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Generalized Meir-Keeler type n-tupled fixed point theorems in ordered partial metric spaces

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Abstract

In this paper, we prove n-tupled fixed point theorems (for even n) for mappings satisfying Meir-Keeler type contractive condition besides enjoying mixed monotone property in ordered partial metric spaces. As applications, some results of integral type are also derived. Our results generalize the corresponding results of Erduran and Imdad (J. Nonlinear Anal. Appl. 2012:jnaa-00169, 2012).

Keywords: partially ordered set; partial metric space; generalized Meir-Keeler type contractive condition; mixed monotone property; n-tupled fixed point

1 Introduction

Existence of a fixed point for contraction type mappings in partially ordered metric spaces with possible applications have been considered recently by many authors (*e.g.* [1–31]). Recently many researchers have obtained fixed and common fixed point results on partially ordered metric spaces (see [5, 10, 11, 18, 22, 32, 33]). In 2006, Bhaskar and Lakshmikantham [13] initiated the idea of coupled fixed point and proved some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property. In this continuation, Lakshmikantham and Ćirić [17] generalized these results for nonlinear ϕ-contraction mappings by introducing two ideas namely: coupled coincidence point and mixed g-monotone property. Thereafter Samet and Vetro [34] extended the idea of coupled fixed point to higher dimensions by introducing the notion of fixed point of n-order (or n-tupled fixed point, where n is natural number greater than or equal to 2) and presented some n-tupled fixed point results in complete metric spaces, using a new concept of F-invariant set. On the other hand, Imdad *et al.* [35] generalized the idea of n-tupled fixed point by considering even-tupled coincidence point besides exploiting the idea of mixed g-monotone property on X^n and proved an even-tupled coincidence point theorem for nonlinear ϕ-contraction mappings satisfying mixed g-monotone property.

The concept of partial metric space was introduced by Matthews [36] in 1994, which is a generalization of usual metric space. In such spaces, the distance of a point to itself may not be zero. The main motivation behind the idea of a partial metric space is to transfer mathematical techniques into computer science. Following this initial work, Matthews [36] generalized the Banach contraction principle in the context of complete partial metric spaces. For more details, we refer the reader to [4, 7–9, 20, 24–26, 37–48].

Samet [27] introduced the concept of generalized Meir-Keeler type contraction function and proved some coupled fixed point theorems in partially ordered metric spaces. In different years, many researchers studied and worked on this contraction condition. Recently, in [12], Berinde and Păcurar gave the concept of symmetric Meir-Keeler type condition and generalized several results in the literature. Very recently, Erduran and Imdad [49] generalized the coupled fixed point theorems in the context of partial metric spaces. In this paper, we established some n-tupled fixed point theorems for generalized Meir-Keeler type contraction condition in ordered partial metric spaces enjoying strict mixed monotone property. In this paper, we prove the existence and uniqueness of some Meir-Keeler type n-tupled fixed point theorems in the context of partially ordered partial metric spaces. The presented theorems extend and improve the recent coupled fixed point theorems due to Erduran and Imdad [49].

2 Preliminaries

In this section, we collect some definitions and properties of partial metric space which are relevant to our presentation.

Definition 2.1 A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p2) $p(x,x) \leq p(x,y)$,

(p3) $p(x,y) = p(y,x)$,

(p4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark 2.1 It is clear that if $p(x,y) = 0$, then from (p1), (p2) and (p3), $x = y$. But if $x = y$, $p(x,y)$ may not be zero.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \epsilon), x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Example 2.1 [36, 37, 43] Consider $X = \mathbb{R}^+$ with $p(x,y) = \max\{x,y\}$. Then $(\mathbb{R}^+, p)$ is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $p^s(x,y) = |x - y|$.

Example 2.2 [50] Let $X = \{[a,b] : a,b \in \mathbb{R}, a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X, p) is a partial metric space.

Example 2.3 [50] Let $X = [0,1] \cup [2,3]$ and define $p : X \times X \to [0, \infty)$ by

$$p(x,y) = \begin{cases} \max\{x,y\}, & \{x,y\} \cap [2,3] \neq \emptyset; \\ |x - y|, & \{x,y\} \subset [0,1]. \end{cases}$$

Then (X, p) is a complete partial metric space.

Example 2.4 [51] Let (X, d) and (X, p) be metric space and partial metric space, respectively. Then the mappings $\rho_i : X \times X \to \mathbb{R}^+$ ($i \in \{1, 2, 3\}$) defined by

$$\rho_1(x, y) = d(x, y) + p(x, y),$$

$$\rho_2(x, y) = d(x, y) + \max\{\omega(x), \omega(y)\},$$

$$\rho_3(x, y) = d(x, y) + a$$

induce partial metrics on X, where $\omega : X \to \mathbb{R}^+$ is an arbitrary function and $a \geq 0$.

Definition 2.2 Let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. Then
  (i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$,
  (ii) $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n,m \to +\infty} p(x_n, x_m)$ exists (and is finite).

Definition 2.3 A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\} \in X$ converges with respect to τ_p, to a point $x \in X$, such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

Lemma 2.1 *Let (X, p) be a partial metric space. Then*
  (i) *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s),*
  (ii) *(X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n \to +\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m).$$

In [52], Meir-Keeler generalized the well-known Banach fixed point theorem by proving the following interesting fixed point theorem.

Theorem 2.1 [52] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. Suppose that for every $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all*

$$x, y \in X \quad \text{with } \epsilon \leq d(x, y) < \epsilon + \delta(\epsilon) \quad \Rightarrow \quad d(Tx, Ty) < \epsilon. \tag{2.1}$$

Then T has a unique fixed point $z \in X$ and for all $x \in X$, the sequence $\{T^n x\}$ converges to z.

In recent years, many authors generalized Meir-Keeler fixed point theorems in various ways in various spaces which include complete metric space as well as ordered metric space. In [27], Samet introduced the concept of generalized Meir-Keeler type contraction function and proved some coupled fixed point results. Samet [27] introduced the definition below to modify the Meir-Keeler contraction and extended its applications.

Definition 2.4 [27] Let (X, d) be a partially ordered metric space and $F : X \times X \to X$ be a given mapping. Then F is a generalized Meir-Keeler type function if for all $\epsilon > 0$ there

exists $\delta(\epsilon) > 0$ such that

$$u \preceq x, y \preceq v, \quad \epsilon \leq \frac{1}{2}\big[d(x,u) + d(y,v)\big] < \epsilon + \delta(\epsilon) \quad \Rightarrow \quad d\big(F(x,y), F(u,v)\big) < \epsilon. \quad (2.2)$$

Very recently Erduran and Imdad [49] generalized the results of Samet [27] for ordered partial metric spaces. For more details, see [12, 27, 53, 54].

Erduran and Imdad [49] proved the following result:

Theorem 2.2 [49] *Let $(X, \preceq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is complete partial metric space. Let $F : X \times X \to X$ be mapping satisfying the following hypotheses:*

(1) *F has the mixed strict monotone property,*

(2) *F is a generalized Meir-Keeler type function,*

(3) *F is continuous or X has the following properties:*

    (a) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

    (b) *if a nonincreasing sequence $\{x_n\} \to x$, then $x \preceq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \prec F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exists $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$. Furthermore, $p(x, x) = p(y, y) = 0$.

Note Throughout the paper we consider n to be an even integer.

Let (X, p) be a partial metric. We endow $X \times X \times \cdots \times X$, n times $(= X^n)$ with the partial metric η defined for $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ by

$$\eta\big((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)\big) = \max\big\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\big\}.$$

Let $F : X^n \to X$ be a given mapping. Then for all $(x^1, x^2, \ldots, x^n) \in X^n$ and for all $m \in \mathbb{N}$, $m \geq 2$, we denote

$$F^m\big(x^1, x^2, \ldots, x^n\big)$$
$$= F\big(F^{m-1}\big(x^1, x^2, \ldots, x^n\big), F^{m-1}\big(x^2, \ldots, x^n, x^1\big), \ldots, F^{m-1}\big(x^n, x^1, \ldots, x^{n-1}\big)\big).$$

In this paper, we used the concept of n-tupled fixed point given by Samet and Vetro [34]. We recall some basic concepts.

Definition 2.5 [35] Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is, if,

(i)   for all $x_1^1, x_2^1 \in X$, $x_1^1 \preceq x_2^1 \quad \Rightarrow \quad F\big(x_1^1, x^2, x^3, \ldots, x^n\big) \preceq F\big(x_2^1, x^2, x^3, \ldots, x^n\big)$,

(ii) for all $x_1^2, x_2^2 \in X$, $x_1^2 \preceq x_2^2 \quad \Rightarrow \quad F\big(x^1, x_2^2, x^3, \ldots, x^n\big) \preceq F\big(x^1, x_1^2, x^3, \ldots, x^n\big)$,

(iii) for all $x_1^3, x_2^3 \in X$, $x_1^3 \preceq x_2^3 \quad \Rightarrow \quad F\big(x^1, x^2, x_1^3, \ldots, x^n\big) \preceq F\big(x^1, x^2, x_2^3, \ldots, x^n\big)$,

     $\vdots$

     for all $x_1^n, x_2^n \in X$, $x_1^n \preceq x_2^n \quad \Rightarrow \quad F\big(x^1, x^2, x^3, \ldots, x_2^n\big) \preceq F\big(x^1, x^2, x^3, \ldots, x_1^n\big)$.

Definition 2.6 [34] An element $(x^1, x^2, \ldots, x^n) \in X^n$ is called an *n*-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{cases} F(x^1, x^2, x^3, \ldots, x^n) = x^1, \\ F(x^2, x^3, \ldots, x^n, x^1) = x^2, \\ F(x^3, \ldots, x^n, x^1, x^2) = x^3, \\ \vdots \\ F(x^n, x^1, x^2, \ldots, x^{n-1}) = x^n. \end{cases}$$

Example 2.5 Let (R, d) be a partially ordered metric space under natural setting and let $F : R^n \to R$ be a mapping defined by $F(x^1, x^2, x^3, \ldots, x^n) = \sin(x^1 \cdot x^2 \cdot x^3 \cdots x^n)$, for any $x^1, x^2, \ldots, x^n \in R$. Then $(0, 0, \ldots, 0)$ is an *n*-tupled fixed point of F.

Remark 2.2 Definition 2.6 with $n = 2, 4$ respectively yields the definition of coupled fixed point [13] and quadrupled fixed point [55].

3 Main results

We begin this section by defining the following definitions:

Definition 3.1 Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$ be a mapping. The mapping F is said to have the mixed strict monotone property if F is nondecreasing in its odd position arguments and nonincreasing in its even position arguments, that is, if,

   (i)   for all $x_1^1, x_2^1 \in X$, $x_1^1 \prec x_2^1$ $\Rightarrow$ $F(x_1^1, x^2, x^3, \ldots, x^n) \prec F(x_2^1, x^2, x^3, \ldots, x^n)$,

  (ii) for all $x_1^2, x_2^2 \in X$, $x_1^2 \prec x_2^2$ $\Rightarrow$ $F(x^1, x_2^2, x^3, \ldots, x^n) \prec F(x^1, x_1^2, x^3, \ldots, x^n)$,

 (iii) for all $x_1^3, x_2^3 \in X$, $x_1^3 \prec x_2^3$ $\Rightarrow$ $F(x^1, x^2, x_1^3, \ldots, x^n) \prec F(x^1, x^2, x_2^3, \ldots, x^n)$,

       $\vdots$

       for all $x_1^n, x_2^n \in X$, $x_1^n \prec x_2^n$ $\Rightarrow$ $F(x^1, x^2, x^3, \ldots, x_2^n) \prec F(x^1, x^2, x^3, \ldots, x_1^n)$.

Definition 3.2 Let (X, p) be a partially ordered partial metric space and $F : X^n \to X$ be a given mapping. We say that F is a generalized Meir-Keeler type function if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ with $x^1 \preceq y^1, y^2 \preceq x^2, x^3 \preceq y^3, \ldots, y^n \preceq x^n$

$$\begin{cases} \epsilon \leq \max\{p(x^1, y^1), p(x^2, y^2), p(x^3, y^3), \ldots, p(x^n, y^n)\} < \epsilon + \delta(\epsilon) \\ \Rightarrow \quad p(F(x^1, x^2, x^3, \ldots, x^n), F(y^1, y^2, y^3, \ldots, y^n)) < \epsilon. \end{cases} \tag{3.1}$$

The aim of this work is to prove the following results:

Lemma 3.1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X^n \to X$ be a given mapping. If F is a generalized Meir-Keeler type function, then for $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$*

$$p(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)) < \max\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\}$$

with $x^1 \prec y^1, y^2 \preceq x^2, x^3 \prec y^3, \ldots, y^n \preceq x^n$ or $x^1 \preceq y^1, y^2 \prec x^2, x^3 \preceq y^3, \ldots, y^n \prec x^n$.

Proof Let $x^1, x^2, \ldots, x^n, y^1, y^2, \ldots, y^n \in X$ such that

$$
\begin{array}{llllll}
x^1 \prec y^1, & y^2 \preceq x^2, & x^3 \prec y^3, & \ldots, & y^n \preceq x^n & \text{or} \\
x^1 \preceq y^1, & y^2 \prec x^2, & x^3 \preceq y^3, & \ldots, & y^n \prec x^n.
\end{array}
\tag{3.2}
$$

Then $\max\{p(x^1, y^1), p(x^2, y^2), p(x^3, y^3), \ldots, p(x^n, y^n)\} > 0$. Since F is a generalized Meir-Keeler type function. Therefore for $\epsilon = \max\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\}$, there exists $\delta(\epsilon) > 0$ such that

$$
\begin{array}{llllll}
x_*^1 \preceq y_*^1, & y_*^2 \preceq x_*^2, & x_*^3 \preceq y_*^3, & \ldots, & y_*^n \preceq x_*^n,
\end{array}
$$

$$
\epsilon \leq \max\{p(x_*^1, y_*^1), p(x_*^2, y_*^2), p(x_*^3, y_*^3), \ldots, p(x_*^n, y_*^n)\} < \epsilon + \delta(\epsilon)
$$

$$
\Rightarrow \quad p\big(F(x_*^1, x_*^2, x_*^3, \ldots, x_*^n), F(y_*^1, y_*^2, y_*^3, \ldots, y_*^n)\big) < \epsilon.
$$

Putting $x_*^1 = x^1, x_*^2 = x^2, \ldots, x_*^n = x^n$ and $y_*^1 = y^1, y_*^2 = y^2, \ldots, y_*^n = y^n$, we obtain the desired result. $\qquad\square$

Lemma 3.2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X^n \to X$ be a given mapping. Assume that the following hypotheses hold*:

(1) *F has the mixed strict monotone property,*

(2) *F is a generalized Meir-Keeler type function,*

(3) *there exist $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ with*
$x^1 \prec y^1, y^2 \preceq x^2, x^3 \prec y^3, \ldots, y^n \preceq x^n.$

Then

$$
\eta\big(\big(F^m(x^1, x^2, x^3, \ldots, x^n), F^m(x^2, x^3, \ldots, x^n, x^1), \ldots, F^m(x^n, x^1, x^2, \ldots, x^{n-1})\big),
$$

$$
\big(F^m(y^1, y^2, y^3, \ldots, y^n), F^m(y^2, y^3, \ldots, y^n, y^1), \ldots, F^m(y^n, y^1, y^2, \ldots, y^{n-1})\big)\big)
$$

$$
\to 0 \quad \text{as } m \to \infty.
$$

Proof We claim that:

$$
\begin{cases}
F^m(x^1, x^2, x^3, \ldots, x^n) \prec F^m(y^1, y^2, y^3, \ldots, y^n), \\
F^m(y^2, y^3, \ldots, y^n, y^1) \prec F^m(x^2, x^3, \ldots, x^n, x^1), \\
\vdots \\
F^m(y^n, y^1, y^2, \ldots, y^{n-1}) \prec F^m(x^n, x^1, x^2, \ldots, x^{n-1}),
\end{cases}
\tag{3.3}
$$

with the notation $F^1 \equiv F$. Then by the mixed strict monotone property of F,

$$
\begin{cases}
x^1 \prec y^1 & \Rightarrow & F(x^1, x^2, x^3, \ldots, x^n) \prec F(y^1, x^2, x^3, \ldots, x^n), \\
y^2 \preceq x^2 & \Rightarrow & F(y^1, x^2, x^3, \ldots, x^n) \preceq F(y^1, y^2, x^3, \ldots, x^n), \\
x^3 \prec y^3 & \Rightarrow & F(y^1, y^2, x^3, x^4, \ldots, x^n) \prec F(y^1, y^2, y^3, x^4, \ldots, x^n), \\
\vdots \\
y^n \preceq x^n & \Rightarrow & F(y^1, y^2, \ldots, y^{n-1}, x^n) \preceq F(y^1, y^2, \ldots, y^{n-1}, y^n).
\end{cases}
$$

Then we have $F(x^1, x^2, x^3, \ldots, x^n) \prec F(y^1, y^2, y^3, \ldots, y^n)$. Also

$$
\begin{cases}
y^2 \preceq x^2 & \Rightarrow \quad F(y^2, y^3, \ldots, y^n, y^1) \prec F(x^2, y^3, \ldots, y^n, y^1), \\
x^3 \prec y^3 & \Rightarrow \quad F(x^2, y^3, y^4, \ldots, y^n, y^1) \preceq F(x^2, x^3, y^4, \ldots, y^n, y^1), \\
\vdots \\
y^n \preceq x^n & \Rightarrow \quad F(x^2, x^3, \ldots, y^n, y^1) \prec F(x^2, x^3, \ldots, x^n, y^1), \\
x^1 \prec y^1 & \Rightarrow \quad F(x^2, x^3, \ldots, x^n, y^1) \preceq F(x^2, x^3, \ldots, x^n, x^1).
\end{cases}
$$

Therefore $F(y^2, y^3, \ldots, y^n, y^1) \prec F(x^2, x^3, \ldots, x^n, x^1)$.

And similarly

$$
\begin{cases}
y^n \preceq x^n & \Rightarrow \quad F(y^n, y^1, y^2, \ldots, y^{n-1}) \prec F(x^n, y^1, y^2, \ldots, y^{n-1}), \\
x^1 \prec y^1 & \Rightarrow \quad F(x^n, y^1, y^2, \ldots, y^{n-1}) \preceq F(x^n, x^1, y^2, \ldots, y^{n-1}), \\
y^2 \preceq x^2 & \Rightarrow \quad F(x^n, x^1, y^2, \ldots, y^{n-1}) \prec F(x^n, x^1, x^2, y^3, \ldots, y^{n-1}), \\
\vdots \\
x^{n-1} \prec y^{n-1} & \Rightarrow \quad F(x^n, x^1, \ldots, x^{n-2}, y^{n-1}) \preceq F(x^n, x^1, \ldots, x^{n-2}, x^{n-1}).
\end{cases}
$$

Therefore $F(y^n, y^1, y^2, \ldots, y^{n-1}) \prec F(x^n, x^1, x^2, \ldots, x^{n-1})$. Thus (3.3) is satisfied for $m = 1$. For $m = 2$, we use the same strategy. We have

$$
\begin{aligned}
& F(x^1, x^2, x^3, \ldots, x^n) \prec F(y^1, y^2, y^3, \ldots, y^n) \\
& \quad \Rightarrow \quad F\big(F(x^1, x^2, x^3, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})\big) \\
& \qquad\qquad \prec F\big(F(y^1, y^2, y^3, \ldots, y^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})\big), \\
& F(y^2, y^3, \ldots, y^n, y^1) \prec F(x^2, x^3, \ldots, x^n, x^1) \\
& \quad \Rightarrow \quad F\big(F(y^1, y^2, y^3, \ldots, y^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})\big) \\
& \qquad\qquad \prec F\big(F(y^1, y^2, y^3, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})\big), \\
& \vdots \\
& F(y^n, y^1, y^2, \ldots, y^{n-1}) \prec F(x^n, y^1, y^2, \ldots, y^{n-1}) \\
& \quad \Rightarrow \quad F\big(F(y^1, y^2, y^3, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})\big) \\
& \qquad\qquad \prec F\big(F(y^1, y^2, y^3, \ldots, y^n), F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, y^2, \ldots, y^{n-1})\big).
\end{aligned}
$$

Thus we get

$$
F^2(x^1, x^2, x^3, \ldots, x^n) \prec F^2(y^1, y^2, y^3, \ldots, y^n).
$$

Now,

$$
\begin{aligned}
& F(y^2, y^3, \ldots, y^n, y^1) \prec F(x^2, x^3, \ldots, x^n, x^1) \\
& \quad \Rightarrow \quad F\big(F(y^2, y^3, \ldots, y^n, y^1), \ldots, F(y^n, y^1, y^2, \ldots, y^{n-1}), F(y^1, y^2, y^3, \ldots, y^n)\big)
\end{aligned}
$$

$$\prec F\big(F\big(x^2,x^3,\ldots,x^n,x^1\big),\ldots,F\big(y^n,y^1,y^2,\ldots,y^{n-1}\big),F\big(y^1,y^2,y^3,\ldots,y^n\big)\big),$$

$$\vdots$$

$$F\big(y^n,y^1,y^2,\ldots,y^{n-1}\big) \prec F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big)$$

$$\Rightarrow \quad F\big(F\big(x^2,x^3,\ldots,x^n,x^1\big),F\big(y^n,y^1,y^2,\ldots,y^{n-1}\big),\ldots,F\big(y^1,y^2,y^3,\ldots,y^n\big)\big)$$

$$\prec F\big(F\big(x^2,x^3,\ldots,x^n,x^1\big),F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big),\ldots,F\big(y^1,y^2,y^3,\ldots,y^n\big)\big),$$

$$F\big(x^1,x^2,x^3,\ldots,x^n\big) \prec F\big(y^1,y^2,y^3,\ldots,y^n\big)$$

$$\Rightarrow \quad F\big(F\big(x^2,x^3,\ldots,x^n,x^1\big),\ldots,F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big),F\big(y^1,y^2,y^3,\ldots,y^n\big)\big)$$

$$\prec F\big(F\big(x^2,x^3,\ldots,x^n,x^1\big),\ldots,F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big),F\big(x^1,x^2,x^3,\ldots,x^n\big)\big).$$

Therefore we get

$$F^2\big(y^2,y^3,\ldots,y^n,y^1\big) \prec F^2\big(x^2,x^3,\ldots,x^n,x^1\big).$$

In the same way,

$$F\big(y^n,y^1,y^2,\ldots,y^{n-1}\big) \prec F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big)$$

$$\Rightarrow \quad F\big(F\big(y^n,y^1,y^2,\ldots,y^{n-1}\big),F\big(y^1,y^2,y^3,\ldots,y^n\big),\ldots,F\big(y^{n-1},\ldots,y^2,y^1,y^n\big)\big)$$

$$\prec F\big(F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big),F\big(y^1,y^2,y^3,\ldots,y^n\big),\ldots,F\big(y^{n-1},\ldots,y^2,y^1,y^n\big)\big),$$

$$\vdots$$

$$F\big(x^1,x^2,x^3,\ldots,x^n\big) \prec F\big(y^1,y^2,y^3,\ldots,y^n\big)$$

$$\Rightarrow \quad F\big(F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big),F\big(y^1,y^2,y^3,\ldots,y^n\big),\ldots,F\big(x^{n-1},\ldots,x^2,x^1,x^n\big)\big)$$

$$\prec F\big(F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big),F\big(x^1,x^2,x^3,\ldots,x^n\big),\ldots,F\big(x^{n-1},\ldots,x^2,x^1,x^n\big)\big),$$

$$F\big(y^2,y^3,\ldots,y^n,y^1\big) \prec F\big(x^2,x^3,\ldots,x^n,x^1\big)$$

$$\Rightarrow \quad F\big(F\big(x^n,x^1,\ldots,x^{n-1}\big),F\big(x^1,x^2,\ldots,x^n\big),F\big(y^2,\ldots,y^n,y^1\big),\ldots,F\big(x^{n-1},\ldots,x^1,x^n\big)\big)$$

$$\prec F\big(F\big(x^n,x^1,\ldots,x^{n-1}\big),F\big(x^1,x^2,\ldots,x^n\big),$$

$$F\big(x^2,\ldots,x^n,x^1\big),\ldots,F\big(x^{n-1},\ldots,x^1,x^n\big)\big).$$

Thus we have

$$F^2\big(y^n,y^1,y^2,\ldots,y^{n-1}\big) \prec F^2\big(x^n,x^1,x^2,\ldots,x^{n-1}\big).$$

Thus (3.3) is satisfied for $m = 2$. Repeating the same argument for each m, we see that (3.3) holds. Now using Lemma 3.1 and (3.3), we get

$$p\big(F^{m+1}\big(x^1,x^2,x^3,\ldots,x^n\big),F^{m+1}\big(y^1,y^2,y^3,\ldots,y^n\big)\big)$$

$$= p\big(F\big(F^m\big(x^1,x^2,x^3,\ldots,x^n\big),F^m\big(x^2,x^3,\ldots,x^n,x^1\big),\ldots,F^m\big(x^n,x^1,x^2,\ldots,x^{n-1}\big)\big),$$

$$F\big(F^m\big(y^1,y^2,y^3,\ldots,y^n\big),F^m\big(y^2,y^3,\ldots,y^n,y^1\big),\ldots,F^m\big(y^n,y^1,y^2,\ldots,y^{n-1}\big)\big)\big)$$

$$< \max\bigl[p\bigl(F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr)\bigr),$$
$$p\bigl(F^m\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),F^m\bigl(y^2,y^3,\ldots,y^n,y^1\bigr)\bigr),\ldots,$$
$$p\bigl(F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr),F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)\bigr]. \tag{3.4}$$

Also we have

$$p\bigl(F^{m+1}\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),F^{m+1}\bigl(y^2,y^3,\ldots,y^n,y^1\bigr)\bigr)$$
$$= p\bigl(F\bigl(F^m\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),\ldots,F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr),F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr)\bigr),$$
$$F\bigl(F^m\bigl(y^2,y^3,\ldots,y^n,y^1\bigr),\ldots,F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr),F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr)\bigr)\bigr)$$
$$< \max\bigl[p\bigl(F^m\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),F^m\bigl(y^2,y^3,\ldots,y^n,y^1\bigr)\bigr),\ldots,p\bigl(F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr),$$
$$F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr),p\bigl(F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr)\bigr)\bigr]. \tag{3.5}$$

Similarly we have,

$$p\bigl(F^{m+1}\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr),F^{m+1}\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)$$
$$< \max\bigl[p\bigl(F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr),F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr),$$
$$p\bigl(F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr)\bigr),\ldots,$$
$$p\bigl(F^m\bigl(x^{n-1},\ldots,x^2,x^1,x^n\bigr),F^m\bigl(y^{n-1},\ldots,y^2,y^1,y^n\bigr)\bigr)\bigr]. \tag{3.6}$$

Combining (3.4), (3.5), and (3.6), we get

$$\eta\bigl(\bigl(F^{m+1}\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^{m+1}\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),\ldots,F^{m+1}\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr)\bigr),$$
$$\bigl(F^{m+1}\bigl(y^1,y^2,y^3,\ldots,y^n\bigr),F^{m+1}\bigl(y^2,y^3,\ldots,y^n,y^1\bigr),\ldots,F^{m+1}\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)\bigr)$$
$$< \eta\bigl(\bigl(F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^m\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),\ldots,F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr)\bigr),$$
$$\bigl(F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr),F^m\bigl(y^2,y^3,\ldots,y^n,y^1\bigr),\ldots,F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)\bigr).$$

This implies that

$$\bigl\{\eta\bigl(\bigl(F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^m\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),\ldots,F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr)\bigr),$$
$$\bigl(F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr),F^m\bigl(y^2,y^3,\ldots,y^n,y^1\bigr),\ldots,F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)\bigr)\bigr\}$$

is a decreasing convergent sequence. Thus there exists $\epsilon \geq 0$ such that

$$\lim_{m \to \infty}\bigl[\eta\bigl(\bigl(F^m\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^m\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),\ldots,F^m\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr)\bigr),$$
$$\bigl(F^m\bigl(y^1,y^2,y^3,\ldots,y^n\bigr),F^m\bigl(y^2,y^3,\ldots,y^n,y^1\bigr),\ldots,F^m\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)\bigr)\bigr] = \epsilon.$$

Now we show that $\epsilon = 0$. Assume that $\epsilon > 0$. This implies that there exists $m_0 \in \mathbb{N}$ such that

$$\epsilon < \eta\bigl(\bigl(F^{m_0}\bigl(x^1,x^2,x^3,\ldots,x^n\bigr),F^{m_0}\bigl(x^2,x^3,\ldots,x^n,x^1\bigr),\ldots,F^{m_0}\bigl(x^n,x^1,x^2,\ldots,x^{n-1}\bigr)\bigr),$$
$$\bigl(F^{m_0}\bigl(y^1,y^2,y^3,\ldots,y^n\bigr),F^{m_0}\bigl(y^2,y^3,\ldots,y^n,y^1\bigr),\ldots,F^{m_0}\bigl(y^n,y^1,y^2,\ldots,y^{n-1}\bigr)\bigr)\bigr)$$
$$< \epsilon + \delta(\epsilon).$$

In this case we have

$$\epsilon \le \max\{p(F^{m_0}(x^1,x^2,x^3,\ldots,x^n),F^{m_0}(y^1,y^2,y^3,\ldots,y^n)),p(F^{m_0}(x^2,x^3,\ldots,x^n,x^1),$$
$$F^{m_0}(y^2,y^3,\ldots,y^n,y^1)),\ldots,p(F^{m_0}(x^n,x^1,x^2,\ldots,x^{n-1}),F^{m_0}(y^n,y^1,y^2,\ldots,y^{n-1}))\}$$
$$< \epsilon + \delta(\epsilon).$$

It follows from (3.3) and hypothesis (2) that

$$p\big((F(F^{m_0}(x^1,x^2,x^3,\ldots,x^n),F^{m_0}(x^2,x^3,\ldots,x^n,x^1),\ldots,F^{m_0}(x^n,x^1,x^2,\ldots,x^{n-1}))),$$
$$(F(F^{m_0}(y^1,y^2,y^3,\ldots,y^n),F^{m_0}(y^2,y^3,\ldots,y^n,y^1),\ldots,F^{m_0}(y^n,y^1,y^2,\ldots,y^{n-1})))\big) < \epsilon,$$

that is,

$$p\big(F^{m_0+1}(x^1,x^2,x^3,\ldots,x^n),F^{m_0+1}(y^1,y^2,y^3,\ldots,y^n)\big) < \epsilon. \tag{3.7}$$

On the other hand, we have

$$\epsilon \le \max\{p(F^{m_0}(x^2,x^3,\ldots,x^n,x^1),F^{m_0}(y^2,y^3,\ldots,y^n,y^1)),\ldots,p(F^{m_0}(x^n,x^1,x^2,\ldots,x^{n-1}),$$
$$F^{m_0}(y^n,y^1,y^2,\ldots,y^{n-1})),p(F^{m_0}(x^1,x^2,x^3,\ldots,x^n),F^{m_0}(y^1,y^2,y^3,\ldots,y^n))\}$$
$$< \epsilon + \delta(\epsilon),$$

which implies that

$$p\big(F^{m_0+1}(x^2,x^3,\ldots,x^n,x^1),F^{m_0+1}(y^2,y^3,\ldots,y^n,y^1)\big) < \epsilon. \tag{3.8}$$

Similarly,

$$p\big(F^{m_0+1}(x^n,x^1,x^2,\ldots,x^{n-1}),F^{m_0+1}(y^n,y^1,y^2,\ldots,y^{n-1})\big) < \epsilon. \tag{3.9}$$

Combining (3.7), (3.8), and (3.9), we have

$$\eta\big((F^{m_0+1}(x^1,x^2,x^3,\ldots,x^n),F^{m_0+1}(x^2,x^3,\ldots,x^n,x^1),\ldots,F^{m_0+1}(x^n,x^1,x^2,\ldots,x^{n-1})),$$
$$(F^{m_0+1}(y^1,y^2,y^3,\ldots,y^n),F^{m_0+1}(y^2,y^3,\ldots,y^n,y^1),\ldots,F^{m_0+1}(y^n,y^1,y^2,\ldots,y^{n-1})))$$
$$< \epsilon,$$

which is a contradiction. Therefore, we have necessarily $\epsilon = 0$. That is,

$$\eta\big((F^m(x^1,x^2,x^3,\ldots,x^n),F^m(x^2,x^3,\ldots,x^n,x^1),\ldots,F^m(x^n,x^1,x^2,\ldots,x^{n-1})),$$
$$(F^m(y^1,y^2,y^3,\ldots,y^n),F^m(y^2,y^3,\ldots,y^n,y^1),\ldots,F^m(y^n,y^1,y^2,\ldots,y^{n-1}))\big) = 0. \qquad \square$$

Remark 3.1 Lemma 3.2 also holds if we replace condition (3) by $\exists(x^1,x^2,x^3,\ldots,x^n),(y^1,y^2,y^3,\ldots,y^n) \in X^n$ such that $x^1 \preceq y^1, y^2 \prec x^2, x^3 \preceq y^3,\ldots,y^n \prec x^n$.

Theorem 3.1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X^n \to X$ be a given mapping satisfying the following hypotheses*:

(1) *F is continuous,*

(2) *F has the mixed strict monotone property,*

(3) *F is a generalized Meir-Keeler type function,*

(4) *there exist $x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that*

$$
\begin{cases}
x_0^1 \prec F(x_0^1, x_0^2, x_0^3, \ldots, x_0^n), \\
F(x_0^2, x_0^3, \ldots, x_0^n, x_0^1) \preceq x_0^2, \\
x_0^3 \prec F(x_0^3, \ldots, x_0^n, x_0^1, x_0^2), \\
\vdots \\
F(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}) \preceq x_0^n.
\end{cases}
\tag{3.10}
$$

Then there exist $(x^1, x^2, x^3, \ldots, x^n) \in X^n$ such that $x^1 = F(x^1, x^2, x^3, \ldots, x^n), x^2 = F(x^2, x^3, \ldots, x^n, x^1), \ldots, x^n = F(x^n, x^1, x^2, \ldots, x^{n-1})$.

Proof Let us define sequences $\{x_m^1\}, \{x_m^2\}, \ldots, \{x_m^n\}$ in X by

$$
\begin{cases}
x_m^1 = F^m(x_0^1, x_0^2, x_0^3, \ldots, x_0^n), \\
x_m^2 = F^m(x_0^2, x_0^3, \ldots, x_0^n, x_0^1), \\
x_m^3 = F^m(x_0^3, \ldots, x_0^n, x_0^1, x_0^2), \\
\vdots \\
x_m^n = F^m(x_0^n, x_0^1, x_0^2, \ldots, x_0^{n-1}), \quad \forall m \in \mathbb{N}.
\end{cases}
$$

Since F has mixed monotone property and from (3.3) we have

$$x_0^1 \prec x_1^1 \prec x_2^1 \prec \cdots \prec x_m^1 \prec x_{m+1}^1 \prec \cdots$$

$$\cdots \preceq x_{m+1}^2 \preceq x_m^2 \preceq \cdots \preceq x_2^2 \preceq x_1^2 \preceq x_0^2,$$

$$x_0^3 \prec x_1^3 \prec x_2^3 \prec \cdots \prec x_m^3 \prec x_{m+1}^3 \prec \cdots$$

$$\vdots$$

$$\cdots \preceq x_{m+1}^n \preceq x_m^n \preceq \cdots \preceq x_2^n \preceq x_1^n \preceq x_0^n.$$

Applying Lemma 3.2 by taking $x^1 = x_0^1, x^2 = x_0^2, \ldots, x^n = x_0^n$ and $y^1 = x_1^1, y^2 = x_1^2, \ldots, y^n = x_1^n$, then we get

$$\eta\big(\big(F^m(x_0^1, x_0^2, \ldots, x_0^n), F^m(x_0^2, \ldots, x_0^n, x_0^1), \ldots, F^m(x_0^n, x_0^1, \ldots, x_0^{n-1})\big),$$

$$\big(F^m(x_1^1, x_1^2, \ldots, x_1^n), F^m(x_1^2, \ldots, x_1^n, x_1^1), \ldots, F^m(x_1^n, x_1^1, \ldots, x_1^{n-1})\big)\big) \to 0 \quad \text{as } m \to \infty,$$

that is,

$$\eta\big(\big(x_m^1, x_m^2, \ldots, x_m^n\big), \big(x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n\big)\big) \to 0 \quad \text{as } m \to \infty. \tag{3.11}$$

Denote

$$\eta^s\big((x_m^1, x_m^2, \ldots, x_m^n), (x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n)\big)$$
$$= 2\max\big\{p^s(x_m^1, x_{m+1}^1), p^s(x_m^2, x_{m+1}^2), \ldots, p^s(x_m^n, x_{m+1}^n)\big\}, \quad \forall m \in \mathbb{N}.$$

From the definition of p^s, it is clear that

$$\eta^s\big((x_m^1, x_m^2, \ldots, x_m^n), (x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n)\big)$$
$$\le 2\eta\big((x_m^1, x_m^2, \ldots, x_m^n), (x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n)\big), \quad \forall m \in \mathbb{N}.$$

Using (3.11), we get

$$\lim_{m \to \infty} \eta^s\big((x_m^1, x_m^2, \ldots, x_m^n), (x_{m+1}^1, x_{m+1}^2, \ldots, x_{m+1}^n)\big)$$
$$= \lim_{m \to \infty} \max\big\{p^s(x_m^1, x_{m+1}^1), p^s(x_m^2, x_{m+1}^2), \ldots, p^s(x_m^n, x_{m+1}^n)\big\} = 0. \tag{3.12}$$

Let $\epsilon > 0$. It follows from (3.12) that there exists $k \in \mathbb{N}$ such that

$$\eta^s\big((x_k^1, x_k^2, \ldots, x_k^n), (x_{k+1}^1, x_{k+1}^2, \ldots, x_{k+1}^n)\big) < \delta(\epsilon). \tag{3.13}$$

Without restriction of generality, we can suppose that $\delta(\epsilon) \le \epsilon$. We introduce the set $\wedge \subset X^n$ defined by

$$\wedge := \big\{(x^1, x^2, x^3, \ldots, x^n) \in X^n : x_k^1 \prec x^1, x^2 \preceq x_k^2, x_k^3 \prec x^3, \ldots, x^n \preceq x_k^n,$$
$$\eta^s\big((x_k^1, x_k^2, x_k^3, \ldots, x_k^n), (x_{k+1}^1, x_{k+1}^2, x_{k+1}^3, \ldots, x_{k+1}^n)\big) < \epsilon + \delta(\epsilon)\big\}.$$

Now we will prove that $\forall (x^1, x^2, x^3, \ldots, x^n) \in \wedge$,

$$\big(F(x^1, x^2, x^3, \ldots, x^n), F(x^2, x^3, \ldots, x^n, x^1), \ldots, F(x^n, x^1, x^2, \ldots, x^{n-1})\big) \in \wedge. \tag{3.14}$$

Let $(x^1, x^2, x^3, \ldots, x^n) \in \wedge$. We have

$$\eta^s\big((x_k^1, x_k^2, \ldots, x_k^n), (F(x^1, x^2, \ldots, x^n), F(x^2, \ldots, x^n, x^1), \ldots, F(x^n, x^1, \ldots, x^{n-1}))\big)$$
$$= \max\big\{p^s(x_k^1, F(x^1, x^2, \ldots, x^n)), p^s(x_k^2, F(x^2, \ldots, x^n, x^1)), \ldots, p^s(x_k^n, F(x^n, x^1, \ldots, x^{n-1}))\big\}$$
$$\le \max\big\{p^s(x_k^1, x_{k+1}^1) + p^s(x_{k+1}^1, F(x^1, x^2, \ldots, x^n)), p^s(x_k^2, x_{k+1}^2)$$
$$+ p^s(x_{k+1}^2, F(x^2, \ldots, x^n, x^1)), \ldots, p^s(x_k^n, x_{k+1}^n) + p^s(x_{k+1}^n, F(x^n, x^1, \ldots, x^{n-1}))\big\}$$
$$= \max\big\{p^s(x_k^1, x_{k+1}^1) + p^s(F(x_k^1, x_k^2, \ldots, x_k^n), F(x^1, x^2, \ldots, x^n)), p^s(x_k^2, x_{k+1}^2)$$
$$+ p^s(F(x_k^2, \ldots, x_k^n, x_k^1), F(x^2, \ldots, x^n, x^1)), \ldots, p^s(x_k^n, x_{k+1}^n)$$
$$+ p^s(F(x_k^n, x_k^1, \ldots, x_k^{n-1}), F(x^n, x^1, \ldots, x^{n-1}))\big\}$$
$$\le \max\big\{p^s(x_k^1, x_{k+1}^1), p^s(x_k^2, x_{k+1}^2), \ldots, p^s(x_k^n, x_{k+1}^n)\big\} + \max\big\{p^s(F(x_k^1, x_k^2, \ldots, x_k^n),$$
$$F(x^1, x^2, \ldots, x^n)), p^s(F(x_k^2, \ldots, x_k^n, x_k^1), F(x^2, \ldots, x^n, x^1)), \ldots,$$
$$p^s(F(x_k^n, x_k^1, \ldots, x_k^{n-1}), F(x^n, x^1, \ldots, x^{n-1}))\big\}$$

$$< \delta(\epsilon) + \max\left\{p^s\left(F\left(x_k^1, x_k^2, \ldots, x_k^n\right), F\left(x^1, x^2, \ldots, x^n\right)\right), p^s\left(F\left(x_k^2, \ldots, x_k^n, x_k^1\right),\right.\right.$$

$$\left.\left. F\left(x^2, \ldots, x^n, x^1\right)\right), \ldots, p^s\left(F\left(x_k^n, x_k^1, \ldots, x_k^{n-1}\right), F\left(x^n, x^1, \ldots, x^{n-1}\right)\right)\right\} \quad \text{(by (3.13))}.$$

We consider the following two cases.

Case I: $\eta^s((x_k^1, x_k^2, \ldots, x_k^n), (x^1, x^2, \ldots, x^n)) \leq \epsilon$.

By Lemma 3.1, we have

$$\eta^s\left(\left(x_k^1, x_k^2, \ldots, x_k^n\right), \left(F\left(x^1, x^2, \ldots, x^n\right), F\left(x^2, \ldots, x^n, x^1\right), \ldots, F\left(x^n, x^1, \ldots, x^{n-1}\right)\right)\right)$$

$$< \delta(\epsilon) + \max\left\{p^s\left(F\left(x_k^1, x_k^2, \ldots, x_k^n\right), F\left(x^1, x^2, \ldots, x^n\right)\right), p^s\left(F\left(x_k^2, \ldots, x_k^n, x_k^1\right),\right.\right.$$

$$\left.\left. F\left(x^2, \ldots, x^n, x^1\right)\right), \ldots, p^s\left(F\left(x_k^n, x_k^1, \ldots, x_k^{n-1}\right), F\left(x^n, x^1, \ldots, x^{n-1}\right)\right)\right\}$$

$$< \delta(\epsilon) + \max\left\{\max\left[p^s\left(x_k^1, x^1\right), p^s\left(x_k^2, x^2\right), \ldots, p^s\left(x_k^n, x^n\right)\right], \max\left[p^s\left(x_k^2, x^2\right), \ldots,\right.\right.$$

$$\left.\left. p^s\left(x_k^n, x^n\right), p^s\left(x_k^1, x^1\right)\right], \ldots, \max\left[p^s\left(x_k^n, x^n\right), p^s\left(x_k^1, x^1\right), \ldots, p^s\left(x_k^{n-1}, x^{n-1}\right)\right]\right\}$$

$$< \delta(\epsilon) + \eta^s\left(\left(x_k^1, x_k^2, \ldots, x_k^n\right), \left(x^1, x^2, \ldots, x^n\right)\right) \leq \delta(\epsilon) + \epsilon.$$

Case II: $\epsilon + \eta^s((x_k^1, x_k^2, \ldots, x_k^n), (x^1, x^2, \ldots, x^n)) \leq \delta(\epsilon) + \epsilon$.

We have

$$\eta^s\left(\left(x_k^1, x_k^2, \ldots, x_k^n\right), \left(F\left(x^1, x^2, \ldots, x^n\right), F\left(x^2, \ldots, x^n, x^1\right), \ldots, F\left(x^n, x^1, \ldots, x^{n-1}\right)\right)\right)$$

$$< \delta(\epsilon) + \max\left\{p^s\left(F\left(x_k^1, x_k^2, \ldots, x_k^n\right), F\left(x^1, x^2, \ldots, x^n\right)\right), p^s\left(F\left(x_k^2, \ldots, x_k^n, x_k^1\right),\right.\right.$$

$$\left.\left. F\left(x^2, \ldots, x^n, x^1\right)\right), \ldots, p^s\left(F\left(x_k^n, x_k^1, \ldots, x_k^{n-1}\right), F\left(x^n, x^1, \ldots, x^{n-1}\right)\right)\right\}. \tag{3.15}$$

In this case, we get

$$\epsilon < \max\left\{p^s\left(x_k^1, x^1\right), p^s\left(x_k^2, x^2\right), \ldots, p^s\left(x_k^n, x^n\right)\right\} < \epsilon + \delta(\epsilon).$$

Since $x_k^1 \prec x^1, x^2 \preceq x_k^2, x_k^3 \prec x^3, \ldots, x^n \preceq x_k^n$, by (3) we get

$$p^s\left(F\left(x^1, x^2, \ldots, x^n\right), F\left(x_k^1, x_k^2, \ldots, x_k^n\right)\right) < \epsilon. \tag{3.16}$$

Also we have

$$\epsilon < \max\left\{p^s\left(x_k^2, x^2\right), \ldots, p^s\left(x_k^n, x^n\right), p^s\left(x_k^1, x^1\right)\right\} < \epsilon + \delta(\epsilon).$$

By (3), this implies that

$$p^s\left(F\left(x_k^2, \ldots, x_k^n, x_k^1\right), F\left(x^2, \ldots, x^n, x^1\right)\right) < \epsilon. \tag{3.17}$$

In the same way we have

$$p^s\left(F\left(x^n, x^1, \ldots, x^{n-1}\right), F\left(x_k^n, x_k^1, \ldots, x_k^{n-1}\right)\right) < \epsilon. \tag{3.18}$$

Hence combining (3.15)-(3.18), we obtain

$$\eta^s\left(\left(x_k^1, x_k^2, \ldots, x_k^n\right), \left(F\left(x_k^1, x_k^2, \ldots, x_k^n\right), F\left(x_k^2, \ldots, x_k^n, x_k^1\right), \ldots, F\left(x_k^n, x_k^1, \ldots, x_k^{n-1}\right)\right)\right) < \epsilon + \delta(\epsilon).$$

On the other hand, using (2), we can check easily that

$$x_k^1 \prec F(x^1, x^2, \ldots, x^n), \qquad F(x^2, \ldots, x^n, x^1) \preceq x_k^2, \qquad \ldots,$$
$$F(x^n, x^1, \ldots, x^{n-1}) \preceq x_k^n.$$

Hence, we deduce that (3.14) holds. By (3.13), we have $(x_{k+1}^1, x_{k+1}^2, \ldots, x_{k+1}^n) \in \wedge$. This implies with (3.14) that

$$\left(x_{k+1}^1, x_{k+1}^2, \ldots, x_{k+1}^n\right) \in \wedge$$
$$\Rightarrow \quad \left(F(x_{k+1}^1, x_{k+1}^2, \ldots, x_{k+1}^n), F(x_{k+1}^2, \ldots, x_{k+1}^n, x_{k+1}^1), \ldots, F(x_{k+1}^n, x_{k+1}^1, \ldots, x_{k+1}^{n-1})\right)$$
$$= \left(x_{k+2}^1, x_{k+2}^2, \ldots, x_{k+2}^n\right) \in \wedge$$
$$\Rightarrow \quad \left(F(x_{k+2}^1, x_{k+2}^2, \ldots, x_{k+2}^n), F(x_{k+2}^2, \ldots, x_{k+2}^n, x_{k+2}^1), \ldots, F(x_{k+2}^n, x_{k+2}^1, \ldots, x_{k+2}^{n-1})\right)$$
$$= \left(x_{k+3}^1, x_{k+3}^2, \ldots, x_{k+3}^n\right) \in \wedge$$
$$\vdots$$
$$\Rightarrow \quad \left(x_m^1, x_m^2, \ldots, x_m^n\right) \in \wedge.$$

Thus for all $m > k$, we have $(x_m^1, x_m^2, \ldots, x_m^n) \in \wedge$. This implies that for all $m, l > k$, we have

$$\eta^s\left(\left(x_m^1, x_m^2, \ldots, x_m^n\right), \left(x_l^1, x_l^2, \ldots, x_l^n\right)\right)$$
$$= \max\left\{p^s(x_m^1, x_l^1), p^s(x_m^2, x_l^2), \ldots, p^s(x_m^n, x_l^n)\right\}$$
$$\leq \max\left\{p^s(x_m^1, x_k^1) + p^s(x_k^1, x_l^1), p^s(x_m^2, x_k^2) + p^s(x_k^2, x_l^2), \ldots, p^s(x_m^n, x_k^n) + p^s(x_k^n, x_l^n)\right\}$$
$$\leq \max\left\{p^s(x_m^1, x_k^1), p^s(x_m^2, x_k^2), \ldots, p^s(x_m^n, x_k^n)\right\}$$
$$\quad + \max\left\{p^s(x_k^1, x_l^1), p^s(x_k^2, x_l^2), \ldots, p^s(x_k^n, x_l^n)\right\}$$
$$= \eta\left(\left(x_m^1, x_m^2, \ldots, x_m^n\right), \left(x_k^1, x_k^2, \ldots, x_k^n\right)\right) + \eta\left(\left(x_k^1, x_k^2, \ldots, x_k^n\right), \left(x_m^1, x_m^2, \ldots, x_m^n\right)\right)$$
$$< 2\left(\epsilon + \delta(\epsilon)\right) < 4\epsilon.$$

We deduce that $\{(x_m^1, x_m^2, \ldots, x_m^n)\}$ is a Cauchy sequence in the metric space (X^n, η^s). Since (X, p) is complete, from Lemma 2.1, (X, p^s) is a complete metric space. Therefore (X^n, η^s) is complete. Hence there exist $(x^1, x^2, \ldots, x^n) \in X^n$ such that

$$\eta^s\left(\left(x_m^1, x_m^2, \ldots, x_m^n\right), \left(x^1, x^2, \ldots, x^n\right)\right) \to 0 \quad \text{as } m \to \infty,$$

which shows that

$$p^s\left(F^m(x_0^1, x_0^2, \ldots, x_0^n), x^1\right) \to 0 \quad \text{as } m \to \infty,$$
$$p^s\left(F^m(x_0^2, \ldots, x_0^n, x_0^1), x^2\right) \to 0 \quad \text{as } m \to \infty,$$
$$\vdots$$
$$p^s\left(F^m(x_0^n, x_0^1, \ldots, x_0^{n-1}), x^n\right) \to 0 \quad \text{as } m \to \infty.$$

Therefore from Lemma 2.1 and using (3.12), we have

$$
\begin{cases}
p(x^1, x^1) = \lim_{m \to \infty} p(x_m^1, x^1) = \lim_{m \to \infty} p(x_m^1, x_m^1) = 0, \\
p(x^2, x^2) = \lim_{m \to \infty} p(x_m^2, x^2) = \lim_{m \to \infty} p(x_m^2, x_m^2) = 0, \\
\vdots \\
p(x^n, x^n) = \lim_{m \to \infty} p(x_m^n, x^n) = \lim_{m \to \infty} p(x_m^n, x_m^n) = 0.
\end{cases}
\tag{3.19}
$$

We will show that $x^1 = F(x^1, x^2, \ldots, x^n), x^2 = F(x^2, \ldots, x^n, x^1), \ldots, x^n = F(x^n, x^1, \ldots, x^{n-1})$. Since F is continuous on X, then F is continuous at $(x^1, x^2, \ldots, x^n)$. Hence for any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that if $(y^1, y^2, \ldots, y^n) \in X^n$ verifying

$$
\eta\big((x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n)\big) < \eta\big((x^1, x^2, \ldots, x^n), (x^1, x^2, \ldots, x^n)\big) + \delta(\epsilon)
$$

means that

$$
\max\big\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\big\}
$$
$$
< \max\big\{p(x^1, x^1), p(x^2, x^2), \ldots, p(x^n, x^n)\big\} + \delta(\epsilon) = \delta(\epsilon),
$$

because $p(x^1, x^1) = p(x^2, x^2) = \cdots = p(x^n, x^n) = 0$, then we have

$$
p\big(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)\big) < p\big(F(x^1, x^2, \ldots, x^n), F(x^1, x^2, \ldots, x^n)\big) + \frac{\epsilon}{2}.
$$

Since

$$
\lim_{m \to \infty} p(x_m^1, x^1) = \lim_{m \to \infty} p(x_m^2, x^2) = \cdots = \lim_{m \to \infty} p(x_m^n, x^n) = 0
$$

for $\alpha = \min\{\frac{\delta(\epsilon)}{2}, \frac{\epsilon}{2}\} > 0$, there exists $m_0, l_0 \in \mathbb{N}$ such that for $m > m_0$, $l > l_0$, $p(x_m^1, x^1) < \alpha, p(x_m^2, x^2) < \alpha, \ldots, p(x_m^n, x^n) < \alpha$. Then for $m \in \mathbb{N}$, $m \geq \max\{m_0, l_0\}$, we have

$$
\max\big\{p(x_m^1, x^1), p(x_m^2, x^2), \ldots, p(x_m^n, x^n)\big\} < \alpha < \frac{\delta(\epsilon)}{2},
$$

so we get

$$
p\big(F(x^1, x^2, \ldots, x^n), F(x_m^1, x_m^2, \ldots, x_m^n)\big) < p\big(F(x^1, x^2, \ldots, x^n), F(x^1, x^2, \ldots, x^n)\big) + \frac{\epsilon}{2}. \tag{3.20}
$$

Now, for any $m \geq \max\{m_0, l_0\}$,

$$
\begin{aligned}
p\big(F(x^1, x^2, \ldots, x^n), x^1\big) &\leq p\big(F(x^1, x^2, \ldots, x^n), x_{m+1}^1\big) + p\big(x_{m+1}^1, x^1\big) \\
&= p\big(F(x^1, x^2, \ldots, x^n), F(x_m^1, x_m^2, \ldots, x_m^n)\big) + p\big(x_{m+1}^1, x^1\big) \\
&< p\big(F(x^1, x^2, \ldots, x^n), F(x^1, x^2, \ldots, x^n)\big) + \frac{\epsilon}{2} + \alpha \quad \text{(by (3.20))} \\
&< p\big(F(x^1, x^2, \ldots, x^n), F(x^1, x^2, \ldots, x^n)\big) + \epsilon.
\end{aligned}
$$

On the other hand, since F is a generalized Meir-Keeler type function, then from Lemma 3.1, we have

$$p\big(F\big(x^1,x^2,\ldots,x^n\big),F\big(x^1,x^2,\ldots,x^n\big)\big) < \max\big\{p\big(x^1,x^1\big),p\big(x^2,x^2\big),\ldots,p\big(x^n,x^n\big)\big\} = 0.$$

In this case, for any $\epsilon > 0$, $p(F(x^1,x^2,\ldots,x^n),x^1) < \epsilon$. This implies that $F(x^1,x^2,\ldots,x^n) = x^1$. Similarly we can show that

$$F\big(x^2,\ldots,x^n,x^1\big) = x^2, \qquad \ldots, \qquad F\big(x^n,x^1,\ldots,x^{n-1}\big) = x^n.$$

Thus we have proved that F has an n-tupled fixed point. $\qquad\square$

Remark 3.2 Theorem 3.1 still holds if we replace (3.10) by $\exists x_0^1,x_0^2,x_0^3,\ldots,x_0^n \in X$ such that

$$\begin{cases} x_0^1 \preceq F(x_0^1,x_0^2,x_0^3,\ldots,x_0^n), \\ F(x_0^2,x_0^3,\ldots,x_0^n,x_0^1) \prec x_0^2, \\ x_0^3 \preceq F(x_0^3,\ldots,x_0^n,x_0^1,x_0^2), \\ \vdots \\ F(x_0^n,x_0^1,x_0^2,\ldots,x_0^{n-1}) \prec x_0^n. \end{cases}$$

Theorem 3.2 *Let* $(X,\preceq)$ *be a partially ordered set and suppose that there is a partial metric* p *on* X *such that* (X,p) *is a complete partial metric space. Assume that* X *has the following properties*:
   (a) *if a nondecreasing sequence* $x_m \to x$ *then* $x_m \preceq x$ *for all* $m \geq 0$,
   (b) *if a nonincreasing sequence* $x_m \to x$ *then* $x \preceq x_m$ *for all* $m \geq 0$.
Let $F : X^n \to X$ *be a given mapping satisfying the following hypotheses*:
   (1) *F is continuous,*
   (2) *F has the mixed strict monotone property,*
   (3) *F is a generalized Meir-Keeler type function,*
   (4) *there exist* $x_0^1,x_0^2,x_0^3,\ldots,x_0^n \in X$ *such that* (3.10) *holds.*
Then there exists $(x^1,x^2,x^3,\ldots,x^n) \in X^n$ *such that* $x^1 = F(x^1,x^2,x^3,\ldots,x^n), x^2 = F(x^2, x^3,\ldots,x^n,x^1),\ldots,x^n = F(x^n,x^1,x^2,\ldots,x^{n-1})$. *Furthermore,* $p(x^1,x^1) = p(x^2,x^2) = \cdots = p(x^n, x^n) = 0$.

Proof Following the proof of Theorem 3.1, we only have to prove that

$$x^1 = F\big(x^1,x^2,x^3,\ldots,x^n\big), \qquad x^2 = F\big(x^2,x^3,\ldots,x^n,x^1\big), \qquad \ldots,$$
$$x^n = F\big(x^n,x^1,x^2,\ldots,x^{n-1}\big).$$

Let $\epsilon > 0$. Since $F^m(x_0^1,x_0^2,\ldots,x_0^n) \to x^1, F^m(x_0^2,\ldots,x_0^n,x_0^1) \to x^2,\ldots,F^m(x_0^n,x_0^1,\ldots,x_0^{n-1}) \to x^n$. Then there exist $m_1,m_2,\ldots,m_n \in \mathbb{N}^*$ such that for all $m \geq m_1, l \geq m_2,\ldots,t \geq m_n$,

$$\begin{cases} p(F^m(x_0^1,x_0^2,\ldots,x_0^n),x^1) < \epsilon, \\ p(F^l(x_0^2,\ldots,x_0^n,x_0^1),x^2) < \epsilon, \\ \vdots \\ p(F^t(x_0^n,x_0^1,\ldots,x_0^{n-1}),x^n) < \epsilon. \end{cases} \qquad (3.21)$$

Taking $m \geq \max\{m_1, m_2, \ldots, m_n\}$ and using

$$F^m(x_0^1, x_0^2, \ldots, x_0^n) \prec x^1, \qquad x^2 \prec F^m(x_0^2, \ldots, x_0^n, x_0^1), \qquad \ldots,$$

$$x^n \prec F^m(x_0^n, x_0^1, \ldots, x_0^{n-1}),$$

by (3.21) and Lemma 3.1, we get

$$
\begin{aligned}
p\big(F(x^1, x^2, \ldots, x^n), x^1\big) &\leq p\big(F(x^1, x^2, \ldots, x^n), F^{m+1}(x_0^1, x_0^2, \ldots, x_0^n)\big) \\
&\quad + p\big(F^{m+1}(x_0^1, x_0^2, \ldots, x_0^n), x^1\big) \\
&= p\big(F(x^1, x^2, \ldots, x^n), F\big(F^m(x_0^1, x_0^2, \ldots, x_0^n), \\
&\quad F^m(x_0^2, \ldots, x_0^n, x_0^1), \ldots, \\
&\quad F^m(x_0^n, x_0^1, \ldots, x_0^{n-1})\big)\big) + p\big(F^{m+1}(x_0^1, x_0^2, \ldots, x_0^n), x^1\big) \\
&< \max\big\{p\big(x^1, F^m(x_0^1, x_0^2, \ldots, x_0^n)\big), p\big(x^2, F^m(x_0^2, \ldots, x_0^n, x_0^1)\big), \ldots, \\
&\quad p\big(x^n, F^m(x_0^n, x_0^1, \ldots, x_0^{n-1})\big)\big\} + p\big(F^{m+1}(x_0^1, x_0^2, \ldots, x_0^n), x^1\big) \\
&< 2\epsilon.
\end{aligned}
$$

This implies that $F(x^1, x^2, \ldots, x^n) = x^1$. Similarly, we can show that

$$p\big(F(x^2, \ldots, x^n, x^1), x^2\big) < 2\epsilon, \qquad \ldots, \qquad p\big(F(x^n, x^1, \ldots, x^{n-1}), x^n\big) < 2\epsilon,$$

which implies that $F(x^2, \ldots, x^n, x^1) = x^2, \ldots, F(x^n, x^1, \ldots, x^{n-1}) = x^n$.

This completes the proof. $\qquad\square$

Now we endow the product space X^n with the following partial order: for $(x^1, x^2, \ldots, x^n)$, $(y^1, y^2, \ldots, y^n) \in X^n$,

$$
\begin{aligned}
&\big(y^1, y^2, \ldots, y^n\big) \preceq \big(x^1, x^2, \ldots, x^n\big) \\
&\quad \Leftrightarrow \quad y^1 \preceq x^1, \qquad x^2 \preceq y^2, \qquad y^3 \preceq x^3, \qquad \ldots, \qquad x^n \preceq y^n.
\end{aligned}
$$

One can prove that n-tupled fixed point is in fact unique and the product space X^n endow with this partial order has the following property:

(A) $\forall (x^1, x^2, \ldots, x^n), (z^1, z^2, \ldots, z^n) \in X^n, \exists (t^1, t^2, \ldots, t^n) \in X^n$ that is comparable to $(x^1, x^2, \ldots, x^n)$ and $(z^1, z^2, \ldots, z^n)$.

Theorem 3.3 *Adding* (A) *to the hypotheses of Theorem* 3.1 (*respectively, Theorem* 3.2), *we obtain the uniqueness of n-tupled fixed point of F.*

Proof Suppose that $(z^1, z^2, \ldots, z^n) \in X^n$ is another n-tupled fixed point of F. We distinguish two cases:

Case I: $(x^1, x^2, \ldots, x^n)$ is comparable to $(z^1, z^2, \ldots, z^n)$ with respect to ordering in X^n, where

$$\lim_{m \to \infty} F^m(x_0^1, x_0^2, \ldots, x_0^n) = x^1, \qquad \lim_{m \to \infty} F^m(x_0^2, \ldots, x_0^n, x_0^1) = x^2, \qquad \ldots,$$

$$\lim_{m \to \infty} F^m(x_0^n, x_0^1, \ldots, x_0^{n-1}) = x^n.$$

Without restriction of generality, we can suppose that

$$F(z^1, z^2, \ldots, z^n) = z^1 \prec x^1 = F(x^1, x^2, \ldots, x^n),$$
$$F(x^2, \ldots, x^n, x^1) = x^2 \preceq z^2 = F(z^2, \ldots, z^n, z^1),$$
$$\vdots$$
$$F(x^n, x^1, \ldots, x^{n-1}) = x^n \preceq z^n = F(z^n, z^1, \ldots, z^{n-1}).$$

We have

$$\eta\big((x^1, x^2, \ldots, x^n), (z^1, z^2, \ldots, z^n)\big)$$
$$= \max\big\{p(x^1, z^1), p(x^2, z^2), \ldots, p(x^n, z^n)\big\}$$
$$= \max\big\{p\big(F(x^1, x^2, \ldots, x^n), F(z^1, z^2, \ldots, z^n)\big), p\big(F(x^2, \ldots, x^n, x^1),$$
$$\quad F(z^2, \ldots, z^n, z^1)\big), \ldots, p\big(F(x^n, x^1, \ldots, x^{n-1}), F(z^n, z^1, \ldots, z^{n-1})\big)\big\}$$
$$< \max\big\{\max\big[p(x^1, z^1), p(x^2, z^2), \ldots, p(x^n, z^n)\big], \max\big[p(x^1, z^1), p(x^2, z^2), \ldots,$$
$$\quad p(x^n, z^n)\big], \ldots, \max\big[p(x^1, z^1), p(x^2, z^2), \ldots, p(x^n, z^n)\big]\big\}$$
$$= \eta\big((x^1, x^2, \ldots, x^n), (z^1, z^2, \ldots, z^n)\big).$$

Case II: $(x^1, x^2, \ldots, x^n)$ is not comparable to $(z^1, z^2, \ldots, z^n)$. Then there exists $(t^1, t^2, \ldots, t^n) \in X^n$ that is comparable to $(x^1, x^2, \ldots, x^n)$ and $(z^1, z^2, \ldots, z^n)$. Without restriction of generality, we can assume that

$$x^1 \prec t^1, \qquad t^2 \preceq x^2, \qquad x^3 \prec t^3, \qquad \ldots, \qquad t^n \preceq x^n \quad \text{and} \tag{3.22}$$
$$z^1 \prec t^1, \qquad t^2 \preceq z^2, \qquad z^3 \prec t^3, \qquad \ldots, \qquad t^n \preceq z^n.$$

From (3.22) and Lemma 3.2, we have

$$\eta\big((F^m(x^1, x^2, \ldots, x^n), F^m(x^2, \ldots, x^n, x^1), \ldots, F^m(x^n, x^1, \ldots, x^{n-1})), (F^m(t^1, t^2, \ldots, t^n),$$
$$F^m(t^2, \ldots, t^n, t^1), \ldots, F^m(t^n, t^1, \ldots, t^{n-1}))\big) \to 0 \quad \text{as } m \to \infty. \tag{3.23}$$

Similarly we have

$$\eta\big((F^m(z^1, z^2, \ldots, z^n), F^m(z^2, \ldots, z^n, z^1), \ldots, F^m(z^n, z^1, \ldots, z^{n-1})), (F^m(t^1, t^2, \ldots, t^n),$$
$$F^m(t^2, \ldots, t^n, t^1), \ldots, F^m(t^n, t^1, \ldots, t^{n-1}))\big) \to 0 \quad \text{as } m \to \infty. \tag{3.24}$$

On the other hand, using the triangular inequality, we get

$$\eta\big((x^1, x^2, \ldots, x^n), (z^1, z^2, \ldots, z^n)\big)$$
$$= \eta\big((F^m(x^1, x^2, \ldots, x^n), F^m(x^2, \ldots, x^n, x^1), \ldots, F^m(x^n, x^1, \ldots, x^{n-1})),$$
$$\quad (F^m(z^1, z^2, \ldots, z^n), F^m(z^2, \ldots, z^n, z^1), \ldots, F^m(z^n, z^1, \ldots, z^{n-1}))\big)$$
$$\leq \eta\big((F^m(x^1, x^2, \ldots, x^n), F^m(x^2, \ldots, x^n, x^1), \ldots, F^m(x^n, x^1, \ldots, x^{n-1})),$$

$$\left(F^m\left(t^1,t^2,\ldots,t^n\right),F^m\left(t^2,\ldots,t^n,t^1\right),\ldots,F^m\left(t^n,t^1,\ldots,t^{n-1}\right)\right)\right)$$

$$+\,\eta\left(\left(F^m\left(t^1,t^2,\ldots,t^n\right),F^m\left(t^2,\ldots,t^n,t^1\right),\ldots,F^m\left(t^n,t^1,\ldots,t^{n-1}\right)\right),\right.$$

$$\left.\left(F^m\left(z^1,z^2,\ldots,z^n\right),F^m\left(z^2,\ldots,z^n,z^1\right),\ldots,F^m\left(z^n,z^1,\ldots,z^{n-1}\right)\right)\right)\right).$$

By (3.22) and (3.23), we have $\eta((x^1,x^2,\ldots,x^n),(z^1,z^2,\ldots,z^n)) = 0$, we get

$$\left(x^1,x^2,\ldots,x^n\right) = \left(z^1,z^2,\ldots,z^n\right).$$

This completes the proof. □

4 Applications

In this section, using the earlier results proved in the preceding section, we obtain some n-tupled fixed point theorem for mappings satisfying a general contractive condition of integral type in partially ordered complete partial metric spaces.

Theorem 4.1 *Let $(X,\preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X,p) is a complete partial metric space. Let $F : X^n \to X$ be a given mapping. Assume that there exists a function θ from $[0,\infty)$ into itself satisfying the following:*

(1) *$\theta(0) = 0$ and $\theta(t) > 0$ for every $t > 0$,*

(2) *θ is nondecreasing and right continuous,*

(3) *for every $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon < \theta\left(\max\left\{p\left(x^1,y^1\right),p\left(x^2,y^2\right),\ldots,p\left(x^n,y^n\right)\right\}\right) < \epsilon + \delta(\epsilon)$$

$$\Rightarrow\quad \theta\left(p\left(F\left(x^1,x^2,\ldots,x^n\right),F\left(y^1,y^2,\ldots,y^n\right)\right)\right) < \epsilon$$

for all $y^1 \prec x^1, x^2 \preceq y^2, y^3 \prec x^3,\ldots,x^n \preceq y^n$.
Then F is a generalized Meir-Keeler type function.

Proof Fix $\epsilon > 0$. Since $\theta(\epsilon) > 0$, there exists $\alpha > 0$ and $(a^1,a^2,\ldots,a^n),(b^1,b^2,\ldots,b^n) \in X^n$ such that

$$\theta(\epsilon) \le \theta\left(\max\left\{p\left(a^1,b^1\right),p\left(a^2,b^2\right),\ldots,p\left(a^n,b^n\right)\right\}\right) < \theta(\epsilon) + \delta(\epsilon)$$

$$\Rightarrow\quad \theta\left(p\left(F\left(a^1,a^2,\ldots,a^n\right),F\left(b^1,b^2,\ldots,b^n\right)\right)\right) < \epsilon. \tag{4.1}$$

From the right continuity of θ, there exists $\delta > 0$ such that $\theta(\epsilon + \delta) < \theta(\epsilon) + \alpha$. Fix $x^1,x^2,\ldots,x^n,y^1,y^2,\ldots,y^n \in X$, then

$$\epsilon \le \max\left\{p\left(x^1,y^1\right),p\left(x^2,y^2\right),\ldots,p\left(x^n,y^n\right)\right\} < \epsilon + \delta.$$

Since θ is a nondecreasing function, we get

$$\theta(\epsilon) \le \theta\left(\max\left\{p\left(x^1,y^1\right),p\left(x^2,y^2\right),\ldots,p\left(x^n,y^n\right)\right\}\right) < \theta(\epsilon + \delta) < \theta(\epsilon) + \alpha.$$

By (4.1) we get

$$\theta\left(p\left(F\left(x^1,x^2,\ldots,x^n\right),F\left(y^1,y^2,\ldots,y^n\right)\right)\right) < \theta(\epsilon),$$

and hence

$$p\big(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)\big) < \epsilon. \qquad \square$$

The following result is an immediate consequence of Theorems 3.1, 3.2 and 4.1.

Corollary 4.1 *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X^n \to X$ be a given mapping satisfying the following hypotheses*:

(1) *F is continuous,*

(2) *F has the mixed strict monotone property,*

(3) *for all $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that*

$$\epsilon \leq \int_0^{\max\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\}} \varphi(t)\, dt < \epsilon + \delta(\epsilon)$$

$$\Rightarrow \int_0^{p(F(x^1, \ldots, x^n), F(y^1, \ldots, y^n))} \varphi(t)\, dt < \epsilon$$

for all $y^1 \prec x^1, x^2 \preceq y^2, y^3 \prec x^3, \ldots, x^n \preceq y^n$, where φ is a locally integrable function from $[0, \infty)$ into itself satisfying

$$\int_0^s \varphi(t)\, dt > 0, \quad \forall s > 0,$$

(4) *$\exists x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that (3.10) holds.*
Then there exists $(x^1, x^2, \ldots, x^n) \in X^n$ such that

$$x^1 = F\big(x^1, x^2, \ldots, x^n\big), \qquad x^2 = F\big(x^2, \ldots, x^n, x^1\big), \qquad \ldots, \qquad x^n = F\big(x^n, x^1, \ldots, x^{n-1}\big).$$

Moreover, if property (A) *is satisfied, then the n-tupled fixed point of F remains unique.*

Remark 4.1 The conclusions of the preceding corollary remain valid if we replace the continuity hypothesis of F by hypotheses (a) and (b) of Theorem 3.2.

Corollary 4.2 *Let $(X, \preceq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X^n \to X$ be a given mapping satisfying the following hypotheses*:

(1) *F is continuous,*

(2) *F has the mixed strict monotone property,*

(3) *for all $y^1 \prec x^1, x^2 \preceq y^2, y^3 \prec x^3, \ldots, x^n \preceq y^n$,*

$$\int_0^{p(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n))} \varphi(t)\, dt \leq k \int_0^{\max\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\}} \varphi(t)\, dt,$$

where $k \in (0, 1)$ and φ is a locally integrable function from $[0, \infty)$ into itself satisfying

$$\int_0^s \varphi(t)\, dt > 0, \quad \forall s > 0,$$

(4) *$\exists x_0^1, x_0^2, x_0^3, \ldots, x_0^n \in X$ such that (3.10) holds.*

Then there exists $(x^1, x^2, \ldots, x^n) \in X^n$ such that

$$x^1 = F(x^1, x^2, \ldots, x^n), \qquad x^2 = F(x^2, \ldots, x^n, x^1), \qquad \ldots, \qquad x^n = F(x^n, x^1, \ldots, x^{n-1}).$$

Moreover, if property (A) *is satisfied, then the n-tupled fixed point of F remains unique.*

Proof For all $\epsilon > 0$, take $\delta(\epsilon) = (\frac{1}{k} - 1)\epsilon$ and apply Corollary 4.1. $\qquad\square$

Remark 4.2 We replace the continuity hypothesis of F by hypotheses (a) and (b) of Theorem 3.2, then this result also remains true.

5 Example

We give the following example to illustrate our main result.

Example 5.1 Let $X = [0, 1]$. Then $(X, \preceq)$ is a partially ordered set under the natural ordering of real numbers. Define $p : [0, 1] \times [0, 1] \to \mathbb{R}^+$ by $p(x, y) = \max\{x, y\}$, $x, y \in [0, 1]$. Then (X, p) is a complete partial metric space.

Now for any fixed even integer $n > 1$, consider the product space $X^n = [0, 1] \times [0, 1] \times \cdots \times [0, 1]$, n times (in short we write $X^n = [0, 1]^n$). Define $F : X^n \to X$ by

$$F(x^1, x^2, x^3, \ldots, x^n) = \frac{x^1}{n} \quad \text{for } x^1, x^2, \ldots, x^n \in [0, 1].$$

Then F has the mixed strict monotone property. Also F is a generalized Meir-Keeler type function. The proof follows in two parts, that is, we prove the following:

For $(x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) \in X^n$ with $x^1 \prec y^1, y^2 \preceq x^2, x^3 \prec y^3, \ldots, y^n \preceq x^n$,

(1) $\quad p(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)) < \max\{p(x^1, y^1), p(x^2, y^2), \ldots, p(x^n, y^n)\}$,

(2) $\quad \eta((F^m(x^1, x^2, x^3, \ldots, x^n), F^m(x^2, x^3, \ldots, x^n, x^1), \ldots, F^m(x^n, x^1, x^2, \ldots, x^{n-1})),$

$\qquad (F^m(y^1, y^2, y^3, \ldots, y^n), F^m(y^2, y^3, \ldots, y^n, y^1), \ldots, F^m(y^n, y^1, y^2, \ldots, y^{n-1})))$

$\qquad \to 0 \quad \text{as } m \to \infty.$

The first part is trivial. For second part, we have

$$\eta((F^m(x^1, x^2, x^3, \ldots, x^n), F^m(x^2, x^3, \ldots, x^n, x^1), \ldots, F^m(x^n, x^1, x^2, \ldots, x^{n-1})),$$

$$(F^m(y^1, y^2, y^3, \ldots, y^n), F^m(y^2, y^3, \ldots, y^n, y^1), \ldots, F^m(y^n, y^1, y^2, \ldots, y^{n-1})))$$

$$= \eta((F(F^{m-1}(x^1, x^2, \ldots, x^n), F^{m-1}(x^2, \ldots, x^n, x^1), \ldots, F^{m-1}(x^n, x^1, \ldots, x^{n-1})),$$

$$F(F^{m-1}(x^2, \ldots, x^n, x^1), \ldots, F^{m-1}(x^n, x^1, \ldots, x^{n-1}), F^{m-1}(x^1, x^2, \ldots, x^n)), \ldots,$$

$$F(F^{m-1}(x^n, x^1, \ldots, x^{n-1}), F^{m-1}(x^1, x^2, \ldots, x^n), \ldots, F^{m-1}(x^{n-1}, \ldots, x^1, x^n))),$$

$$(F(F^{m-1}(y^1, y^2, \ldots, y^n), F^{m-1}(y^2, \ldots, y^n, y^1), \ldots, F^{m-1}(y^n, y^1, \ldots, y^{n-1})),$$

$$F(F^{m-1}(y^2, \ldots, y^n, y^1), \ldots, F^{m-1}(y^n, y^1, \ldots, y^{n-1}), F^{m-1}(y^1, y^2, \ldots, y^n)), \ldots,$$

$$F(F^{m-1}(y^n, y^1, \ldots, y^{n-1}), F^{m-1}(y^1, y^2, \ldots, y^n), \ldots, F^{m-1}(y^{n-1}, \ldots, y^1, y^n))))$$

$$\vdots$$

$$
= \eta \left(\left(F\left(\frac{x^1}{n^{m-1}}, \frac{x^2}{n^{m-1}}, \frac{x^3}{n^{m-1}}, \ldots, \frac{x^n}{n^{m-1}} \right), F\left(\frac{x^2}{n^{m-1}}, \frac{x^3}{n^{m-1}}, \ldots, \frac{x^n}{n^{m-1}}, \frac{x^1}{n^{m-1}} \right), \ldots, \right. \right.
$$

$$
\left. F\left(\frac{x^n}{n^{m-1}}, \frac{x^1}{n^{m-1}}, \frac{x^2}{n^{m-1}}, \ldots, \frac{x^{n-1}}{n^{m-1}} \right) \right), \left(F\left(\frac{y^1}{n^{m-1}}, \frac{y^2}{n^{m-1}}, \frac{y^3}{n^{m-1}}, \ldots, \frac{y^n}{n^{m-1}} \right), \right.
$$

$$
\left. \left. F\left(\frac{y^2}{n^{m-1}}, \frac{y^3}{n^{m-1}}, \ldots, \frac{y^n}{n^{m-1}}, \frac{y^1}{n^{m-1}} \right), \ldots, F\left(\frac{y^n}{n^{m-1}}, \frac{y^1}{n^{m-1}}, \frac{y^2}{n^{m-1}}, \ldots, \frac{y^{n-1}}{n^{m-1}} \right) \right) \right)
$$

$$
= \eta \left(\left(\frac{x^1}{n^m}, \frac{x^2}{n^m}, \frac{x^3}{n^m}, \ldots, \frac{x^n}{n^m} \right), \left(\frac{y^1}{n^m}, \frac{y^2}{n^m}, \frac{y^3}{n^m}, \ldots, \frac{y^n}{n^m} \right) \right)
$$

$$
= \max \left\{ p\left(\frac{x^1}{n^m}, \frac{y^1}{n^m} \right), p\left(\frac{x^2}{n^m}, \frac{y^2}{n^m} \right), p\left(\frac{x^3}{n^m}, \frac{y^3}{n^m} \right), \ldots, p\left(\frac{x^n}{n^m}, \frac{y^n}{n^m} \right) \right\}
$$

$$
= \max \left\{ \frac{y^1}{n^m}, \frac{x^2}{n^m}, \frac{y^3}{n^m}, \ldots, \frac{x^n}{n^m} \right\} \to 0 \quad \text{as } m \to \infty.
$$

Hence all the hypotheses of Theorem 3.1 are satisfied. Therefore, F has a unique n-tupled fixed point. Here $(0, 0, \ldots, 0)$ is an n-tupled fixed point of F.

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