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A note on 'A best proximity point theorem for Geraghty-contractions'

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Abstract

In Caballero *et al.* (Fixed Point Theory Appl. (2012). doi:10.1186/1687-1812-2012-231), the authors prove a best proximity point theorem for Geraghty nonself contraction. In this note, not only *P*-property has been weakened, but also an improved best proximity point theorem will be presented by a short and simple proof. An example which satisfies weak *P*-property but not *P*-property has been presented to demonstrate our results.

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1 Introduction and preliminaries

Let A and B be nonempty subsets of a metric space (X,d). An operator $T:A\to B$ is said to be contractive if there exists $k\in[0,1)$ such that $d(Tx,Ty)\le kd(x,y)$ for any $x,y\in A$. The well-known Banach contraction principle says: Let (X,d) be a complete metric space, and $T:X\to X$ be a contraction of X into itself. Then T has a unique fixed point in X.

In 1973, Geraghty introduced the Geraghty-contraction and obtained Theorem 1.2.

Definition 1.1 ([1]) Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be a *Geraghty-contraction* if there exists $\beta\in\Gamma$ such that for any $x,y\in X$

$$d(Tx,Ty)\le\beta\bigl(d(x,y)\bigr)\cdot d(x,y),$$

where the class Γ denotes those functions $\beta:[0,\infty)\to[0,1)$ satisfying the following condition:

$$\beta(t_n)\to 1\quad\Rightarrow\quad t_n\to 0.$$

Theorem 1.2 ([1]) *Let (X,d) be a complete metric space and $T:X\to X$ be an operator. Suppose that there exists $\beta\in\Gamma$ such that for any $x,y\in X$,*

$$d(Tx,Ty)\le\beta\bigl(d(x,y)\bigr)\cdot d(x,y).$$

Then T has a unique fixed point.

Obviously, Theorem 1.2 is an extensive version of Banach contraction principle. In 2012, Caballero *et al.* introduced generalized Geraghty-contraction as follows.

Definition 1.3 ([2]) Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a *Geraghty-contraction* if there exists $\beta \in \Gamma$ such that for any $x, y \in A$

$$d(Tx, Ty) \le \beta\bigl(d(x,y)\bigr) \cdot d(x,y),$$

where the class Γ denotes those functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \Rightarrow \quad t_n \to 0.$$

Now we need the following notations and basic facts.

Let A and B be two nonempty subsets of a metric space (X,d). We denote by A_0 and B_0 the following sets:

$$A_0 = \bigl\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\bigr\},$$
$$B_0 = \bigl\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\bigr\},$$

where $d(A,B) = \inf\{d(x,y) : x \in A \text{ and } y \in B\}$.

In [3], the authors give sufficient conditions for when the sets A_0 and B_0 are nonempty. In [4], the author presents the following definition and proves that any pair (A,B) of nonempty, closed and convex subsets of a real Hilbert space H satisfies the P-property.

Definition 1.4 ([2]) Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the *P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2).$$

Let A, B be two nonempty subsets of a complete metric space and consider a mapping $T : A \to B$. The best proximity point problem is whether we can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \ge d(A,B)$ for any $x \in A$, in fact, the optimal solution to this problem is the one for which the value $d(A,B)$ is attained.

In [2], the authors give a generalization of Theorem 1.2 by considering a nonself map and they get the following theorem.

Theorem 1.5 ([2]) *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A,B) has the P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.*

Remark In [2], the proof of Theorem 1.5 is unnecessarily complex. In this note, not only P-property has been weakened, but also an improved best proximity point theorem will be presented by a short and simple proof. An example which satisfies weak P-property but not P-property has been presented to demonstrate our results.

2 Main results

Before giving our main results, we first introduce the notion of weak P-property.

Weak *P*-property Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the *weak P-property* if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$\begin{cases} d(x_1,y_1) = d(A,B), \\ d(x_2,y_2) = d(A,B) \end{cases} \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2).$$

Now we are in a position to give our main results.

Theorem 2.1 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that $A_0 \neq \emptyset$. Let $T : A \to B$ be a Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A,B) has the weak P-property. Then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.*

Proof We first prove that B_0 is closed. Let $\{y_n\} \subseteq B_0$ be a sequence such that $y_n \to q \in B$. It follows from the weak P-property that

$$d(y_n,y_m) \to 0 \quad \Rightarrow \quad d(x_n,x_m) \to 0,$$

as $n,m \to \infty$, where $x_n, x_m \in A_0$ and $d(x_n,y_n) = d(A,B)$, $d(x_m,y_m) = d(A,B)$. Then $\{x_n\}$ is a Cauchy sequence so that $\{x_n\}$ converges strongly to a point $p \in A$. By the continuity of metric d we have $d(p,q) = d(A,B)$, that is, $q \in B_0$, and hence B_0 is closed.

Let $\overline{A}_0$ be the closure of A_0, we claim that $T(\overline{A}_0) \subseteq B_0$. In fact, if $x \in \overline{A}_0 \setminus A_0$, then there exists a sequence $\{x_n\} \subseteq A_0$ such that $x_n \to x$. By the continuity of T and the closeness of B_0, we have $Tx = \lim_{n\to\infty} Tx_n \in B_0$. That is $T(\overline{A}_0) \subseteq B_0$.

Define an operator $P_{A_0} : T(\overline{A}_0) \to A_0$, by $P_{A_0}y = \{x \in A_0 : d(x,y) = d(A,B)\}$. Since the pair (A,B) has weak P-property and T is a Geraghty-contraction, we have

$$d(P_{A_0}Tx_1, P_{A_0}Tx_2) \leq d(Tx_1, Tx_2) \leq \beta\big(d(x_1,x_2)\big) \cdot d(x_1,x_2)$$

for any $x_1, x_2 \in \overline{A}_0$. This shows that $P_{A_0}T : \overline{A}_0 \to \overline{A}_0$ is a Geraghty-contraction from complete metric subspace $\overline{A}_0$ into itself. Using Theorem 1.2, we can get $P_{A_0}T$ has a unique fixed point x^*. That is $P_{A_0}Tx^* = x^* \in A_0$. It implies that

$$d\big(x^*, Tx^*\big) = d(A,B).$$

Therefore, x^* is the unique one in A_0 such that $d(x^*, Tx^*) = d(A,B)$. It is easy to see that x^* is also the unique one in A such that $d(x^*, Tx^*) = d(A,B)$. □

Remark In Theorem 2.1, P-property is weakened to weak P-property. Therefore, Theorem 2.1 is an improved result of Theorem 1.5. In addition, our proof is shorter and simpler than that in [2]. In fact, our proof process is less than one page. However, the proof process in [2] is three pages.

3 Example

Now we present an example which satisfies weak P-property but not P-property.

Consider (R^2, d), where d is the Euclidean distance and the subsets $A = \{(0,0)\}$ and $B = \{y = 1 + \sqrt{1 - x^2}\}$.

Obviously, $A_0 = \{(0,0)\}$, $B_0 = \{(-1,1),(1,1)\}$ and $d(A,B) = \sqrt{2}$. Furthermore,

$$d\bigl((0,0),(-1,1)\bigr) = d\bigl((0,0),(1,1)\bigr) = \sqrt{2},$$

however,

$$0 = d\bigl((0,0),(0,0)\bigr) < d\bigl((-1,1),(1,1)\bigr) = 2.$$

We can see that the pair (A,B) satisfies the weak P-property but not the P-property.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All the authors contributed equally to the writing of the present article. All authors read and approved the final manuscript.

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