Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH — Open Access

Strong convergence of approximated iterations for asymptotically pseudocontractive mappings

Yonghong Yao[1], Mihai Postolache[2] and Shin Min Kang[3*]

*Correspondence: smkang@gnu.ac.kr
[3]Department of Mathematics and RINS, Gyeongsang National University, Jinju, 660-701, Korea
Full list of author information is available at the end of the article

Abstract

The asymptotically nonexpansive mappings have been introduced by Goebel and Kirk in 1972. Since then, a large number of authors have studied the weak and strong convergence problems of the iterative algorithms for such a class of mappings. It is well known that the asymptotically nonexpansive mappings is a proper subclass of the class of asymptotically pseudocontractive mappings. In the present paper, we devote our study to the iterative algorithms for finding the fixed points of asymptotically pseudocontractive mappings in Hilbert spaces. We suggest an iterative algorithm and prove that it converges strongly to the fixed points of asymptotically pseudocontractive mappings.

MSC: 47J25; 47H09; 65J15

Keywords: asymptotically pseudocontractive mappings; iterative algorithms; fixed point; Hilbert space

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty, closed, and convex subset of H. Let $T: C \to C$ be a nonlinear mapping. We use $F(T)$ to denote the fixed point set of T.

Recall that T is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|$$

for all $x, y \in C$. In this case, if $L < 1$, then we call T an *L-contraction*. If $L = 1$, we call T *nonexpansive*. T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n\|x - y\| \tag{1.1}$$

for all $x, y \in C$ and all $n \ge 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] in 1972. They proved that, if C is a nonempty bounded, closed, and convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex.

Since then, a large number of authors have studied the following algorithms for the iterative approximation of fixed points of asymptotically nonexpansive mappings (see, *e.g.*, [2–29] and the references therein).

(A) *The modified Mann iterative algorithm.* For arbitrary $x_0 \in C$, the modified Mann iteration generates a sequence $\{x_n\}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad \forall n \geq 1. \tag{1.2}$$

(B) *The modified Ishikawa iterative algorithm.* For arbitrary $x_0 \in C$, the modified Ishikawa iteration generates a sequence $\{x_n\}$ by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T^n x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1. \end{cases} \tag{1.3}$$

(C) *The CQ algorithm.* For arbitrary $x_0 \in C$, the *CQ* algorithm generates a sequence $\{x_n\}$ by

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad \forall n \geq 1. \end{cases} \tag{1.4}$$

An important class of asymptotically pseudocontractive mappings generalizing the class of asymptotically nonexpansive mapping has been introduced and studied by Schu in 1991; see [19].

Recall that $T : C \to C$ is called an *asymptotically pseudocontractive mapping* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ for which the following inequality holds:

$$\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2 \tag{1.5}$$

for all $x, y \in C$ and all $n \geq 1$. It is clear that (1.5) is equivalent to

$$\|T^n x - T^n y\|^2 \leq k_n \|x - y\|^2 + \|(x - T^n x) - (y - T^n y)\|^2 \tag{1.6}$$

for all $x, y \in C$ and all $n \geq 1$.

Recall also that T is called *uniformly L-Lipschitzian* if there exists $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|$$

for all $x, y \in C$ and all $n \geq 1$.

Now, we know that the class of asymptotically nonexpansive mappings is a proper subclass of the class of asymptotically pseudocontractive mappings. If we define a mapping $T : [0, 1] \to [0, 1]$ by the formula $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$, then we can verify that T is asymptotically pseudocontractive but it is not asymptotically nonexpansive.

In order to approximate the fixed point of asymptotically pseudocontractive mappings, the following two results are interesting.

One is due to Schu [19], who proved the following convergence theorem.

Theorem 1.1 *Let H be a Hilbert space, C be a nonempty closed bounded and convex subset of H. Let T be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive self-mapping of C with $\{k_n\} \subset [1,\infty)$ and $\sum_{n=1}^{\infty}(q_n^2 - 1) < \infty$, where $q_n = (2k_n - 1)$ for all $n \geq 1$. Let $\{\alpha_n\} \subset [0,1]$ and $\{\beta_n\} \subset [0,1]$ be two sequences satisfying $0 < \epsilon \leq \alpha_n \leq \beta_n \leq b < L^{-2}(\sqrt{1+L^2} - 1)$ for all $n \geq 1$. Then the sequence $\{x_n\}$ generated by the modified Ishikawa iteration* (1.3) *converges strongly to some fixed point of T.*

Another one is due to Chidume and Zegeye [30] who introduced the following algorithm in 2003.

Let a sequence $\{x_n\}$ be generated from $x_1 \in C$ by

$$x_{n+1} = \lambda_n\theta_n x_1 + (1 - \lambda_n - \lambda_n\theta_n)x_n + \lambda_n T^n x_n, \quad \forall n \geq 1, \tag{1.7}$$

where the sequences $\{\lambda_n\}$ and $\{\theta_n\}$ satisfy

(i) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty$ and $\lambda_n(1 + \theta_n) \leq 1$;
(ii) $\frac{\lambda_n}{\theta_n} \to 0$, $\theta_n \to 0$ and $\frac{(\frac{\theta_{n-1}}{\theta_n} - 1)}{\lambda_n\theta_n} \to 0$;
(iii) $\frac{k_n - k_{n-1}}{\lambda_n\theta_n^2} \to 0$;
(iv) $\frac{k_n - 1}{\theta_n} \to 0$.

They gave the strong convergence analysis for the above algorithm (1.7) with some further assumptions on the mapping T in Banach spaces.

Remark 1.2 Note that there are some additional assumptions imposed on the underlying space C and the mapping T in the above two results. In (1.7), the parameter control is also restricted.

Inspired by the results above, the main purpose of this article is to construct an iterative method for finding the fixed points of asymptotically pseudocontractive mappings. We construct an algorithm which is based on the algorithms (1.2) and (1.7). Under some mild conditions, we prove that the suggested algorithm converges strongly to the fixed point of asymptotically pseudocontractive mapping T.

2 Preliminaries

It is well known that in a real Hilbert space H, the following inequality and equality hold:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H \tag{2.1}$$

and

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $t \in [0,1]$.

Lemma 2.1 ([31]) *Let C be a nonempty bounded and closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschtzian and asymptotically pseudocontraction. Then $I - T$ is demiclosed at zero.*

Lemma 2.2 ([32]) *Let $\{r_n\}$ be a sequence of real numbers. Assume $\{r_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $r_{n_k} \leq r_{n_k+1}$ for* *all $k \geq 0$. For every $n \geq N$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : r_{n_i} < r_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$, and for all $n \geq N$

$$\max\{r_{\tau(n)}, r_n\} \leq r_{\tau(n)+1}.$$

Lemma 2.3 ([33]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + \xi_n,$$

where $\{\delta_n\}$ is a sequence in $(0,1)$ and $\{\xi_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \delta_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{\xi_n}{\delta_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\xi_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

Now we introduce the following iterative algorithm for asymptotically pseudocontractive mappings.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $f : C \to C$ be a ρ-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real number sequences in $[0,1]$.

Algorithm 3.1 For $x_0 \in C$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = (1-\gamma_n)x_n + \gamma_n T^n x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n-\beta_n)x_n + \beta_n T^n y_n, \quad \forall n \geq 1. \end{cases} \tag{3.1}$$

Next, we prove our main result as follows.

Theorem 3.2 *Suppose that $F(T) \neq \emptyset$. Assume the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions*:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\alpha_n + \beta_n \leq \gamma_n$ *and* $0 < \liminf_{n\to\infty} \beta_n$;
(iii) $0 < a \leq \gamma_n \leq b < \frac{2}{\sqrt{(1+k_n)^2+4L^2}+1+k_n}$ *for all* $n \geq 1$.
Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $u = P_{F(T)}f(u)$, which is the unique solution of the variational inequality $\langle (I-f)x^*, x-x^* \rangle \geq 0$ for all $x \in F(T)$.

Proof From (3.1), we have

$$\begin{aligned} \|x_{n+1} - u\| &= \big\|\alpha_n f(x_n) + (1-\alpha_n-\beta_n)x_n + \beta_n T^n y_n - u\big\| \\ &\leq \big\|\alpha_n\big(f(x_n) - u\big) + (1-\alpha_n-\beta_n)(x_n - u) + \beta_n\big(T^n y_n - u\big)\big\| \end{aligned}$$

$$\begin{aligned}
&= \left\| \alpha_n\bigl(f(x_n) - u\bigr) + (1-\alpha_n)\left(\frac{1-\alpha_n-\beta_n}{1-\alpha_n}(x_n - u) + \frac{\beta_n}{1-\alpha_n}\bigl(T^n y_n - u\bigr)\right)\right\| \\
&\le (1-\alpha_n)\left\| \frac{(1-\alpha_n-\beta_n)(x_n-u)}{1-\alpha_n} + \frac{\beta_n(T^n y_n - u)}{1-\alpha_n}\right\| + \alpha_n \bigl\|f(x_n) - u\bigr\|. \qquad (3.2)
\end{aligned}$$

Using the equality (2.2), we get

$$\begin{aligned}
&\left\| \frac{(1-\alpha_n-\beta_n)(x_n-u)}{1-\alpha_n} + \frac{\beta_n(T^n y_n - u)}{1-\alpha_n}\right\|^2 \\
&\quad = \frac{1-\alpha_n-\beta_n}{1-\alpha_n}\|x_n - u\|^2 + \frac{\beta_n}{1-\alpha_n}\bigl\|T^n y_n - u\bigr\|^2 \\
&\qquad - \frac{\beta_n(1-\alpha_n-\beta_n)}{(1-\alpha_n)^2}\bigl\|x_n - T^n y_n\bigr\|^2. \qquad (3.3)
\end{aligned}$$

Picking up $y = u$ in (1.6) we deduce

$$\bigl\|T^n x - u\bigr\|^2 \le k_n\|x - u\|^2 + \bigl\|x - T^n x\bigr\|^2 \qquad (3.4)$$

for all $x \in C$.

From (3.1), (3.4), and (2.2), we obtain

$$\begin{aligned}
&\bigl\|T^n y_n - u\bigr\|^2 \\
&\quad \le k_n\|y_n - u\|^2 + \bigl\|y_n - T^n y_n\bigr\|^2 \\
&\quad = k_n\bigl\|(1-\gamma_n)x_n + \gamma_n T^n x_n - u\bigr\|^2 + \bigl\|(1-\gamma_n)x_n + \gamma_n T^n x_n - T^n y_n\bigr\|^2 \\
&\quad = k_n\bigl\|(1-\gamma_n)(x_n - u) + \gamma_n\bigl(T^n x_n - u\bigr)\bigr\|^2 \\
&\qquad + \bigl\|(1-\gamma_n)\bigl(x_n - T^n y_n\bigr) + \gamma_n\bigl(T^n x_n - T^n y_n\bigr)\bigr\|^2 \\
&\quad = k_n\bigl[(1-\gamma_n)\|x_n - u\|^2 + \gamma_n\bigl\|T^n x_n - u\bigr\|^2 - \gamma_n(1-\gamma_n)\bigl\|x_n - T^n x_n\bigr\|^2\bigr] \\
&\qquad + (1-\gamma_n)\bigl\|x_n - T^n y_n\bigr\|^2 + \gamma_n\bigl\|T^n x_n - T^n y_n\bigr\|^2 - \gamma_n(1-\gamma_n)\bigl\|x_n - T^n x_n\bigr\|^2 \\
&\quad \le k_n\bigl[(1-\gamma_n)\|x_n - u\|^2 + \gamma_n\bigl(k_n\|x_n - u\|^2 + \bigl\|x_n - T^n x_n\bigr\|^2\bigr) \\
&\qquad - \gamma_n(1-\gamma_n)\bigl\|x_n - T^n x_n\bigr\|^2\bigr] + (1-\gamma_n)\bigl\|x_n - T^n y_n\bigr\|^2 \\
&\qquad + \gamma_n\bigl\|T^n x_n - T^n y_n\bigr\|^2 - \gamma_n(1-\gamma_n)\bigl\|x_n - T^n x_n\bigr\|^2. \qquad (3.5)
\end{aligned}$$

By (3.1), we have

$$\|x_n - y_n\| = \gamma_n\bigl\|x_n - T^n x_n\bigr\|. \qquad (3.6)$$

Noting that T is uniformly L-Lipschitzian, from (3.5) and (3.6), we deduce

$$\begin{aligned}
&\bigl\|T^n y_n - u\bigr\|^2 \\
&\quad \le k_n\bigl[(1-\gamma_n)\|x_n - u\|^2 + \gamma_n\bigl(k_n\|x_n - u\|^2 + \bigl\|x_n - T^n x_n\bigr\|^2\bigr) \\
&\qquad - \gamma_n(1-\gamma_n)\bigl\|x_n - T^n x_n\bigr\|^2\bigr] + (1-\gamma_n)\bigl\|x_n - T^n y_n\bigr\|^2
\end{aligned}$$

$$\begin{aligned}
&+ \gamma_n L^2 \|x_n - y_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2 \\
&= k_n\big[(1-\gamma_n)\|x_n - u\|^2 + \gamma_n\big(k_n\|x_n - u\|^2 + \|x_n - T^n x_n\|^2\big) \\
&\quad - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2\big] + (1-\gamma_n)\|x_n - T^n y_n\|^2 \\
&\quad + \gamma_n^3 L^2\|x_n - T^n x_n\|^2 - \gamma_n(1-\gamma_n)\|x_n - T^n x_n\|^2 \\
&= \big[1 + (k_n\gamma_n + 1)(k_n - 1)\big]\|x_n - u\|^2 + (1-\gamma_n)\|x_n - T^n y_n\|^2 \\
&\quad - \gamma_n\big(1 - \gamma_n - k_n\gamma_n - \gamma_n^2 L^2\big)\|x_n - T^n x_n\|^2.
\end{aligned} \tag{3.7}$$

By condition (iii), we know that $\gamma_n \le b < \frac{2}{\sqrt{(1+k_n)^2+4L^2}+k_n+1}$ for all n. Then we deduce that $1 - \gamma_n - k_n\gamma_n - \gamma_n^2 L^2 > 0$ for all $n \ge 0$.

Therefore, from (3.7), we derive

$$\begin{aligned}
\|T^n y_n - u\|^2 &\le \big[1 + (k_n\gamma_n + 1)(k_n - 1)\big]\|x_n - u\|^2 \\
&\quad + (1-\gamma_n)\|x_n - T^n y_n\|^2.
\end{aligned} \tag{3.8}$$

Note that $\beta_n \le 1 - \alpha_n$ and substituting (3.8) to (3.3), we obtain

$$\begin{aligned}
&\left\| \frac{(1-\alpha_n-\beta_n)(x_n - u)}{1-\alpha_n} + \frac{\beta_n(T^n y_n - u)}{1-\alpha_n} \right\|^2 \\
&\le \frac{\beta_n}{1-\alpha_n}\big\{\big[1 + (k_n\gamma_n + 1)(k_n - 1)\big]\|x_n - u\|^2 + (1-\gamma_n)\|x_n - T^n y_n\|^2\big\} \\
&\quad + \frac{1-\alpha_n-\beta_n}{1-\alpha_n}\|x_n - u\|^2 - \frac{\beta_n(1-\alpha_n-\beta_n)}{(1-\alpha_n)^2}\|x_n - T^n y_n\|^2 \\
&= \left[1 + \frac{\beta_n}{1-\alpha_n}(k_n\gamma_n + 1)(k_n - 1)\right]\|x_n - u\|^2 + \frac{\beta_n(\alpha_n+\beta_n-\gamma_n)}{(1-\alpha_n)^2}\|x_n - T^n y_n\|^2 \\
&\le \left[1 + \frac{\beta_n}{1-\alpha_n}(k_n\gamma_n + 1)(k_n - 1)\right]\|x_n - u\|^2 \\
&\le \big[1 + (k_n\gamma_n + 1)(k_n - 1)\big]\|x_n - u\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\| \frac{(1-\alpha_n-\beta_n)(x_n - u)}{1-\alpha_n} + \frac{\beta_n(T^n y_n - u)}{1-\alpha_n} \right\| \\
&\le \sqrt{1 + (k_n\gamma_n + 1)(k_n - 1)}\|x_n - u\| \\
&\le \big[1 + (k_n\gamma_n + 1)(k_n - 1)\big]\|x_n - u\|.
\end{aligned} \tag{3.9}$$

Since $k_n \to 1$, without loss of generality, we assume that $k_n \le 2$ for all $n \ge 1$. It follows from (3.2) and (3.9) that

$$\begin{aligned}
&\|nx_{n+1} - u\| \\
&\le \alpha_n\|f(x_n) - u\| + (1-\alpha_n)\big[1 + (k_n\gamma_n + 1)(k_n - 1)\big]\|nx_n - u\| \\
&\le \alpha_n\|f(x_n) - f(u)\| + \alpha_n\|f(u) - u\|
\end{aligned}$$

$$
\begin{aligned}
&\quad + (1-\alpha_n)\big[1+(k_n\gamma_n+1)(k_n-1)\big]\|x_n-u\| \\
&\le \alpha_n\rho\|x_n-u\| + \alpha_n\big\|f(u)-u\big\| \\
&\quad + (1-\alpha_n)\big[1+(k_n\gamma_n+1)(k_n-1)\big]\|x_n-u\| \\
&\le \alpha_n\big\|f(u)-u\big\| + \big[1-(1-\rho)\alpha_n\big]\|x_n-u\| \\
&\quad + (k_n\gamma_n+1)(k_n-1)\|x_n-u\| \\
&\le (1-\rho)\alpha_n\frac{\|f(u)-u\|}{1-\rho} + \big[1-(1-\rho)\alpha_n\big]\|x_n-u\| + 3(k_n-1)\|x_n-u\|.
\end{aligned}
$$

An induction induces

$$
\begin{aligned}
\|x_{n+1}-u\| &\le \big[1+3(k_n-1)\big]\max\left\{\|x_n-u\|, \frac{\|f(u)-u\|}{1-\rho}\right\} \\
&\le \prod_{j=1}^{n}\big[1+3(k_j-1)\big]\max\left\{\|x_0-u\|, \frac{\|f(u)-u\|}{1-\rho}\right\}.
\end{aligned}
$$

This implies that the sequence $\{x_n\}$ is bounded by the condition $\sum_{n=1}^{\infty}(k_n-1)<\infty$.

From (2.1) and (3.1), we have

$$
\begin{aligned}
&\|x_{n+1}-u\|^2 \\
&\quad = \big\|(1-\alpha_n)(x_n-u)-\beta_n\big(x_n-T^n y_n\big)+\alpha_n\big(f(x_n)-u\big)\big\|^2 \\
&\quad \le \big\|(1-\alpha_n)(x_n-u)-\beta_n\big(x_n-T^n y_n\big)\big\|^2 + 2\alpha_n\big\langle f(x_n)-u, x_{n+1}-u\big\rangle \\
&\quad = \big\|(1-\alpha_n)(x_n-u)\big\|^2 - 2\beta_n(1-\alpha_n)\big\langle x_n-T^n y_n, x_n-u\big\rangle \\
&\qquad + \beta_n^2\big\|x_n-T^n y_n\big\|^2 + 2\alpha_n\big\langle f(x_n)-u, x_{n+1}-u\big\rangle.
\end{aligned}
\tag{3.10}
$$

From (3.7), we deduce

$$
\begin{aligned}
&2\big\langle x_n-T^n y_n, x_n-u\big\rangle \\
&\quad \ge \gamma_n\big\|x_n-T^n y_n\big\|^2 + \gamma_n\big(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\big)\big\|x_n-T^n x_n\big\|^2 \\
&\qquad - (k_n\gamma_n+1)(k_n-1)\|x_n-u\|^2 \\
&\quad \ge \gamma_n\big(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\big)\big\|x_n-T^n x_n\big\|^2 + \gamma_n\big\|x_n-T^n y_n\big\|^2.
\end{aligned}
\tag{3.11}
$$

By condition (ii), we have $1\ge\gamma_n\ge\alpha_n+\beta_n\ge\alpha_n\gamma_n+\beta_n$ for all $n\ge 1$. Hence, by (3.10) and (3.11), we get

$$
\begin{aligned}
&\|x_{n+1}-u\|^2 \\
&\quad \le (1-\alpha_n)\|x_n-u\|^2 - \beta_n(1-\alpha_n)\gamma_n\big\|x_n-T^n y_n\big\|^2 + \beta_n^2\big\|x_n-T^n y_n\big\|^2 \\
&\qquad - \beta_n(1-\alpha_n)\gamma_n\big(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\big)\big\|x_n-T^n x_n\big\|^2 \\
&\qquad + 2\alpha_n\big\langle f(x_n)-u, x_{n+1}-u\big\rangle \\
&\quad \le (1-\alpha_n)\|x_n-u\|^2 + 2\alpha_n\big\langle f(x_n)-u, x_{n+1}-u\big\rangle \\
&\qquad - \beta_n(1-\alpha_n)\gamma_n\big(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\big)\big\|x_n-T^n x_n\big\|^2.
\end{aligned}
\tag{3.12}
$$

It follows that

$$\begin{aligned}&\|x_{n+1}-u\|^2-\|x_n-u\|^2+\beta_n(1-\alpha_n)\gamma_n\bigl(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\bigr)\bigl\|x_n-T^nx_n\bigr\|^2\\&\quad\le\alpha_n\bigl(2\langle f(x_n)-u,x_{n+1}-u\rangle-\|x_n-u\|^2\bigr).\end{aligned}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, there exists $M>0$ such that $\sup_n\{2\langle f(x_n)-u,x_{n+1}-u\rangle-\|x_n-u\|^2\}\le M$. So,

$$\begin{aligned}&\beta_n(1-\alpha_n)\gamma_n\bigl(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\bigr)\bigl\|x_n-T^nx_n\bigr\|^2\\&\quad+\|x_{n+1}-u\|^2-\|x_n-u\|^2\le\alpha_nM.\end{aligned}\tag{3.13}$$

Next, we consider two possible cases.

Case 1. Assume there exists some integer $m>0$ such that $\{\|x_n-u\|\}$ is decreasing for all $n\ge m$.

In this case, we know that $\lim_{n\to\infty}\|x_n-u\|$ exists. From (3.13), we deduce

$$\begin{aligned}&\beta_n(1-\alpha_n)\gamma_n\bigl(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2\bigr)\bigl\|x_n-T^nx_n\bigr\|^2\\&\quad\le\|x_n-u\|^2-\|x_{n+1}-u\|^2+M\alpha_n.\end{aligned}\tag{3.14}$$

By conditions (ii) and (iii), we have $\liminf_{n\to\infty}\beta_n(1-\alpha_n)\gamma_n(1-\gamma_n-k_n\gamma_n-\gamma_n^2L^2)>0$. Thus, from (3.14), we get

$$\lim_{n\to\infty}\bigl\|x_n-T^nx_n\bigr\|=0.\tag{3.15}$$

It follows from (3.6) and (3.15) that

$$\lim_{n\to\infty}\|x_n-y_n\|=0.\tag{3.16}$$

Since T is uniformly L-Lipschitzian, we have $\|T^ny_n-T^nx_n\|\le L\|x_n-y_n\|$. This together with (3.16) implies that

$$\lim_{n\to\infty}\bigl\|T^ny_n-T^nx_n\bigr\|=0.\tag{3.17}$$

Note that

$$\bigl\|x_n-T^ny_n\bigr\|\le\bigl\|x_n-T^nx_n\bigr\|+\bigl\|T^nx_n-T^ny_n\bigr\|.\tag{3.18}$$

Combining (3.15), (3.17), and (3.18), we have

$$\lim_{n\to\infty}\bigl\|x_n-T^ny_n\bigr\|=0.\tag{3.19}$$

From (3.1), we deduce

$$\|x_{n+1}-x_n\|\le\alpha_n\bigl\|f(x_n)-x_n\bigr\|+\beta_n\bigl\|T^ny_n-x_n\bigr\|.$$

Therefore,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.20}$$

Since T is uniformly L-Lipschitzian, we derive

$$\begin{aligned} &\|x_n - Tx_n\| \\ &\quad \le \|x_n - T^n x_n\| + \|T^n x_n - Tx_n\| \\ &\quad \le \|x_n - T^n x_n\| + L\|T^{n-1}x_n - x_n\| \\ &\quad \le \|x_n - T^n x_n\| + L\|T^{n-1}x_n - T^{n-1}x_{n-1}\| \\ &\qquad + L\|T^{n-1}x_{n-1} - x_{n-1}\| + L\|x_{n-1} - x_n\| \\ &\quad \le \|x_n - T^n x_n\| + (L^2 + L)\|x_n - x_{n-1}\| + L\|T^{n-1}x_{n-1} - x_{n-1}\|. \end{aligned} \tag{3.21}$$

By (3.15), (3.20), and (3.21), we have immediately

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.22}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$x_{n_k} \to \tilde{x} \in C$$

and

$$\limsup_{n\to\infty} \langle f(u) - u, x_n - u\rangle = \lim_{k\to\infty} \langle f(u) - u, x_{n_k} - u\rangle.$$

Thus, we use the demiclosed principle of T (Lemma 2.1) and (3.22) to deduce

$$\tilde{x} \in F(T).$$

So,

$$\begin{aligned} \limsup_{n\to\infty} \langle f(u) - u, x_n - u\rangle &= \lim_{k\to\infty} \langle f(u) - u, x_{n_k} - u\rangle \\ &= \langle f(u) - u, \tilde{x} - u\rangle \\ &\le 0. \end{aligned}$$

Returning to (3.12) to obtain

$$\begin{aligned} \|x_{n+1} - u\|^2 &\le (1-\alpha_n)\|x_n - u\|^2 + 2\alpha_n\langle f(x_n) - u, x_{n+1} - u\rangle \\ &= (1-\alpha_n)\|x_n - u\|^2 + 2\alpha_n\langle f(x_n) - f(u), x_{n+1} - u\rangle \\ &\quad + 2\alpha_n\langle f(u) - u, x_{n+1} - u\rangle \\ &\le (1-\alpha_n)\|x_n - u\|^2 + 2\alpha_n\rho\|x_n - u\|\|x_{n+1} - u\| \\ &\quad + 2\alpha_n\langle f(u) - u, x_{n+1} - u\rangle \end{aligned}$$

$$\begin{aligned} &\le (1-\alpha_n)\|x_n - u\|^2 + \alpha_n\rho\big(\|x_n - u\|^2 + \|x_{n+1} - u\|^2\big) \\ &\quad + 2\alpha_n\big\langle f(u) - u, x_{n+1} - u\big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - u\|^2 &\le \big[1 - (1-\rho)\alpha_n\big]\|x_n - u\|^2 \\ &\quad + \frac{2\alpha_n}{1-\alpha_n\rho}\big\langle f(u) - u, x_{n+1} - u\big\rangle. \end{aligned} \tag{3.23}$$

In Lemma 2.3, we take $a_n = \|x_{n+1} - u\|^2$, $\delta_n = (1-\rho)\alpha_n$, and $\xi_n = \frac{2\alpha_n}{1-\alpha_n\rho}\langle f(u) - u, x_{n+1} - u\rangle$. We can easily check that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\frac{\xi_n}{\delta_n} \le 0$. Thus, we deduce that $x_n \to u$.

Case 2. Assume there exists an integer n_0 such that $\|x_{n_0} - u\| \le \|x_{n_0+1} - u\|$. At this case, we set $\omega_n = \{\|x_n - u\|\}$. Then we have $\omega_{n_0} \le \omega_{n_0+1}$. Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \le l \le n, \omega_l \le \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty}\tau(n) = \infty$$

and

$$\omega_{\tau(n)} \le \omega_{\tau(n)+1}$$

for all $n \ge n_0$. From (3.22), we get

$$\lim_{n\to\infty}\|x_{\tau(n)} - Tx_{\tau(n)}\| = 0.$$

This implies that $\omega_w(x_{\tau(n)}) \subset F(T)$. Thus, we obtain

$$\limsup_{n\to\infty}\big\langle f(u) - u, x_{\tau(n)} - u\big\rangle \le 0. \tag{3.24}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from (3.23) that

$$\omega_{\tau(n)}^2 \le \omega_{\tau(n)+1}^2 \le \big[1 - (1-\rho)\alpha_{\tau(n)}\big]\omega_{\tau(n)}^2 + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\rho}\big\langle f(u) - u, x_{\tau(n)+1} - u\big\rangle.$$

It follows that

$$\omega_{\tau(n)}^2 \le \frac{2}{(1-\alpha_{\tau(n)}\rho)(1-\rho)}\big\langle f(u) - u, x_{\tau(n)+1} - u\big\rangle. \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\limsup_{n\to\infty}\omega_{\tau(n)} \le 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.26}$$

From (3.23), we obtain

$$\|x_{\tau(n)+1} - u\|^2 \leq \bigl[1 - (1-\rho)\alpha_\tau(n)\bigr]\|x_{\tau(n)} - u\|^2 + \frac{2\alpha_{\tau(n)}}{1-\alpha_{\tau(n)}\rho}\bigl\langle f(u) - u, x_{\tau(n)+1} - u\bigr\rangle.$$

It follows that

$$\limsup_{n\to\infty} \omega_{\tau(n)+1} \leq \limsup_{n\to\infty} \omega_{\tau(n)}.$$

This together with (3.26) imply that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.2 to get

$$0 \leq \omega_n \leq \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to u$. The proof is completed. □

Since the class of asymptotically nonexpansive mappings is a proper subclass of the class of asymptotically pseudocontractive mappings and asymptotically nonexpansive mapping T is L-Lipschitzian with $L = \sup_n k_n$. Thus, from Theorem 3.2, we get the following corollary.

Corollary 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be an asymptotically nonexpansive mapping satisfying $\sum_{n=1}^\infty (k_n - 1) < \infty$. Suppose that $F(T) \neq \emptyset$. Let $f : C \to C$ be a ρ-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three real number sequences in $[0,1]$. Assume the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $\alpha_n + \beta_n \leq \gamma_n$ *and* $0 < \liminf_{n\to\infty} \beta_n$;

(iii) $0 < a \leq \gamma_n \leq b < \frac{2}{\sqrt{(1+L)^2 + 4L^2} + 1 + L}$ *for all* $n \geq 1$, *where* $L = \sup_n k_n$.

Then the sequence $\{x_n\}$ defined by (3.1) *converges strongly to $u = P_{F(T)}f(u)$, which is the unique solution of the variational inequality $\langle (I - f)x^*, x - x^*\rangle \geq 0$ for all $x \in F(T)$.*

Remark 3.4 Our Theorem 3.2 does not impose any boundedness or compactness assumption on the space C or the mapping T. The parameter control conditions (i)-(iii) are mild.

Remark 3.5 Our Corollary 3.3 is also a new result.

4 Conclusion

This work contains our dedicated study to develop and improve iterative algorithms for finding the fixed points of asymptotically pseudocontractive mappings in Hilbert spaces. We introduced our iterative algorithm for this class of problems, and we have proven its strong convergence. This study is motivated by relevant applications for solving classes of real-world problems, which give rise to mathematical models in the sphere of nonlinear analysis.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China. [2]Faculty of Applied Sciences, University Politehnica of Bucharest, Splaiul Independentei 313, Bucharest, 060042, Romania. [3]Department of Mathematics and RINS, Gyeongsang National University, Jinju, 660-701, Korea.

Received: 19 February 2014 Accepted: 21 April 2014 Published: 02 May 2014

References

1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972). doi:10.1090/S0002-9939-1972-0298500-3
2. Ceng, LC, Sahu, DR, Yao, JC: Implicit iterative algorithms for asymptotically nonexpansive mappings in the intermediate sense and Lipschitz-continuous monotone mappings. J. Comput. Appl. Math. **233**, 2902-2915 (2010). doi:10.1016/j.cam.2009.11.035
3. Ceng, LC, Wong, NC, Yao, JC: Fixed point solutions of variational inequalities for a finite family of asymptotically nonexpansive mappings without common fixed point assumption. Comput. Math. Appl. **56**, 2312-2322 (2008). doi:10.1016/j.camwa.2008.05.002
4. Ceng, LC, Xu, HK, Yao, JC: The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces. Nonlinear Anal. **69**, 1402-1412 (2008). doi:10.1016/j.na.2007.06.040
5. Chang, SS, Lee, HWJ, Chan, CK, Kim, JK: Approximating solutions of variational inequalities for asymptotically nonexpansive mappings. Appl. Math. Comput. **212**, 51-59 (2009). doi:10.1016/j.amc.2009.01.078
6. Chidume, CE, Ali, B: Weak and strong convergence theorems for finite families of asymptotically nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **330**, 377-387 (2007). doi:10.1016/j.jmaa.2006.07.060
7. Cho, YJ, Zhou, H, Guo, G: Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. **47**, 707-717 (2004). doi:10.1016/S0898-1221(04)90058-2
8. Dehghan, H, Shahzad, N: Strong convergence of a CQ method for *k*-strictly asymptotically pseudocontractive mappings. Fixed Point Theory Appl. **2012**, 208 (2012). doi:10.1186/1687-1812-2012-208
9. Gornicki, J: Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces. Comment. Math. Univ. Carol. **30**, 249-252 (1989)
10. Guo, WP, Cho, YJ, Guo, W: Convergence theorems for mixed type asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2012**, 224 (2012). doi:10.1186/1687-1812-2012-224
11. Huang, Z: Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. **37**, 1-7 (1999). doi:10.1016/S0898-1221(99)00040-1
12. Kim, TH, Xu, HK: Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigrous. Nonlinear Anal. **64**, 1140-1152 (2006). doi:10.1016/j.na.2005.05.059
13. Lim, TC, Xu, HK: Fixed point theorems for asymptotically nonexpansive mappings. Nonlinear Anal. **22**, 1345-1355 (1994). doi:10.1016/0362-546X(94)90116-3
14. Liu, LQ: Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings. Nonlinear Anal. **26**, 1835-1842 (1996). doi:10.1016/0362-546X(94)00351-H
15. Osilike, MO, Aniagbosor, SC: Weak and strong convergence theorems for fixed points of asymptotically nonexpansive mappings. Math. Comput. Model. **32**, 1181-1191 (2000). doi:10.1016/S0895-7177(00)00199-0
16. Plubtieng, S, Wangkeeree, R, Punpaeng, R: On the convergence of modified Noor iterations with errors for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **322**, 1018-1029 (2006). doi:10.1016/j.jmaa.2005.09.078
17. Qin, X, Agarwal, RP, Cho, SY, Kang, SM: Convergence of algorithms for fixed points of generalized asymptotically quasi-ϕ-nonexpansive mappings with applications. Fixed Point Theory Appl. **2012**, 58 (2012). doi:10.1186/1687-1812-2012-58
18. Qin, X, Cho, SY, Kang, SM: A weak convergence theorem for total asymptotically pseudocontractive mappings in Hilbert spaces. Fixed Point Theory Appl. **2012**, Article ID 859795 (2011). doi:10.1155/2011/859795
19. Schu, J: Iteration construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991). doi:10.1016/0022-247X(91)90245-U
20. Schu, J: Approximation of fixed points of asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **112**, 143-151 (1991). doi:10.1090/S0002-9939-1991-1039264-7
21. Shahzad, N, Udomene, A: Fixed point solutions of variational inequalities for asymptotically nonexpansive mappings in Banach spaces. Nonlinear Anal. **64**, 558-567 (2006). doi:10.1016/j.na.2005.03.114

22. Shimizu, T, Takahashi, W: Strong convergence theorem for asymptotically nonexpansive mappings. Nonlinear Anal. **26**, 265-272 (1996). doi:10.1016/0362-546X(94)00278-P
23. Shioji, N, Takahashi, W: Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces. Nonlinear Anal. **34**, 87-99 (1998). doi:10.1016/S0362-546X(97)00682-2
24. Shioji, N, Takahashi, W: Strong convergence of averaged approximants for asymptotically nonexpansive mappings in Banach spaces. J. Approx. Theory **97**, 53-64 (1999). doi:10.1006/jath.1996.3251
25. Sunthrayuth, P, Kumam, P: Fixed point solutions of variational inequalities for a semigroup of asymptotically nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2012**, 177 (2012). doi:10.1186/1687-1812-2012-177
26. Yao, Y, Ćirić, L, Liou, YC, Chen, R: Iterative algorithms for a finite family of asymptotically nonexpansive mappings. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **18**, 27-39 (2011)
27. Yao, Y, Liou, YC: Strong convergence to common fixed points of a finite family of asymptotically nonexpansive mappings. Taiwan. J. Math. **11**, 849-865 (2007)
28. Zegeye, H, Shahzad, N: Strong convergence theorems for continuous semigroups of asymptotically nonexpansive mappings. Numer. Funct. Anal. Optim. **30**, 833-848 (2009). doi:10.1080/01630560903123197
29. Zegeye, H, Shahzad, N: Approximation of the common minimum-norm fixed point of a finite family of asymptotically nonexpansive mappings. Fixed Point Theory Appl. **2013**, 1 (2013). doi:10.1186/1687-1812-2013-1
30. Chidume, CE, Zegeye, H: Approximate fixed point sequences and convergence theorems for asymptotically pseudocontractive mappings. J. Math. Anal. Appl. **278**, 354-366 (2003). doi:10.1016/S0022-247X(02)00572-3
31. Zhou, H: Demiclosedness principle with applications for asymptotically pseudo-contractions in Hilbert spaces. Nonlinear Anal. **70**, 3140-3145 (2009). doi:10.1016/j.na.2008.04.017
32. Mainge, PE: Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **325**, 469-479 (2007). doi:10.1016/j.jmaa.2005.12.066
33. Xu, HK: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240-256 (2002). doi:10.1112/S0024610702003332

10.1186/1687-1812-2014-100
Cite this article as: Yao et al.: Strong convergence of approximated iterations for asymptotically pseudocontractive mappings. *Fixed Point Theory and Applications* 2014, 2014:100