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A note on recent fixed point results involving g-quasicontractive type mappings in partially ordered metric spaces

Erdal Karapınar[1,2*] and Bessem Samet[3]

*Correspondence: erdalkarapinar@yahoo.com; ekarapinar@atilim.edu.tr
[1]Department of Mathematics, Atilim University, İncek, Ankara 06836, Turkey
[2]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this note, we establish the equivalence between recent fixed point theorems involving quasicontractive type mappings in metric spaces endowed with a partial order.
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1 Introduction

Let (X,d) be a metric space and let $f,g:X\to X$ be two self-maps on X. Let

$$M(f,g,x,y):=\max\{d(gx,gy),d(gx,fx),d(gy,fy),d(gx,fy),d(gy,fx)\}\quad\text{for all } x,y\in X.$$

Suppose that X is endowed with a partial order $\preceq$. We say that f is an ordered g-quasicontraction (see [1, 2]) if

$$d(fx,fy)\le\lambda M(f,g,x,y)\quad\text{for all } x,y\in X \text{ such that } gy\preceq gx$$

for some constant $\lambda\in(0,1)$. If $g=id_X$ (the identity map on X), then f is said to be an ordered quasicontraction.

In [1], the authors established the following result.

Theorem 1.1 *Let (X,d) be a metric space endowed with a certain partial order $\preceq$. Let $f,g:X\to X$ be two self-maps on X satisfying the following conditions*:

(i) $fX\subseteq gX$;
(ii) *gX is complete*;
(iii) *f is g-nondecreasing, i.e.,* $gx\preceq gy\Longrightarrow fx\preceq fy$;
(iv) *f is an ordered g-quasicontraction*;
(v) *there exists $x_0\in X$ such that* $gx_0\preceq fx_0$;
(vi) *if $\{gx_n\}$ is a nondecreasing sequence (w.r.t. $\preceq$) that converges to some $gz\in gX$, then $gx_n\preceq gz$ for each $n\in\mathbb{N}$.*

Then f and g have a coincidence point, i.e., there exists $z\in X$ such that $fz=gz$.

Taking $g = id_X$ in Theorem 1.1, we obtain immediately the following result.

Theorem 1.2 *Let (X,d) be a complete metric space endowed with a certain partial order $\preceq$. Let $f: X \to X$ be a self-map on X satisfying the following conditions*:

(iii) *f is nondecreasing, i.e., $x \preceq y \Longrightarrow fx \preceq fy$*;
(iv) *f is an ordered quasicontraction*;
(v) *there exists $x_0 \in X$ such that $x_0 \preceq fx_0$*;
(vi) *if $\{x_n\}$ is a nondecreasing sequence (w.r.t. $\preceq$) that converges to some $z \in X$, then $x_n \preceq z$ for each $n \in \mathbb{N}$.*

Then f has a fixed point.

Let us denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(Ψ_1) ψ is nondecreasing;
(Ψ_2) ψ is subadditive, *i.e.*, $\psi(s+t) \leq \psi(s) + \psi(t)$, for every $s,t \geq 0$;
(Ψ_3) ψ is continuous;
(Ψ_4) $\psi(t) = 0 \Longleftrightarrow t = 0$.

In [3], the authors established the following result.

Theorem 1.3 *Let (X,d) be a metric space endowed with a certain partial order $\preceq$. Let $f,g: X \to X$ be two self-maps on X satisfying the following conditions*:

(i) *$fX \subseteq gX$*;
(ii) *gX is complete*;
(iii) *f is g-nondecreasing*;
(iv) *there exists $\psi \in \Psi$ such that*

$$\psi\bigl(d(fx,fy)\bigr) \leq \lambda \max\bigl\{\psi\bigl(d(gx,gy)\bigr), \psi\bigl(d(gx,fx)\bigr), \psi\bigl(d(gy,fy)\bigr), \psi\bigl(d(gx,fy)\bigr), \psi\bigl(d(gy,fx)\bigr)\bigr\}$$

for all $x,y \in X$ such that $gy \preceq gx$;
(v) *there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$*;
(vi) *if $\{gx_n\}$ is a nondecreasing sequence that converges to some $gz \in gX$, then $gx_n \preceq gz$ for each $n \in \mathbb{N}$.*

Then f and g have a coincidence point.

The aim of this note is to prove that Theorems 1.1, 1.2 and 1.3 are equivalent.

2 Main result

Our main result in this note is the following.

Theorem 2.1 *We have the following equivalence*:

$$\textit{Theorem } 1.2 \quad \Longleftrightarrow \quad \textit{Theorem } 1.1 \quad \Longleftrightarrow \quad \textit{Theorem } 1.3.$$

Proof We consider three steps in the proof.

◇ Step 1. Theorem 1.2 $\Longrightarrow$ Theorem 1.1.

Suppose that all the assumptions of Theorem 1.1 are satisfied. Recall that if $S : X \to X$ is a given map, then there exists a subset E of X such that $SE = SX$ and $S : E \to X$ is one-to-one. For the proof of this result, we refer to [4]. Due to this remark, there exists $E \subseteq X$ such that $gE = gX$ and $g : E \to X$ is one-to-one. Let us define the map $T : gE \to gE$ by

$$T(gx) = fx, \quad x \in E.$$

Notice that the mapping T is well defined since g is one-to-one on E. From condition (ii) of Theorem 1.1, the metric space (gE, d) is complete. From condition (iii) of Theorem 1.1, the mapping T is nondecreasing. Observe also that T is an ordered quasicontraction. Indeed, if $u, v \in gE$ such that $v \preceq u$, from condition (iv) of Theorem 1.1 and the definition of gE, there exist $x, y \in E$ with $v = gy \preceq gx = u$ such that

$$\begin{aligned} d(Tu, Tv) &= d(fx, fy) \\ &\le \lambda M(f, g, x, y) \\ &= \lambda \max\bigl\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\bigr\} \\ &= \lambda \max\bigl\{d(u, v), d(u, Tu), d(v, Tv), d(u, Tv), d(v, Tu)\bigr\}. \end{aligned}$$

From condition (v) of Theorem 1.1, there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$. Let $u_0 = gx_0 \in gE$, we have $u_0 \preceq Tu_0$. Finally, from condition (iv) of Theorem 1.1, if $\{u_n\} \subset gE$ is a nondecreasing sequence that converges to some $u \in gE$, then $u_n \preceq u$ for each $n \in \mathbb{N}$. Thus we proved that T satisfies all the conditions of Theorem 1.2. Then we deduce that T has a fixed point $u^* \in gE$. This means that there exists some $x^* \in X$ such that $fx^* = T(gx^*) = gx^*$, that is, $x^* \in X$ is a coincidence point of f and g.

◇ Step 2. Theorem 1.1 $\Longrightarrow$ Theorem 1.3.

Suppose that all the assumptions of Theorem 1.3 are satisfied. Define the function $d_\psi : X \times X \to [0, \infty)$ by

$$d_\psi(x, y) := \psi\bigl(d(x, y)\bigr) \quad \text{for all } x, y \in X.$$

In [5], we proved that d_ψ is a metric on X. Moreover, (X, d) is complete if and only if (X, d_ψ) is complete. Then from condition (iv) of Theorem 1.3, we deduce that f is an ordered g-quasicontraction with respect to the new metric d_ψ. More precisely, we have

$$d_\psi(fx, fy) \le \lambda \max\bigl\{d_\psi(gx, gy), d_\psi(gx, fx), d_\psi(gy, fy), d_\psi(gx, fy), d_\psi(gy, fx)\bigr\}$$

for all $x, y \in X$ such that $gy \preceq gx$. Now, applying Theorem 1.1 with the metric space (X, d_ψ), we obtain the result of Theorem 1.3.

◇ Step 3. Theorem 1.3 $\Longrightarrow$ Theorem 1.2.

Taking $g = id_X$ and $\psi(t) = t$ in Theorem 1.3, we obtain immediately the result of Theorem 1.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Atilim University, İncek, Ankara 06836, Turkey. [2]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia. [3]Department of Mathematics, King Saud University, Riyadh, Saudi Arabia.

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