n,x_n),F(y_{n-1},x_{n-1})\big)\\&=G\big(F(y_{n-1},x_{x-1}),F(y_n,x_n),F(y_n,x_n)\big)\\&\leq\phi\left(\frac{G(g(y_{n-1}),g(y_n),g(y_n))+G(g(x_{n-1}),g(x_n),g(x_n))}{2}\right).\end{aligned}\tag{5}$$

Let

$$t_n=G\big(g(x_{n+1}),g(x_{n+1}),g(x_n)\big)+G\big(g(y_{n+1}),g(y_{n+1}),g(y_n)\big).\tag{6}$$

Adding (4) with (5) which implies that

$$t_n\leq 2\phi\left(\frac{t_{n-1}}{2}\right).\tag{7}$$

Since $\phi(t)<t$ for all $t>0$, it follows that $\{t_n\}$ is decreasing sequence. Therefore, there is some $\delta\geq 0$ such that $\lim_{n\to\infty}t_n=\delta$.

We shall prove that $\delta=0$. Assume, to the contrary, that $\delta>0$. Then by letting $n\to\infty$ in (7) and using the properties of the map ϕ, we get

$$\delta=\lim_{n\to\infty}t_n\leq\lim_{n\to\infty}2\phi\left(\frac{t_{n-1}}{2}\right)=2\lim_{t_{n-1}\to\delta^+}\phi(t_{n-1})<\delta.$$

A contradiction, thus $\delta=0$ and hence

$$\lim_{n\to\infty}t_n=\lim_{n\to\infty}\big[G\big(g(x_{n+1}),g(x_{n+1}),g(x_n)\big)+G\big(g(y_{n+1}),g(y_{n+1}),g(y_n)\big)\big]=0.\tag{8}$$

Next, we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequence in the G-metric space (X,G). Suppose, to the contrary, that is the least of $\{g(x_n)\}$ and $\{g(y_n)\}$ is not a Cauchy sequence in (X,G). Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{g(x_{m(k)})\}$ and $\{g(x_{n(k)})\}$ of $\{g(x_n)\}$, $\{g(y_{m(k)})\}$ and $\{g(y_{n(k)})\}$ of $\{g(y_n)\}$ with $m(k) > n(k) \geq K$ such that

$$G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{n(k)})\big) + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{n(k)})\big) \geq \varepsilon. \tag{9}$$

Further, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k) \geq K$ and satisfying (9). Then

$$G\big(g(x_{m(k)-1}),g(x_{m(k)-1}),g(x_{n(k)})\big) + G\big(g(y_{m(k)-1}),g(y_{m(k)-1}),g(y_{n(k)})\big) < \varepsilon. \tag{10}$$

Using the rectangle inequality, we get

$$\begin{aligned}
\varepsilon \leq r_k &:= G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{n(k)})\big) + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{n(k)})\big) \\
&\leq G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{m(k)-1})\big) + G\big(g(x_{m(k)-1}),g(x_{m(k)-1}),g(x_{n(k)})\big) \\
&\quad + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{m(k)-1})\big) + G\big(g(y_{m(k)-1}),g(y_{m(k)-1}),g(y_{n(k)})\big) \\
&< t_{m(k)-1} + \varepsilon.
\end{aligned} \tag{11}$$

Letting $k \to +\infty$ in the above inequality and using (8), we get

$$\lim_{k\to\infty} r_k = \lim_{k\to+\infty} G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{n(k)})\big) + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{n(k)})\big) = \varepsilon. \tag{12}$$

Again, by the rectangle inequality, we have

$$\begin{aligned}
r_k &:= G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{n(k)})\big) + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{n(k)})\big) \\
&\leq G\big(g(x_{n(k)+1}),g(x_{n(k)+1}),g(x_{n(k)})\big) + G\big(g(y_{n(k)+1}),g(y_{n(k)+1}),g(y_{n(k)})\big) \\
&\quad + G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{m(k)+1})\big) + G\big(g(x_{m(k)+1}),g(x_{m(k)+1}),g(x_{n(k)+1})\big) \\
&\quad + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{m(k)+1})\big) + G\big(g(y_{m(k)+1}),g(y_{m(k)+1}),g(y_{n(k)+1})\big) \\
&= t_{n(k)} \\
&\quad + G\big(g(x_{m(k)}),g(x_{m(k)}),g(x_{m(k)+1})\big) + G\big(g(x_{m(k)+1}),g(x_{m(k)+1}),g(x_{n(k)+1})\big) \\
&\quad + G\big(g(y_{m(k)}),g(y_{m(k)}),g(y_{m(k)+1})\big) + G\big(g(y_{m(k)+1}),g(y_{m(k)+1}),g(y_{n(k)+1})\big).
\end{aligned}$$

Using the fact that $G(x,x,y) \leq 2G(x,y,y)$ for any $x,y \in X$, we obtain

$$\begin{aligned}
r_k &\leq t_{n(k)} + 2t_{m(k)} + G\big(g(x_{m(k)+1}),g(x_{m(k)+1}),g(x_{n(k)+1})\big) \\
&\quad + G\big(g(y_{m(k)+1}),g(y_{m(k)+1}),g(y_{n(k)+1})\big).
\end{aligned} \tag{13}$$

Since $m(k) > n(k)$, using (3), we have

$$\big(g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)-1}),g(y_{m(k)-1})\big) \in M$$

and

$$\big(g(x_{m(k)-1}),g(y_{m(k)-1}),g(x_{m(k)-1}),g(y_{m(k)-1}),g(x_{m(k)-2}),g(y_{m(k)-2})\big) \in M.$$

From the fact that M is an (F^*,g)-invariant set which satisfies the transitive property, we have

$$\big(g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)-2}),g(y_{m(k)-2})\big) \in M.$$

Again from

$$\big(g(x_{m(k)-2}),g(y_{m(k)-2}),g(x_{m(k)-2}),g(y_{m(k)-2}),g(x_{m(k)-3}),g(y_{m(k)-3})\big) \in M$$

we get

$$\big(g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)-3}),g(y_{m(k)-3})\big) \in M.$$

By this process, we can get

$$\big(g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)}),g(y_{m(k)}),g(x_{n(k)}),g(y_{n(k)})\big) \in M.$$

Now, using (1), we have

$$\begin{aligned} &G\big(g(x_{m(k)+1}),g(x_{m(k)+1}),g(x_{n(k)+1})\big) \\ &\quad = G\big(F(x_{m(k)},y_{m(k)}),F(x_{m(k)},y_{m(k)}),F(x_{n(k)},y_{n(k)})\big) \\ &\quad \le \phi\left(\frac{G(g(x_{m(k)}),g(x_{m(k)}),g(x_{n(k)})) + G(g(y_{m(k)}),g(y_{m(k)}),g(y_{n(k)}))}{2}\right). \end{aligned} \tag{14}$$

Since $(g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)}),g(y_{m(k)}),g(x_{n(k)}),g(y_{n(k)})) \in M$ and M is an (F^*,g)-invariant set, we have

$$\big(g(y_{n(k)}),g(x_{n(k)}),g(y_{m(k)}),g(x_{m(k)}),g(y_{m(k)}),g(x_{m(k)})\big) \in M$$

and

$$\begin{aligned} &G\big(g(y_{m(k)+1}),g(y_{m(k)+1}),g(y_{n(k)+1})\big) \\ &\quad = G\big(F(y_{m(k)},x_{m(k)}),F(y_{m(k)},x_{m(k)}),F(y_{n(k)},x_{n(k)})\big) \\ &\quad = G\big(F(y_{n(k)},x_{n(k)}),F(y_{m(k)},x_{m(k)}),F(y_{m(k)},x_{m(k)})\big) \\ &\quad \le \phi\left(\frac{G(g(y_{n(k)}),g(y_{m(k)}),g(y_{m(k)})) + G(g(x_{n(k)}),g(x_{m(k)}),g(x_{m(k)}))}{2}\right). \end{aligned} \tag{15}$$

Adding (14) to (15), we get

$$\begin{aligned} &G\big(g(x_{m(k)+1}),g(x_{m(k)+1}),g(x_{n(k)+1})\big) + G\big(g(y_{m(k)+1}),g(y_{m(k)+1}),g(y_{n(k)+1})\big) \\ &\quad = G\big(F(x_{m(k)},y_{m(k)}),F(x_{m(k)},y_{m(k)}),F(x_{n(k)},y_{n(k)})\big) \end{aligned}$$

$$\begin{aligned}
&+ G\big(F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)})\big) \\
&\le 2\phi\left(\frac{G(g(x_{m(k)}), g(x_{m(k)}), g(x_{n(k)})) + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{n(k)}))}{2}\right) \\
&\le 2\phi\left(\frac{r_k}{2}\right).
\end{aligned} \tag{16}$$

From (13) and (16), it follows that

$$r_k \le t_{n(k)} + 2t_{m(k)} + 2\phi\left(\frac{r_k}{2}\right). \tag{17}$$

Letting $k \to +\infty$ in (17) and using (8), (12) and $\lim_{r\to t^+} \phi(r) < t$ for all $t > 0$, we have

$$\varepsilon = \lim_{k\to\infty} r_k \le 2 \lim_{n\to\infty} \phi\left(\frac{r_k}{2}\right) = 2 \lim_{r_k\to\varepsilon^+} \phi\left(\frac{r_k}{2}\right) < \varepsilon.$$

This is a contradiction. This shows that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequence in the G-metric space (X,G). Since (X,G) is complete, $\{g(x_n)\}$ and $\{g(y_n)\}$ are G-convergent, there exist $x, y \in X$ such that $\lim_{n\to\infty} g(x_n) = x$ and $\lim_{n\to\infty} g(y_n) = y$. That is, from Proposition 2.4, we have

$$\lim_{n\to\infty} G\big(g(x_n), g(x_n), x\big) = \lim_{n\to\infty} G\big(g(x_n), x, x\big) = 0, \tag{18}$$

$$\lim_{n\to\infty} G\big(g(y_n), g(y_n), y\big) = \lim_{n\to\infty} G\big(g(y_n), y, y\big) = 0. \tag{19}$$

From (18), (19), continuity of g, and Proposition (2.7), we get

$$\lim_{n\to\infty} G\big(g\big(g(x_n)\big), g\big(g(x_n), g(x)\big)\big) = \lim_{n\to\infty} G\big(g\big(g(x_n)\big), g(x), g(x)\big) = 0, \tag{20}$$

$$\lim_{n\to\infty} G\big(g\big(g(y_n)\big), g\big(g(y_n)\big), g(y)\big) = \lim_{n\to\infty} G\big(g\big(g(y_n)\big), g(y), g(y)\big) = 0. \tag{21}$$

From (2) and commutativity of F and g, we have

$$g\big(g(x_{n+1})\big) = g\big(F(x_n, y_n)\big) = F\big(g(x_n), g(y_n)\big), \tag{22}$$

$$g\big(g(y_{n+1})\big) = g\big(F(y_n, x_n)\big) = F\big(g(y_n), g(x_n)\big). \tag{23}$$

We now show that $F(x,y) = g(x)$ and $F(y,x) = g(y)$.

Taking the limit as $n \to +\infty$ in (22) and (23), by (20), (21), and continuity of F, we get

$$\begin{aligned}
g(x) &= g\Big(\lim_{n\to\infty} g(x_{n+1})\Big) = \lim_{n\to\infty} g\big(g(x_{n+1})\big) = \lim_{n\to\infty} g\big(F(x_n, y_n)\big) \\
&= \lim_{n\to\infty} F\big(g(x_n), g(y_n)\big) = F(x,y)
\end{aligned}$$

and

$$\begin{aligned}
g(y) &= g\Big(\lim_{n\to\infty} g(x_{y+1})\Big) = \lim_{n\to\infty} g\big(g(y_{n+1})\big) = \lim_{n\to\infty} g\big(F(y_n, x_n)\big) \\
&= \lim_{n\to\infty} F\big(g(y_n), g(x_n)\big) = F(y,x).
\end{aligned}$$

Thus we prove that $F(x,y) = g(x)$ and $F(y,x) = g(y)$. □

In the next theorem, we omit the continuity hypothesis of F.

Theorem 3.2 *Let $(X, \preceq)$ be a partially ordered set and G be a G-metric on X such that (X, G) is a complete G-metric space and M be a nonempty subset of X^6. Assume that there exists $\phi \in \Phi$ and also suppose that $F : X \times X \to X$ and $g : X \to X$ such that*

$$G\big(F(x,u), F(y,v), F(z,w)\big) \le \phi\left(\frac{G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))}{2}\right)$$

for all $(g(x), g(u), g(y), g(v), g(z), g(w)) \in M$.

Suppose also that $(g(X), G)$ is complete $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F; if we have any two sequences $\{x_n\}$, $\{y_n\}$ with

$$(x_{n+1}, y_{n+1}, x_{n+1}, y_{n+1}, x_n, y_n) \in M,$$

$\{x_n\} \to x$ and $\{y_n\} \to y$ for all $n \ge 1$, then $(x, y, x_n, y_n, x_n, y_n) \in M$ for all $n \ge 1$. If there exists $(x_0, y_0) \in X \times X$ such that

$$\big(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), g(x_0), g(y_0)\big) \in M$$

and M is an (F^*, g)-invariant set which satisfies the transitive property, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof Proceeding exactly as in Theorem 3.1, we find that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the complete G-metric space $(g(X), G)$. Then there exist $x, y \in X$ such that $\{g(x_n)\} \to g(x)$ and $\{g(y_n)\} \to g(y)$ and

$$\big(g(x_{n+1}), g(y_{n+1}), g(x_{n+1}), g(y_{n+1}), g(x_n), g(y_n)\big) \in M;$$

by the assumption, we have

$$\big(g(x), g(y), g(x_n), g(y_n), g(x_n), g(y_n)\big) \in M$$

for all $n \ge 1$.

Since we have the (F^*, g)-invariant set property,

$$\big(g(y_n), g(x_n), g(y_n), g(x_n), g(y), g(x)\big) \in M$$

for all $n \ge 1$. By the rectangle inequality, (1), and $\phi(t) < t$ for all $t > 0$, we get

$$\begin{aligned}
&G\big(F(x,y), g(x), g(x)\big) + G\big(F(y,x), g(y), g(y)\big) \\
&\quad \le G\big(F(x,y), g(x_{n+1}), g(x_{n+1})\big) + G\big(g(x_{n+1}), g(x), g(x)\big) \\
&\qquad + G\big(F(y,x), g(y_{n+1}), g(y_{n+1})\big) + G\big(g(y_{n+1}), g(y), g(y)\big) \\
&\quad = G\big(F(x,y), F(x_n, y_n), F(x_n, y_n)\big) + G\big(g(x_{n+1}), g(x), g(x)\big) \\
&\qquad + G\big(F(y,x), F(y_n, x_n), F(y_n, x_n)\big) + G\big(g(y_{n+1}), g(y), g(y)\big)
\end{aligned}$$

$$\begin{aligned}
&= G\big(F(x,y),F(x_n,y_n),F(x_n,y_n)\big) + G\big(g(x_{n+1}),g(x),g(x)\big) \\
&\quad + G\big(F(y_n,x_n),F(y_n,x_n),F(y,x)\big) + G\big(g(y_{n+1}),g(y),g(y)\big) \\
&\leq \phi\left(\frac{G(g(x),g(x_n),g(x_n)) + G(g(y),g(y_n),g(y_n))}{2}\right) \\
&\quad + \phi\left(\frac{G(g(y_n),g(y_n),g(y)) + G(g(x_n),g(x_n),g(x))}{2}\right) \\
&\quad + G\big(g(x_{n+1}),g(x),g(x)\big) + G\big(g(y_{n+1}),g(y),g(y)\big) \\
&< G\big(g(x),g(x_n),g(x_n)\big) + G\big(g(y),g(y_n),g(y_n)\big) \\
&\quad + G\big(g(x_{n+1}),g(x),g(x)\big) + G\big(g(y_{n+1}),g(y),g(y)\big).
\end{aligned}$$

Taking the limit as $n \to \infty$ in the above inequality, we obtain

$$G\big(F(x,y),g(x),g(x)\big) + G\big(F(y,x),g(y),g(y)\big) = 0.$$

This implies that $g(x) = F(x,y)$ and $g(y) = F(y,x)$. Thus we prove that (x,y) is a coupled coincidence point of F and g. □

The following example is valid for Theorem 3.1.

Example 3.3 Let $X = \mathbb{R}$. Define $G : X \times X \times X \to [0,+\infty)$ by $G(x,y,z) = |x-y| + |x-z| + |y-z|$ and let $F : X \times X \to X$ be defined by

$$F(x,y) = \frac{x+y}{8}, \quad (x,y) \in X^2,$$

and $g : X \to X$ by $g(x) = \frac{x}{2}$. Let $y_1 = 2$ and $y_2 = 4$. Then we have $g(y_1) \leq g(y_2)$, but $F(x,y_1) \leq F(x,y_2)$, and so the mapping F does not satisfy the mixed g-monotone property.

Letting $x,u,y,v,z,w \in X$, we have

$$\begin{aligned}
&G\big(F(x,u),F(y,v),F(z,w)\big) \\
&\quad = \left|\frac{x+u}{8} - \frac{y+v}{8}\right| + \left|\frac{x+u}{8} - \frac{z+w}{8}\right| + \left|\frac{y+v}{8} - \frac{z+w}{8}\right| \\
&\quad \leq \left|\frac{x-y}{8}\right| + \left|\frac{x-z}{8}\right| + \left|\frac{y-z}{8}\right| + \left|\frac{u-v}{8}\right| + \left|\frac{u-w}{8}\right| + \left|\frac{v-w}{8}\right| \\
&\quad = \frac{1}{8}\big(|x-y| + |x-z| + |y-z|\big) + \frac{1}{8}\big(|u-v| + |u-w| + |v-w|\big)
\end{aligned}$$

and we have

$$\begin{aligned}
\frac{G(g(x),g(y),g(z)) + G(g(u),g(v),g(w))}{2} &= \frac{G(\frac{x}{2},\frac{y}{2},\frac{z}{2})}{2} + \frac{G(\frac{u}{2},\frac{v}{2},\frac{w}{2})}{2} \\
&= \frac{1}{4}\big(|x-y| + |x-z| + |y-z|\big) \\
&\quad + \frac{1}{4}\big(|u-v| + |u-w| + |v-w|\big).
\end{aligned}$$

Now, let $\phi \in \Phi$ such that $\phi(t) = t/2$, then

$$G\bigl(F(x,u),F(y,v),F(z,w)\bigr) \leq \phi\left(\frac{G(g(x),g(y),g(z)) + G(g(u),g(v),g(w))}{2}\right).$$

Therefore, if we apply Theorem 3.1 with $M = X^6$, we know that F has a coupled coincidence point $(0,0)$.

Next, we give a sufficient condition for the uniqueness of the coupled coincidence point in Theorem 3.1.

Theorem 3.4 *In addition to the hypotheses of Theorem* 3.1, *suppose that for every* $(x,y),(x^*,y^*) \in X \times X$ *there exists* $(u,v) \in X \times X$ *such that*

$$\bigl(g(u),g(v),g(x),g(y),g(x),g(y)\bigr) \in M \quad \textit{and}$$
$$\bigl(g(u),g(v),g(x^*),g(y^*),g(x^*),g(y^*)\bigr) \in M.$$

Suppose also that ϕ *is a non-decreasing function. Then* F *and* g *have a unique coupled common fixed point, that is, there exists a unique* $(x,y) \in X \times X$ *such that* $x = g(x) = F(x,y)$ *and* $y = g(y) = F(y,x)$.

Proof From Theorem 3.1, the set of coupled coincidence points is nonempty. Suppose (x,y) and (x^*,y^*) are coupled coincidence points of F, that is,

$$g(x) = F(x,y), \qquad g(y) = F(y,x), \qquad g(x^*) = F(x^*,y^*) \quad \text{and} \quad g(y^*) = F(y^*,x^*).$$

We shall show that

$$g(x^*) = g(x) \quad \text{and} \quad g(y^*) = g(y). \tag{24}$$

By assumption there is $(u,v) \in X \times X$ such that

$$\bigl(g(u),g(v),g(x),g(y),g(x),g(y)\bigr) \in M \quad \text{and}$$
$$\bigl(g(u),g(v),g(x^*),g(y^*),g(x^*),g(y^*)\bigr) \in M.$$

Put $u_0 = u$, $v_0 = v$ and choose $u_1, v_1 \in X$, such that $g(u_1) = F(u_0,v_0)$ and $g(v_1) = F(v_0,u_0)$. Then similarly as in the proof of Theorem 3.1, we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_n) = F(u_{n-1},v_{n-1}) \quad \text{and} \quad g(v_n) = F(v_{n-1},u_{n-1}) \quad \text{for all } n \geq 1.$$

Since M is (F^*,g)-invariant and $(g(u_0),g(v_0),g(x),g(y),g(x),g(y)) \in M$, we have

$$\bigl(F(u_0,v_0),F(v_0,u_0),F(x,y),F(y,x),F(x,y),F(y,x)\bigr) \in M.$$

That is, $(g(u_1),g(v_1),g(x),g(y),g(x),g(y)) \in M$.

From $(g(u_1), g(v_1), g(x), g(y), g(x), g(y)) \in M$, if we use again the property of (F^*, g)-invariance, then it follows that

$$\begin{aligned}&\big(F(u_1,v_1),F(v_1,u_1),F(x,y),F(y,x),F(x,y),F(y,x)\big)\\&\quad=\big(g(u_2),g(v_2),g(x),g(y),g(x),g(y)\big)\in M.\end{aligned}$$

By repeating this process, we get

$$\big(g(u_n),g(v_n),g(x),g(y),g(x),g(y)\big)\in M\quad\text{for all } n\geq 1. \tag{25}$$

Since M is (F^*, g)-invariant, we get

$$\big(g(y),g(x),g(y),g(x),g(v_n),g(u_n)\big)\in M\quad\text{for all } n\geq 1. \tag{26}$$

Thus from (1), (25), and (26), we have

$$\begin{aligned}&G\big(g(u_{n+1}),g(x),g(x)\big)+G\big(g(v_{n+1}),g(y),g(y)\big)\\&\quad=G\big(F(u_n,v_n),F(x,y),F(x,y)\big)+G\big(F(v_n,u_n),F(y,x),F(y,x)\big)\\&\quad=G\big(F(u_n,v_n),F(x,y),F(x,y)\big)+G\big(F(y,x),F(y,x),F(v_n,u_n)\big)\\&\quad\leq\phi\bigg(\frac{G(g(u_n),g(x),g(x))+G(g(v_n),g(y),g(y))}{2}\bigg)\\&\qquad+\phi\bigg(\frac{G(g(y),g(y),g(v_n))+G(g(x),g(x),g(u_n))}{2}\bigg)\\&\quad=2\phi\bigg(\frac{G(g(u_n),g(x),g(x))+G(g(v_n),g(y),g(y))}{2}\bigg).\end{aligned} \tag{27}$$

Thus from(27), we have

$$\begin{aligned}&\frac{G(g(u_{n+1}),g(x),g(x))+G(g(v_{n+1}),g(y),g(y))}{2}\\&\quad\leq\phi\bigg(\frac{G(g(u_n),g(x),g(x))+G(g(v_n),g(y),g(y))}{2}\bigg).\end{aligned} \tag{28}$$

Since ϕ is non-decreasing and (28), we get

$$\begin{aligned}&\frac{G(g(u_{n+1}),g(x),g(x))+G(g(v_{n+1}),g(y),g(y))}{2}\\&\quad\leq\phi^n\bigg(\frac{G(g(u_1),g(x),g(x))+G(g(v_1),g(y),g(y))}{2}\bigg)\end{aligned} \tag{29}$$

for each $n \geq 1$. Letting $n \to +\infty$ in (29) and using Lemma 2.15, this implies

$$\lim_{n\to\infty}G\big(g(u_{n+1}),g(x),g(x)\big)=\lim_{n\to\infty}G\big(g(v_{n+1}),g(y),g(y)\big)=0. \tag{30}$$

Similarly, we obtain

$$\lim_{n\to\infty}G\big(g(u_{n+1}),g\big(x^*\big),g\big(x^*\big)\big)=\lim_{n\to\infty}G\big(g(v_{n+1}),g\big(y^*\big),g\big(y^*\big)\big)=0. \tag{31}$$

Hence, from (30), (31), and Proposition 2.4, we get $g(x^*) = g(x)$ and $g(y^*) = g(y)$.

Since $g(x) = F(x,y)$ and $g(y) = F(y,x)$, by commutativity of F and g, we have

$$g\bigl(g(x)\bigr) = g\bigl(F(x,y)\bigr) = F\bigl(g(x),g(y)\bigr) \quad \text{and} \quad g\bigl(g(y)\bigr) = g\bigl(F(y,x)\bigr) = F\bigl(g(y),g(x)\bigr). \tag{32}$$

Denote $g(x) = z$ and $g(y) = w$. Then from (32)

$$g(z) = F(z,w) \quad \text{and} \quad g(w) = F(w,z)). \tag{33}$$

Therefore, (z,w) is a coupled coincidence fixed point of F and g. Then from (24) with $x^* = z$ and $y^* = w$, it follows that $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \quad \text{and} \quad g(w) = w. \tag{34}$$

From (33) and (34), $z = g(z) = F(z,w)$ and $w = g(w) = F(w,z)$. Therefore, (z,w) is a coupled common fixed point of F and g.

To prove the uniqueness, assume that (p,q) is another coupled common fixed point. Then by (24) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. □

Next, we give a simple application of our results to coupled coincidence point theorems in partially ordered metric spaces with the mixed g-monotone property.

Corollary 3.5 *Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X such that (X,G) is a complete G-metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$, and $g : X \to X$ such that*

$$G\bigl(F(x,u),F(y,v),F(z,w)\bigr) \le \phi\left(\frac{G(g(x),g(y),g(z)) + G(g(u),g(v),g(w))}{2}\right)$$

for all $x,y,z,u,v,w \in X$ for which $g(x) \succeq g(y) \succeq g(z)$ and $g(u) \preceq g(v) \preceq g(w)$.

Suppose also that F is continuous and has the mixed g-monotone property, $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F. If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0,y_0) \quad \textit{and} \quad g(y_0) \succeq F(y_0,x_0),$$

then there exists $(x,y) \in X \times X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Proof We define the subset $M \subseteq X^6$ by $M = \{(x,u,y,v,z,w) \in X^6 : x \succeq y \succeq z, u \preceq v \preceq w\}$. From Example 2.23, M is an (F^*,g)-invariant set which satisfies the transitive property. By (1), we have

$$G\bigl(F(x,u),F(y,v),F(z,w)\bigr) \le \phi\left(\frac{G(g(x),g(y),g(z)) + G(g(u),g(v),g(w))}{2}\right)$$

for all $(g(x),g(u),g(y),g(v),g(z),g(w)) \in M$.

Since $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0,y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0,x_0).$$

We have $(F(x_0,y_0),F(y_0,x_0),F(x_0,y_0),F(y_0,x_0),g(x_0),g(y_0)) \in M$. Since F is continuous, all the hypotheses of Theorem 3.1 hold, and we have $x = F(x,y)$ and $y = F(y,x)$. □

Corollary 3.6 *Let $(X,\preceq)$ be a partially ordered set and G be a G-metric on X such that $(X,G,\preceq)$ is regular. Suppose that there exist $\phi \in \Phi$, $F : X \times X \to X$, and $g : X \to X$ such that*

$$G\big(F(x,u),F(y,v),F(z,w)\big) \le \phi\left(\frac{G(g(x),g(y),g(z)) + G(g(u),g(v),g(w))}{2}\right)$$

for all $x,y,z,u,v,w \in X$ for which $g(x) \succeq g(y) \succeq g(z)$ and $g(u) \preceq g(v) \preceq g(w)$.

Suppose also that $(g(X),G)$ is complete, F has the mixed g-monotone property, $F(X \times X) \subseteq G(X)$, and g is continuous and commutes with F. If there exist $x_0,y_0 \in X$ such that

$$g(x_0) \preceq F(x_0,y_0) \quad \textit{and} \quad g(y_0) \succeq F(y_0,x_0),$$

then there exists $(x,y) \in X \times X$ such that $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Proof As in Corollary 3.5, we get

$$\big(F(x_0,y_0),F(y_0,x_0),F(x_0,y_0),F(y_0,x_0),g(x_0),g(y_0)\big) \in M.$$

Since any two sequences $\{g(x_n)\}$, $\{g(y_n)\}$ in X such that $\{g(x_n)\}$ is non-decreasing sequence with $\{g(x_n)\} \to g(x)$ and $\{g(y_n)\}$ is non-increasing sequence with $\{g(y_n)\} \to g(y)$, for all $n \ge 1$.

Since $(X,G,\preceq)$ is regular, we have

$$g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_n) \preceq g(x)$$

and

$$g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_n) \succeq g(y).$$

Therefore, we have $(g(x),g(y),g(x_n),g(y_n),g(x_n),g(y_n)) \in M$ for all $n \ge 1$, and so the whole assumption of Theorem 3.2 holds, thus F has a coupled coincidence point. □

Next, we show the uniqueness of a coupled fixed point of F.

Corollary 3.7 *In addition to the hypothesis of Corollary* 3.5 (*Corollary* 3.6), *suppose that for every $(x,y),(x^*,y^*) \in X \times X$ there exists a $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Suppose also that ϕ is a non-decreasing function. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x,y) \in X \times X$ such that $x = g(x) = F(x,y)$ and $y = g(y) = F(y,x)$.*

Proof We define the subset $M \subseteq X^6$ by $M = \{(x,u,y,v,z,w) \in X^6 : x \succeq y \succeq z, u \preceq v \preceq w\}$. From Example 2.23, M is an (F^*,g)-invariant set which satisfies the transitive property. Thus, the proof of the existence of a coupled fixed point is straightforward by following the same lines as in the proof of Corollary 3.5 (Corollary 3.6).

Next, we show the uniqueness of a coupled fixed point of F.

Since for all $(x,y),(x^*,y^*) \in X \times X$, there exists $(u,v) \in X \times X$ such that $g(x) \preceq g(u), g(y) \succeq g(v)$ and $g(x^*) \preceq g(u)$, $g(y^*) \succeq g(v)$ we can conclude that

$$\bigl(g(u), g(v), g(x), g(y), g(x), g(y)\bigr) \in M$$

and

$$\bigl(g(u), g(v), g\bigl(x^*\bigr), g\bigl(y^*\bigr), g\bigl(x^*\bigr), g\bigl(y^*\bigr)\bigr) \in M.$$

Therefore, since all the hypotheses of Theorem 3.4 hold, and F has a unique coupled fixed point. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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