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The new modified Ishikawa iteration method for the approximate solution of different types of differential equations

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Abstract

In this article, the new Ishikawa iteration method is presented to find the approximate solution of an ordinary differential equation with an initial condition. Additionally, some numerical examples with initial conditions are given to show the properties of the iteration method. Furthermore, the results of absolute errors are compared with Euler, Runge-Kutta and Picard iteration methods. Finally, the present method, namely the new modified Ishikawa iteration method, is seen to be very effective and efficient in solving different type of the problem.

MSC: 65K15; 65L07; 65L06; 65L70

Keywords: ordinary differential equation; Euler method; fixed point; numerical analysis; modified Ishikawa iteration; Picard successive iteration method

Introduction

Various kinds of numerical methods, especially iterative methods [1–3], were used to solve different types of differential equations. In recent years, there has been a growing interest in the treatment of iterative approximation of fixed point theory on normed linear spaces [4–13], Banach spaces [14–18] and Hilbert spaces [19, 20], respectively.

A new modified Ishikawa iteration method has been developed to find an approximate solution for different types of differential equations with initial conditions [13, 15–29] on metric spaces. The solutions are also obtained in terms of the Picard iteration. Also in Section 2, examples of these kinds of equations are solved using this new method which is called new modified Ishikawa iteration method and also the results are discussed and comparison using Euler [1, 2], Runge-Kutta [30, 31], and Picard iteration methods [1, 2, 27] is presented in tables and figures. Additionally, this approximated method can solve various different differential equations such as integral, difference, integro-differential and functional differential equations. Finally, numerical results shows that the new modified Ishikawa iteration method is more or less effective and also convenient for solving different types of differential equations.

Now, let us give some of the important theorems and definitions in order to solve linear and nonlinear differential equations using a contraction mapping.

1 Preliminaries

Theorem 1.1 (Banach contraction principle) *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction with the Lipschitzian constant L. Then T has a unique fixed point $u \in X$. Furthermore, for any $x \in X$ we have*

$$\lim_{n\to\infty} T^n(x) = u$$

with

$$d\bigl(T^n(x), u\bigr) \le \frac{L^n}{1-L} d\bigl(x, F(x)\bigr)$$

(*see* [13]).

Proof We first show uniqueness. Suppose there exist $x, y \in X$ with $T(x) = x$ and $T(y) = y$. Then

$$d(x, y) = d\bigl(T(x), T(y)\bigr) \le Ld(x, y),$$

therefore $d(x, y) = 0$.

To show existence select, $x \in X$. We first show that $\{T^n(x)\}$ is a Cauchy sequence. Notice for $n \in \{0, 1, \ldots\}$ that

$$d\bigl(T^n(x), T^{n+1}(x)\bigr) \le Ld\bigl(T^{n-1}(x), T^n x\bigr) \le \cdots \le L^n d\bigl(x, T(x)\bigr).$$

Thus, for $m > n$ where $n \in \{0, 1, \ldots\}$,

$$\begin{aligned} d\bigl(T^n(x), T^m(x)\bigr) &\le d\bigl(T^n(x), T^{n+1}(x)\bigr) + d\bigl(T^{n+1}(x), T^{n+2}(x)\bigr) + \cdots + \\ &\le \cdots \le d\bigl(T^{m-1}(x), T^m(x)\bigr) \\ &\le L^n d\bigl(x, T(x)\bigr) + \cdots + L^{m-1} d\bigl(x, T(x)\bigr) \\ &\le L^n d\bigl(x, T(x)\bigr)\bigl[1 + L + L^2 + \cdots\bigr] \\ &= \frac{L^n}{1-L} d\bigl(x, T(x)\bigr). \end{aligned}$$

That is, for $m > n$, $n \in \{0, 1, \ldots\}$,

$$d\bigl(T^n(x), T^m(x)\bigr) \le \frac{L^n}{1-L} d\bigl(x, T(x)\bigr). \tag{1.1}$$

This shows that $\{T^n(x)\}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ with $\lim_{n\to\infty} T^n(x) = u$. Moreover, the continuity of T yields

$$u = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T\bigl(T^n(x)\bigr) = T(u),$$

therefore u is a fixed point of T. Finally, letting $m \to \infty$ in (1.1) yields

$$d\bigl(T^n(x), u\bigr) \le \frac{L^n}{1-L} d\bigl(x, T(x)\bigr). \qquad \square$$

Corollary 1.2 *Let (X,d) be a complete metric space and let $B(x_0,r) = \{x \in X : d(x,x_0) < r\}$, where $x_0 \in X$ and $r > 0$. Suppose $T : B(x_0,r) \to X$ is a contraction (that is, $d(T(x),T(y)) \leq Ld(x,y)$ for all $x,y \in B(x_0,r)$ with $0 \leq L < 1$) with $d(T(x_0),x_0) < (1-L)r$. Then T has a unique fixed point in $B(x_0,r)$ (see* [13]).

Definition 1.3 If the sequence $\{x_n\}_{n=0}^{\infty}$ provides the condition $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots,$ then this is called the Picard iteration [1, 27].

Definition 1.4 Let $x_0 \in X$ be arbitrary. If the $\{x_n\}_{n=0}^{\infty}$ sequence provides the condition

$$x_{n+1} = (1-\alpha_n)x_n + \{\alpha_n\}Ty_n,$$
$$y_n = (1-\beta_n)x_n + \{\beta_n\}Tx_n$$

for $n = 0,1,2,\ldots,$ then this is called the Ishikawa iteration [16, 24] where (α_n) and (β_n) are sequences of positive numbers that satisfy the following conditions:

(i) $0 \leq \alpha_n \leq \beta_n < 1$, for all positive integers n,
(ii) $\lim_{n\to\infty} \beta_n = 0$,
(iii) $\sum_{n\geq 0} \alpha_n\beta_n = \infty$.

Definition 1.5 If $\lambda \in [0,1]$, $\gamma \in [0,1]$ and $y_0 \in X$, T is defined as a contraction mapping with regard to Picard iteration and also the $\{y_n\}_{n=0}^{\infty}$ sequence provides the conditions

$$y_{n+1} = \lambda y_{n-1} + (1-\lambda)Ty_{n-1},$$
$$y_n = (1-\gamma)y_{n-2} + \gamma Ty_{n-2}, \quad n = 2,4,\ldots,$$
$$y_{n+1} = y_0 + \int_{x_0}^{x} F\bigl(t,y_n(t)\bigr)\,dt, \quad n = 0,$$
$$Ty_{n-1} = y_n, \quad 0 < \lambda,\mu < 1$$

then this is called a new modified Ishikawa iteration where $T = \int_{x_0}^{x} F(t,y_n(t))\,dt$.

In order to illustrate the performance of the new modified Ishikawa iteration method in solving linear and nonlinear differential equations and justify the accuracy and efficiency of the method presented in this paper, we consider the following examples. In all examples, we used four types of iteration methods and the comparison is shown in figures and tables respectively.

2 Application of methods

Example 2.1 Let us consider the differential equation $y' = \sqrt{|y|}$ subject to the initial condition $y(0) = 1$.

Firstly, we obtained the exact solution of the equation as $|y| = \frac{1}{4}(x+2)^2 = 1 + x + \frac{x^2}{4}$.

By Theorem 1.1 and Corollary 1.2, since $T = \int_{x_0}^{x} F(t,y_n(t))\,dt$, then

$$\bigl|T(x)-T(y)\bigr| = \left|\int_0^x \sqrt{t}\,dt - \int_0^y \sqrt{t}\,dt\right| = \left|\frac{2}{3}\sqrt{t^3}\Big|_0^x - \frac{2}{3}\sqrt{t^3}\Big|_0^y\right|$$
$$\leq \frac{2}{3}\bigl|\sqrt{x^3}-\sqrt{y^3}\bigr| \leq \frac{2}{3}|x-y|.$$

So, $|T(x) - T(y)| \leq \frac{2}{3}|x - y|$ is found. Thus T has a unique fixed point, which is the unique solution of the integral equation $T = \int_{x_0}^{x} F(t, y_n(t))\,dt$ or the differential equation $y' = \sqrt{|y|}$, $y(0) = 1$.

Firstly, we approach the approximate solution using by the Picard iteration method. Thus

$$y_1 = 1 + x,$$
$$y_2 = \frac{1}{3} + \frac{2}{3}(x + 1)^{3/2}$$

are obtained. If we take the series expansion of the function $(x + 1)^{3/2}$ for the seven terms, then $y_2 = 1 + x + \frac{x^2}{4} + \frac{x^3}{24} + \frac{x^4}{64} + \frac{x^5}{128} + \frac{7x^6}{1536}$ is found. Now, applying the new modified Ishikawa iteration method to the equation for $\lambda = 0.5$, $\gamma = 0.5$,

$$y_1 = 1 + x,$$
$$y_2 = 1 + 0.5x,$$
$$y_3 = 1 + 0.75x,$$
$$y_4 = 1 + 0.625x,$$
$$y_5 = 1 + 0.6875x,$$
$$y_6 = 1 + 0.65625x,$$
$$y_7 = 1 + 0.671875x,$$
$$y_8 = 1 + 0.6640625x,$$
$$y_9 = 1 + 0.66796875x,$$
$$y_{10} = 1 + 0.666015625x,$$
$$y_{11} = 1 + 0.666992187x$$

are found and also for $\lambda = 0.5$, $\gamma = 0.25$,

$$y_1 = 1 + x,$$
$$y_2 = 1 + 0.25x,$$
$$y_3 = 1 + 0.625x,$$
$$y_4 = 1 + 0.34375x,$$
$$y_5 = 1 + 0.484375x,$$
$$y_6 = 1 + 0.37890625x,$$
$$y_7 = 1 + 0.431640625x,$$
$$y_8 = 1 + 0.392089843x,$$
$$y_9 = 1 + 0.411865234x,$$
$$y_{10} = 1 + 0.39703369x,$$
$$y_{11} = 1 + 0.404449462x$$

are obtained. On the other hand, for $\lambda = 0.25$, $\gamma = 0.5$,

$$\begin{aligned}
y_1 &= 1 + x, \\
y_2 &= 1 + 0.5x, \\
y_3 &= 1 + 0.625x, \\
y_4 &= 1 + 0.5625x, \\
y_5 &= 1 + 0.578125x, \\
y_6 &= 1 + 0.5703125x, \\
y_7 &= 1 + 0.572265625x, \\
y_8 &= 1 + 0.571289062x, \\
y_9 &= 1 + 0.571533203x, \\
y_{10} &= 1 + 0.571411132x, \\
y_{11} &= 1 + 0.57144165x
\end{aligned}$$

are calculated. In the same way, for $\lambda = 0.25$, $\gamma = 0.25$,

$$\begin{aligned}
y_1 &= 1 + x, \\
y_2 &= 1 + 0.25x, \\
y_3 &= 1 + 0.4375x, \\
y_4 &= 1 + 0.296875x, \\
y_5 &= 1 + 0.33203125x, \\
y_6 &= 1 + 0.305664062x, \\
y_7 &= 1 + 0.312255859x, \\
y_8 &= 1 + 0.307312011x, \\
y_9 &= 1 + 0.308547973x, \\
y_{10} &= 1 + 0.307621001x, \\
y_{11} &= 1 + 0.307852744x
\end{aligned}$$

are found and also for $\lambda = 0.75$, $\gamma = 0.25$,

$$\begin{aligned}
y_1 &= 1 + x, \\
y_2 &= 1 + 0.25x, \\
y_3 &= 1 + 0.8125x, \\
y_4 &= 1 + 0.390625x, \\
y_5 &= 1 + 0.70703125x, \\
y_6 &= 1 + 0.469726562x,
\end{aligned}$$

$$y_7 = 1 + 0.647705078x,$$

$$y_8 = 1 + 0.514221191x,$$

$$y_9 = 1 + 0.614334106x,$$

$$y_{10} = 1 + 0.539249419x,$$

$$y_{11} = 1 + 0.595562934x$$

are obtained. Similarly, for $\lambda = 0.25$, $\gamma = 0.75$,

$$y_1 = 1 + x,$$

$$y_2 = 1 + 0.75x,$$

$$y_3 = 1 + 0.8125x,$$

$$y_4 = 1 + 0.796875x,$$

$$y_5 = 1 + 0.80078125x,$$

$$y_6 = 1 + 0.799804687x,$$

$$y_7 = 1 + 0.800048828x,$$

$$y_8 = 1 + 0.799987792x,$$

$$y_9 = 1 + 0.800003051x,$$

$$y_{10} = 1 + 0.799999236x,$$

$$y_{11} = 1 + 0.80000019x$$

are calculated. Finally, for $\lambda = 0.75$, $\gamma = 0.75$,

$$y_1 = 1 + x,$$

$$y_2 = 1 + 0.75x,$$

$$y_3 = 1 + 0.9375x,$$

$$y_4 = 1 + 0.890625x,$$

$$y_5 = 1 + 0.92578125x,$$

$$y_6 = 1 + 0.916992187x,$$

$$y_7 = 1 + 0.923583984x,$$

$$y_8 = 1 + 0.921936035x,$$

$$y_9 = 1 + 0.923171996x,$$

$$y_{10} = 1 + 0.922863006x,$$

$$y_{11} = 1 + 0.922940253x$$

are obtained.

Now we calculate the approximate solution by the Euler method. At first we use the formula

$$y_{n+1} = y_n + hF(x_n, y_n)$$

with $F(x,y) = \sqrt{|\,y\,|}$, $h = 0.2$ and $x_0 = 0$, $y_0 = 1$. From the initial condition $y(0) = 1$, we have $F(0,1) = 1$. We now proceed with the calculations as follows:

$$F_0 = F(0,1) = 1,$$
$$y_1 = y_0 + hF_0 = 1 + 0.2,$$
$$y_1 = 1.2,$$
$$F_1 = F(0.2, 1.2) = 1.095445115,$$
$$y_2 = 1.2 + 0.2F_1,$$
$$y_2 = 1.419089023,$$
$$F_2 = F(0.4, 1.419089023) = 1.19125523,$$
$$y_3 = 1.419089023 + 0.2F_2,$$
$$y_3 = 1.657340069.$$

Finally, applying the Runge-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point and round off the final results at each step to four such places. Here $F(x,y) = \sqrt{|y|}x_0 = 0$, $y_0 = 1$ and we are to use $h = 0.2$. Using these quantities, we calculated successively k_1, k_2, k_3, k_4 and K_0 defined by

$$k_1 = hg(y_0, x_0),$$
$$k_2 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right),$$
$$k_3 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right),$$
$$k_4 = hg(y_0 + h, x_0 + k_3)$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, $y_{n+1} = y_n + K_0$. Thus we find k_1, k_2, k_3, k_4 for $n = 0$ as follows:

$$k_1 = 0.2F(x_0, y_0) = 0.2F(0,1) = 0.2,$$
$$k_2 = 0.2F(x_0 + 0.1, y_0 + 0.1) = 0.209761769,$$
$$k_3 = 0.2F(x_0 + 0.1, y_0 + 0.104880884) = 0.210226628,$$
$$k_4 = 0.2F(x_0 + 0.2, y_0 + 0.210226628) = 0.220020601.$$

So, $y_1 = 1.209999565$ is obtained for $x_1 = 0.2$. On the other hand, we calculated k_1, k_2, k_3, k_4 for $n = 1$ as follows:

$$k_1 = 0.2F(x_1, y_1) = 0.2F(0.2, 1.209999565) = 0.21999996,$$

$$k_2 = 0.2F(x_1 + 0.1, y_1 + 0.10999998) = 0.229782466,$$

$$k_3 = 0.2F(x_1 + 0.1, y_1 + 0.114891233) = 0.230207801,$$

$$k_4 = 0.2F(x_1 + 0.2, y_1 + 0.230207801) = 0.240017279.$$

Hence $y_2 = 1.4399999194$ is calculated for $x_2 = 0.4$. Finally we get k_1, k_2, k_3, k_4 for $n = 2$ as follows:

$$k_1 = hF(x_2, y_2) = 0.239999993,$$

$$k_2 = hF\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = 0.249799913,$$

$$k_3 = hF\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = 0.2250191916,$$

$$k_4 = hF(x_2 + h, y_2 + k_3) = 0.260014756.$$

Thus, $y_3 = 1.689999654$ is obtained for $x_3 = 0.6$.

After the necessary calculation which is done above, the comparison is shown schematically in Figure 1.

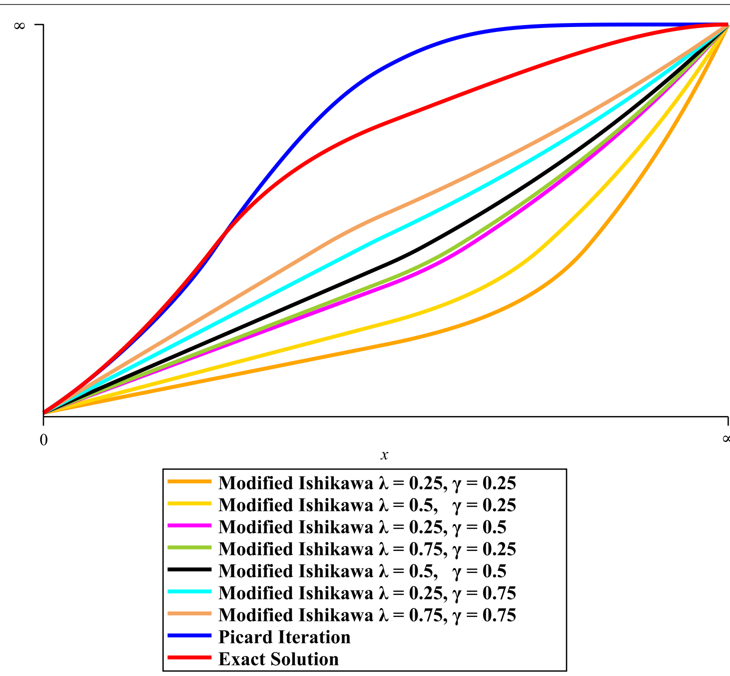

Figure 1 The comparison of exact solution and approximate solution of Example 2.1 for different values of λ and γ.

Table 1 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	λ = 0.5, γ = 0.5	λ = 0.5, γ = 0.25	λ = 0.25, γ = 0.5	λ = 0.25, γ = 0.25
$x = 0.2$	$y_1 = 1.2$	$y_1 = 1.2$	$y_1 = 1.2$	$y_1 = 1.2$
	$y_2 = 1.1$	$y_2 = 1.05$	$y_2 = 1.1$	$y_2 = 1.05$
	$y_3 = 1.15$	$y_3 = 1.125$	$y_3 = 1.125$	$y_3 = 1.0875$
	$y_4 = 1.125$	$y_4 = 1.06875$	$y_4 = 1.1125$	$y_4 = 1.059375$
	$y_5 = 1.1375$	$y_5 = 1.096875$	$y_5 = 1.115625$	$y_5 = 1.06640625$
	$y_6 = 1.13125$	$y_6 = 1.07578125$	$y_6 = 1.1140625$	$y_6 = 1.06113281$
	$y_7 = 1.134375$	$y_7 = 1.086328125$	$y_7 = 1.114453125$	$y_7 = 1.062451171$
	$y_8 = 1.1328125$	$y_8 = 1.078417968$	$y_8 = 1.114257812$	$y_8 = 1.061462402$
	$y_9 = 1.13359375$	$y_9 = 1.082373046$	$y_9 = 1.11430664$	$y_9 = 1.061709594$
	$y_{10} = 1.133203125$	$y_{10} = 1.079406738$	$y_{10} = 1.114282226$	$y_{10} = 1.0615242$
	$y_{11} = 1.13398437$	$y_{11} = 1.080889892$	$y_{11} = 1.11428833$	$y_{11} = 1.061570548$
$x = 0.4$	$y_1 = 1.4$	$y_1 = 1.4$	$y_1 = 1.4$	$y_1 = 1.4$
	$y_2 = 1.2$	$y_2 = 1.1$	$y_2 = 1.2$	$y_2 = 1.1$
	$y_3 = 1.3$	$y_3 = 1.25$	$y_3 = 1.25$	$y_3 = 1.175$
	$y_4 = 1.25$	$y_4 = 1.1375$	$y_4 = 1.225$	$y_4 = 1.11875$
	$y_5 = 1.275$	$y_5 = 1.19375$	$y_5 = 1.23125$	$y_5 = 1.1328125$
	$y_6 = 1.2625$	$y_6 = 1.1515625$	$y_6 = 1.228125$	$y_6 = 1.122265624$
	$y_7 = 1.26875$	$y_7 = 1.17265625$	$y_7 = 1.22890625$	$y_7 = 1.124902343$
	$y_8 = 1.265625$	$y_8 = 1.156835937$	$y_8 = 1.228515624$	$y_8 = 1.122924804$
	$y_9 = 1.2671875$	$y_9 = 1.164746093$	$y_9 = 1.228613281$	$y_9 = 1.123419189$
	$y_{10} = 1.26640625$	$y_{10} = 1.158813476$	$y_{10} = 1.228564452$	$y_{10} = 1.1230484$
	$y_{11} = 1.266796874$	$y_{11} = 1.161779784$	$y_{11} = 1.22857666$	$y_{11} = 1.123141097$
$x = 0.5$	$y_1 = 1.5$	$y_1 = 1.5$	$y_1 = 1.5$	$y_1 = 1.5$
	$y_2 = 1.25$	$y_2 = 1.125$	$y_2 = 1.25$	$y_2 = 1.125$
	$y_3 = 1.375$	$y_3 = 1.3125$	$y_3 = 1.3125$	$y_3 = 1.21875$
	$y_4 = 1.3125$	$y_4 = 1.171875$	$y_4 = 1.28125$	$y_4 = 1.1484375$
	$y_5 = 1.34375$	$y_5 = 1.2421875$	$y_5 = 1.2890625$	$y_5 = 1.166015625$
	$y_6 = 1.328125$	$y_6 = 1.189453125$	$y_6 = 1.28515625$	$y_6 = 1.152832031$
	$y_7 = 1.3359375$	$y_7 = 1.215820312$	$y_7 = 1.286132812$	$y_7 = 1.156127929$
	$y_8 = 1.33203125$	$y_8 = 1.196044921$	$y_8 = 1.285644531$	$y_8 = 1.153656005$
	$y_9 = 1.333984375$	$y_9 = 1.205932617$	$y_9 = 1.285766601$	$y_9 = 1.154273986$
	$y_{10} = 1.333007812$	$y_{10} = 1.198516845$	$y_{10} = 1.285705566$	$y_{10} = 1.1538105$
	$y_{11} = 1.333496093$	$y_{11} = 1.202224731$	$y_{11} = 1.285720825$	$y_{11} = 1.153926372$

On the other hand, we may give Table 1, Table 2, Table 3 and Table 4 using the modified Ishikawa iteration method for different values of λ and γ. Now we may give Table 5 which is expressed that absolute error of Example 2.1 for different values of λ and γ with $x = 0.2$, $x = 0.4$ and $x = 0.6$ respectively.

Corollary 2.1 *If we compare the approximate solution with the different values of* λ *and* γ*, then the conclusion may be indicated using Table* 1, *Table* 2, *Table* 3 *and Table* 4 *as follows.*

The best approximation is obtained taking the different values of λ *and* γ *and using the new modified Ishikawa iteration method for* $x = 0.2$, $x = 0.4$ *and* $x = 0.5$ *getting* (λ = 0.25, γ = 0.25; λ = 0.5, γ = 0.25; λ = 0.25, γ = 0.5; λ = 0.75, γ = 0.25; λ = 0.5, γ = 0.5; λ = 0.25, γ = 0.75; λ = 0.75, γ = 0.75) *respectively.*

Similarly, we calculated the solution for $x = 0.6$, $x = 1$ *and* $x = 1.5$ *then the approximation is found more sensitive taking* (λ = 0.25, γ = 0.25; λ = 0.5, γ = 0.25; λ = 0.25, γ = 0.5; λ = 0.75, γ = 0.25; λ = 0.5, γ = 0.5; λ = 0.25, γ = 0.75; λ = 0.75, γ = 0.75) *respectively.*

Corollary 2.2 *Absolute error of the modified Ishikawa iteration method is computed taking different values of* λ *and* γ ($x = 0.2$, $x = 0.4$ *and* $x = 0.6$), *which is not more effective than Runge-Kutta*, *Picard and Euler iteration methods*.

Table 2 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	$\lambda = 0.75$, $\gamma = 0.25$	$\lambda = 0.25$, $\gamma = 0.75$	$\lambda = 0.75$, $\gamma = 0.75$
$x = 0.2$	$y_1 = 1.2$	$y_1 = 1.2$	$y_1 = 1.2$
	$y_2 = 1.05$	$y_2 = 1.15$	$y_2 = 1.15$
	$y_3 = 1.1625$	$y_3 = 1.1625$	$y_3 = 1.1875$
	$y_4 = 1.078125$	$y_4 = 1.159375$	$y_4 = 1.178125$
	$y_5 = 1.14140625$	$y_5 = 1.16015625$	$y_5 = 1.18515625$
	$y_6 = 1.093945312$	$y_6 = 1.159960937$	$y_6 = 1.183398437$
	$y_7 = 1.129541015$	$y_7 = 1.160009765$	$y_7 = 1.184716796$
	$y_8 = 1.102844238$	$y_8 = 1.159997558$	$y_8 = 1.184387207$
	$y_9 = 1.122866821$	$y_9 = 1.16000061$	$y_9 = 1.184634399$
	$y_{10} = 1.107849883$	$y_{10} = 1.159999847$	$y_{10} = 1.184572601$
	$y_{11} = 1.119112586$	$y_{11} = 1.160000038$	$y_{11} = 1.18458805$
$x = 0.4$	$y_1 = 1.4$	$y_1 = 1.4$	$y_1 = 1.4$
	$y_2 = 1.1$	$y_2 = 1.3$	$y_2 = 1.3$
	$y_3 = 1.325$	$y_3 = 1.325$	$y_3 = 1.375$
	$y_4 = 1.15625$	$y_4 = 1.31875$	$y_4 = 1.35625$
	$y_5 = 1.2828125$	$y_5 = 1.3203125$	$y_5 = 1.3703125$
	$y_6 = 1.187890624$	$y_6 = 1.319921874$	$y_6 = 1.366796874$
	$y_7 = 1.259082031$	$y_7 = 1.320019531$	$y_7 = 1.369433593$
	$y_8 = 1.205688476$	$y_8 = 1.319995116$	$y_8 = 1.368774414$
	$y_9 = 1.245733642$	$y_9 = 1.32000122$	$y_9 = 1.369268798$
	$y_{10} = 1.215699767$	$y_{10} = 1.319999694$	$y_{10} = 1.369145202$
	$y_{11} = 1.238225173$	$y_{11} = 1.320000076$	$y_{11} = 1.369176101$
$x = 0.5$	$y_1 = 1.5$	$y_1 = 1.5$	$y_1 = 1.5$
	$y_2 = 1.125$	$y_2 = 1.375$	$y_2 = 1.375$
	$y_3 = 1.40625$	$y_3 = 1.40625$	$y_3 = 1.46875$
	$y_4 = 1.1953125$	$y_4 = 1.3984375$	$y_4 = 1.4453125$
	$y_5 = 1.353515625$	$y_5 = 1.400390625$	$y_5 = 1.462890625$
	$y_6 = 1.234863281$	$y_6 = 1.399902343$	$y_6 = 1.458496093$
	$y_7 = 1.323852539$	$y_7 = 1.400024414$	$y_7 = 1.461791992$
	$y_8 = 1.257110595$	$y_8 = 1.399993896$	$y_8 = 1.460968017$
	$y_9 = 1.307167053$	$y_9 = 1.400001525$	$y_9 = 1.461585998$
	$y_{10} = 1.269624709$	$y_{10} = 1.399999618$	$y_{10} = 1.461431503$
	$y_{11} = 1.297781467$	$y_{11} = 1.400000095$	$y_{11} = 1.461470126$

Example 2.2 Let us consider the differential equation

$$y' = y + x^2$$

subject to the initial condition

$$y(0) = 0.$$

Firstly, we obtained the exact solution of the equation as $y = 2e^x - x^2 - 2x - 2$.

Using Theorem 1.1 and Corollary 1.2, since $T = \int_{x_0}^{x} F(t, y_n(t))\,dt$, then T has a unique fixed point, which is the unique solution of the differential equation $y' = y + x^2$ with the initial condition $y(0) = 0$.

At first, we approach the approximate solution by the Picard iteration method as follows:

$$y_1 = \frac{x^3}{3},$$

$$y_2 = \frac{x^3}{3} + \frac{x^4}{12},$$

Table 3 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	$\lambda = 0.75, \gamma = 0.25$	$\lambda = 0.25, \gamma = 0.75$	$\lambda = 0.75, \gamma = 0.75$
$x = 0.6$	$y_1 = 1.6$	$y_1 = 1.6$	$y_1 = 1.6$
	$y_2 = 1.15$	$y_2 = 1.45$	$y_2 = 1.45$
	$y_3 = 1.4875$	$y_3 = 1.4875$	$y_3 = 1.5625$
	$y_4 = 1.234375$	$y_4 = 1.478125$	$y_4 = 1.534375$
	$y_5 = 1.42421875$	$y_5 = 1.48046875$	$y_5 = 1.55546875$
	$y_6 = 1.281835937$	$y_6 = 1.479882812$	$y_6 = 1.550195312$
	$y_7 = 1.388623046$	$y_7 = 1.480029296$	$y_7 = 1.55415039$
	$y_8 = 1.308532714$	$y_8 = 1.479992675$	$y_8 = 1.553161621$
	$y_9 = 1.368600463$	$y_9 = 1.48000183$	$y_9 = 1.553903197$
	$y_{10} = 1.323549651$	$y_{10} = 1.479999541$	$y_{10} = 1.553717803$
	$y_{11} = 1.35733776$	$y_{11} = 1.480000114$	$y_{11} = 1.553764151$
$x = 1$	$y_1 = 2$	$y_1 = 2$	$y_1 = 2$
	$y_2 = 1.25$	$y_2 = 1.75$	$y_2 = 1.75$
	$y_3 = 1.8125$	$y_3 = 1.8125$	$y_3 = 1.9375$
	$y_4 = 1.390625$	$y_4 = 1.796875$	$y_4 = 1.890625$
	$y_5 = 1.70703125$	$y_5 = 1.80078125$	$y_5 = 1.92578125$
	$y_6 = 1.469726562$	$y_6 = 1.799804687$	$y_6 = 1.916992187$
	$y_7 = 1.647705078$	$y_7 = 1.800048828$	$y_7 = 1.923583984$
	$y_8 = 1.514221191$	$y_8 = 1.799987792$	$y_8 = 1.921936035$
	$y_9 = 1.614334106$	$y_9 = 1.800003051$	$y_9 = 1.923171996$
	$y_{10} = 1.539249419$	$y_{10} = 1.799999236$	$y_{10} = 1.922863006$
	$y_{11} = 1.595562934$	$y_{11} = 1.80000019$	$y_{11} = 1.922940253$
$x = 1.5$	$y_1 = 2.5$	$y_1 = 2.5$	$y_1 = 2.5$
	$y_2 = 1.375$	$y_2 = 2.125$	$y_2 = 2.125$
	$y_3 = 2.21875$	$y_3 = 2.21875$	$y_3 = 2.40625$
	$y_4 = 1.5859375$	$y_4 = 2.1953125$	$y_4 = 2.3359375$
	$y_5 = 2.060546875$	$y_5 = 2.201171875$	$y_5 = 2.388671875$
	$y_6 = 1.704589843$	$y_6 = 2.199707031$	$y_6 = 2.375488281$
	$y_7 = 1.971557617$	$y_7 = 2.200073242$	$y_7 = 2.385375976$
	$y_8 = 1.771331786$	$y_8 = 2.199981688$	$y_8 = 2.382904053$
	$y_9 = 1.921501159$	$y_9 = 2.200004577$	$y_9 = 2.384757994$
	$y_{10} = 1.808874128$	$y_{10} = 2.199998854$	$y_{10} = 2.384294509$
	$y_{11} = 1.893344401$	$y_{11} = 2.200000285$	$y_{11} = 2.38441038$

$$y_3 = \frac{x^3}{3} + \frac{x^4}{12}\frac{x^5}{60},$$

$$y_4 = \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360}.$$

Now applying the new modified Ishikawa iteration method to the equation for $\lambda = 0.5$, $\gamma = 0.5$, then

$$y_1 = \frac{x^3}{3},$$

$$y_2 = \frac{x^3}{6},$$

$$y_3 = \frac{x^3}{4},$$

$$y_4 = 0.208333x^3,$$

$$y_5 = 0.2291666x^3,$$

$$y_6 = 0.218749983x^3,$$

$$y_7 = 0.223958321x^3,$$

Table 4 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	λ = 0.5, γ = 0.5	λ = 0.5, γ = 0.25	λ = 0.25, γ = 0.5	λ = 0.25, γ = 0.25
x = 0.6	$y_1 = 1.6$	$y_1 = 1.6$	$y_1 = 1.6$	$y_1 = 1.6$
	$y_2 = 1.3$	$y_2 = 1.15$	$y_2 = 1.3$	$y_2 = 1.15$
	$y_3 = 1.45$	$y_3 = 1.375$	$y_3 = 1.375$	$y_3 = 1.2625$
	$y_4 = 1.375$	$y_4 = 1.20625$	$y_4 = 1.3375$	$y_4 = 1.178125$
	$y_5 = 1.4125$	$y_5 = 1.290625$	$y_5 = 1.346875$	$y_5 = 1.19921875$
	$y_6 = 1.39375$	$y_6 = 1.22734375$	$y_6 = 1.3421875$	$y_6 = 1.183398437$
	$y_7 = 1.403125$	$y_7 = 1.258984375$	$y_7 = 1.343359375$	$y_7 = 1.187353515$
	$y_8 = 1.3984375$	$y_8 = 1.235253905$	$y_8 = 1.342773437$	$y_8 = 1.184387206$
	$y_9 = 1.40078125$	$y_9 = 1.24711914$	$y_9 = 1.342919921$	$y_9 = 1.185128783$
	$y_{10} = 1.399609375$	$y_{10} = 1.238220214$	$y_{10} = 1.342846679$	$y_{10} = 1.1845726$
	$y_{11} = 1.400195312$	$y_{11} = 1.242669677$	$y_{11} = 1.34286499$	$y_{11} = 1.184711646$
x = 1	$y_1 = 2$	$y_1 = 2$	$y_1 = 2$	$y_1 = 2$
	$y_2 = 1.5$	$y_2 = 1.25$	$y_2 = 1.5$	$y_2 = 1.25$
	$y_3 = 1.75$	$y_3 = 1.625$	$y_3 = 1.625$	$y_3 = 1.4375$
	$y_4 = 1.625$	$y_4 = 1.34375$	$y_4 = 1.5625$	$y_4 = 1.296875$
	$y_5 = 1.6875$	$y_5 = 1.484375$	$y_5 = 1.578125$	$y_5 = 1.33203125$
	$y_6 = 1.65625$	$y_6 = 1.37890625$	$y_6 = 1.5703125$	$y_6 = 1.305664062$
	$y_7 = 1.671875$	$y_7 = 1.431640625$	$y_7 = 1.572265625$	$y_7 = 1.312255859$
	$y_8 = 1.6640625$	$y_8 = 1.392089843$	$y_8 = 1.571289062$	$y_8 = 1.307312011$
	$y_9 = 1.66796875$	$y_9 = 1.411865234$	$y_9 = 1.571533203$	$y_9 = 1.308547973$
	$y_{10} = 1.666015625$	$y_{10} = 1.39703369$	$y_{10} = 1.571411132$	$y_{10} = 1.307621001$
	$y_{11} = 1.666992187$	$y_{11} = 1.404449462$	$y_{11} = 1.57144165$	$y_{11} = 1.307852744$
x = 1.5	$y_1 = 2.5$	$y_1 = 2.5$	$y_1 = 2.5$	$y_1 = 2.5$
	$y_2 = 1.75$	$y_2 = 1.375$	$y_2 = 1.75$	$y_2 = 1.375$
	$y_3 = 2.125$	$y_3 = 1.9375$	$y_3 = 1.9375$	$y_3 = 1.65625$
	$y_4 = 1.9375$	$y_4 = 1.515625$	$y_4 = 1.84375$	$y_4 = 1.4453125$
	$y_5 = 2.03125$	$y_5 = 1.7265625$	$y_5 = 1.8671875$	$y_5 = 1.498046875$
	$y_6 = 1.984375$	$y_6 = 1.568359375$	$y_6 = 1.85546875$	$y_6 = 1.458496093$
	$y_7 = 2.0078125$	$y_7 = 1.647460937$	$y_7 = 1.858398437$	$y_7 = 1.468383788$
	$y_8 = 1.99609375$	$y_8 = 1.588134764$	$y_8 = 1.856933593$	$y_8 = 1.460968016$
	$y_9 = 2.001953125$	$y_9 = 1.617797851$	$y_9 = 1.857299804$	$y_9 = 1.462821959$
	$y_{10} = 1.999023437$	$y_{10} = 1.595550535$	$y_{10} = 1.857116698$	$y_{10} = 1.461431501$
	$y_{11} = 2.000488281$	$y_{11} = 1.606674193$	$y_{11} = 1.857162475$	$y_{11} = 1.461779116$

Table 5 Absolute error of Example 2.1 for different values of λ and γ (x = 0.2, x = 0.4 and x = 0.6 respectively)

	x = 0.2	x = 0.4	x = 0.6
λ = 0.5, γ = 0.5	0.12398437	0.173203126	0.289804688
λ = 0.5, γ = 0.25	0.129110108	0.278220216	0.447330323
λ = 0.25, γ = 0.5	0.09571167	0.21142334	0.34713501
λ = 0.25, γ = 0.25	0.148429452	0.316858903	0.505288354
λ = 0.75, γ = 0.25	0.090887414	0.201774827	0.33266224
λ = 0.25, γ = 0.75	0.049999962	0.119999924	0.209999886
λ = 0.75, γ = 0.75	0.02541195	0.070823899	0.136235849
Picard	0.0003	0.002328	0.007369875
Runge-Kutta	0.000000435	0.000000081	0.00000046
Euler	0.01	0.020910977	0.032659931

$$y_8 = 0.221354152x^3,$$

$$y_9 = 0.222656236x^3,$$

$$y_{10} = 0.222005194x^3,$$

$$y_{11} = 0.222330715x^3$$

are found and also for $\lambda = 0.5$, $\gamma = 0.25$,

$$
\begin{aligned}
y_1 &= \frac{x^3}{3}, \\
y_2 &= 0.08333333x^3, \\
y_3 &= 0.208333316x^3, \\
y_4 &= 0.114583304x^3, \\
y_5 &= 0.16145831x^3, \\
y_6 &= 0.126302055x^3, \\
y_7 &= 0.143880182x^3, \\
y_8 &= 0.130696586x^3, \\
y_9 &= 0.137288384x^3, \\
y_{10} &= 0.132344535x^3, \\
y_{11} &= 0.134816459x^3
\end{aligned}
$$

are obtained. On the other hand, for $\lambda = 0.25$, $\gamma = 0.5$,

$$
\begin{aligned}
y_1 &= \frac{x^3}{3}, \\
y_2 &= \frac{x^3}{6}, \\
y_3 &= 0.208333333x^3, \\
y_4 &= 0.375x^3, \\
y_5 &= 0.192708333x^3, \\
y_6 &= 0.190104166x^3, \\
y_7 &= 0.190755207x^3, \\
y_8 &= 0.190429686x^3, \\
y_9 &= 0.190348306x^3, \\
y_{10} &= 0.190388996x^3, \\
y_{11} &= 0.190378823x^3
\end{aligned}
$$

are calculated. In the same way, for $\lambda = 0.25$, $\gamma = 0.25$,

$$
\begin{aligned}
y_1 &= \frac{x^3}{3}, \\
y_2 &= 0.083333333x^3, \\
y_3 &= 0.1458333333x^3, \\
y_4 &= 0.098958333x^3,
\end{aligned}
$$

$$y_5 = 0.110677083x^3,$$
$$y_6 = 0.10188802x^3,$$
$$y_7 = 0.104085285x^3,$$
$$y_8 = 0.102437336x^3,$$
$$y_9 = 0.102849323x^3,$$
$$y_{10} = 0.102540332x^3,$$
$$y_{11} = 0.10261758x^3$$

are found and also for $\lambda = 0.75$, $\gamma = 0.25$,

$$y_1 = \frac{x^3}{3},$$
$$y_2 = 0.083333333x^3,$$
$$y_3 = 0.270833333x^3,$$
$$y_4 = 0.130208333x^3,$$
$$y_5 = 0.235677083x^3,$$
$$y_6 = 0.15657552x^3,$$
$$y_7 = 0.215901692x^3,$$
$$y_8 = 0.171407063x^3,$$
$$y_9 = 0.204778034x^3,$$
$$y_{10} = 0.179749805x^3,$$
$$y_{11} = 0.198520977x^3$$

are obtained. Similarly, for $\lambda = 0.25$, $\gamma = 0.75$,

$$y_1 = \frac{x^3}{3},$$
$$y_2 = 0.25x^3,$$
$$y_3 = 0.270833333x^3,$$
$$y_4 = 0.265625x^3,$$
$$y_5 = 0.266927083x^3,$$
$$y_6 = 0.266601562x^3,$$
$$y_7 = 0.266682942x^3,$$
$$y_8 = 0.266662597x^3,$$
$$y_9 = 0.266667683x^3,$$
$$y_{10} = 0.266666411x^3,$$
$$y_{11} = 0.266666729x^3$$

are calculated. Finally, for $\lambda = 0.75$, $\gamma = 0.75$,

$$y_1 = \frac{x^3}{3},$$
$$y_2 = 0.25x^3,$$
$$y_3 = 0.3125x^3,$$
$$y_4 = 0.296875x^3,$$
$$y_5 = 0.30859375x^3,$$
$$y_6 = 0.305664062x^3,$$
$$y_7 = 0.307861328x^3,$$
$$y_8 = 0.307312011x^3,$$
$$y_9 = 0.307723998x^3,$$
$$y_{10} = 0.307621001x^3,$$
$$y_{11} = 0.307698249x^3$$

are found. Now, we get the approximate solution using by the Euler method. Firstly, we use the formula

$$y_{n+1} = y_n + hF(x_n, y_n)$$

with $F(x,y) = y + x^2$, $h = 0.2$ and $x_0 = 0$, $y_0 = 0$. From the initial condition $y(0) = 0$, we have $F(0,0) = 0$. We now proceed with the calculations as follows:

$$y_1 = y_0 + hF(y_0, x_0) = 0 + 0.0 = 0.0,$$
$$x_1 = x_0 + h = 0.0 + 0.2 = 0.2,$$
$$y_2 = y_1 + hF(y_1, x_1) = 0.0 + 0.2 \cdot 0.04 = 0.008,$$
$$x_2 = x_1 + h = 0.2 + 0.2 = 0.4,$$
$$y_3 = y_2 + hF(y_2, x_2) = 0.008 + 0.0336 = 0.0416,$$
$$x_3 = x_2 + h = 0.4 + 0.2 = 0.6.$$

Finally, applying the Runge-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after decimal point and round off the final results each at step to four such places. $F(x,y) = y + x^2$, $x_0 = 0$, $y_0 = 0$ and we are to use $h = 0.2$. Using these quantities, we calculated successively k_1, k_2, k_3, k_4 and K_0 defined by

$$k_1 = hg(y_0, x_0),$$
$$k_2 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right),$$

$$k_3 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right),$$

$$k_4 = hg(y_0 + h, x_0 + k_3)$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, $y_{n+1} = y_n + K_0$. Thus we find k_1, k_2, k_3, k_4 for $n = 0$ as follows:

$$k_1 = 0.2f(x_0, y_0) = 0,$$

$$k_2 = 0.2f\left(x_0 + 0.1, y_0 + \frac{k_1}{2}\right) = 0.002,$$

$$k_3 = 0.2f\left(x_0 + 0.1, y_0 + \frac{k_2}{2}\right) = 0.0022,$$

$$k_4 = 0.2f(x_0 + 0.2, y_0 + k_3) = 0.00844.$$

So, $y_1 = 0.0028066666666667$ is obtained for $x_1 = 0.2$. On the other hand, we calculated k_1, k_2, k_3, k_4 for $n = 1$ as follows:

$$k_1 = 0.2f(x_1, y_1) = 0.0085613333333333,$$

$$k_2 = 0.2f\left(x_1 + 0.1, y_1 + \frac{k_1}{2}\right) = 0.0194174666666667,$$

$$k_3 = 0.2f\left(x_1 + 0.1, y_1 + \frac{k_2}{2}\right) = 0.02050308,$$

$$k_4 = 0.2f(x_1 + 0.2, y_1 + k_3) = 0.0366619493333333.$$

Hence $y_2 = 0.0236507293333334$ is calculated for $x_2 = 0.4$. Finally, we get k_1, k_2, k_3, k_4 for $n = 2$ as follows:

$$k_1 = 0.2f(x_2, y_2) = 0.03673014586666668,$$

$$k_2 = 0.2f\left(x_2 + 0.1, y_2 + \frac{k_1}{2}\right) = 0.058403160453333348,$$

$$k_3 = 0.2f\left(x_2 + 0.1, y_2 + \frac{k_2}{2}\right) = 0.0605704619120000148,$$

$$k_4 = 0.2f(x_2 + 0.2, y_2 + k_3) = 0.08884423704906680296.$$

Thus $y_3 = 0.0842376672744001014266666666667$ is obtained for $x_3 = 0.6$.

After the necessary calculation which is done above, the comparison is shown schematically in Figure 2.

On the other hand, we may give Table 6, Table 7, Table 8 and Table 9 by the new modified Ishikawa iteration method for different values of λ and γ. Now we may give Table 10 which is expressed that absolute error of Example 2.2 for different values of λ and γ with $x = 0.2$, $x = 0.4$ and $x = 0.6$ respectively.

Corollary 2.3 *If we compare the approximate solution with the different values of* λ *and* γ, *then the conclusion may be indicated using Table* 6, *Table* 7, *Table* 8 *and Table* 9 *as follows*.

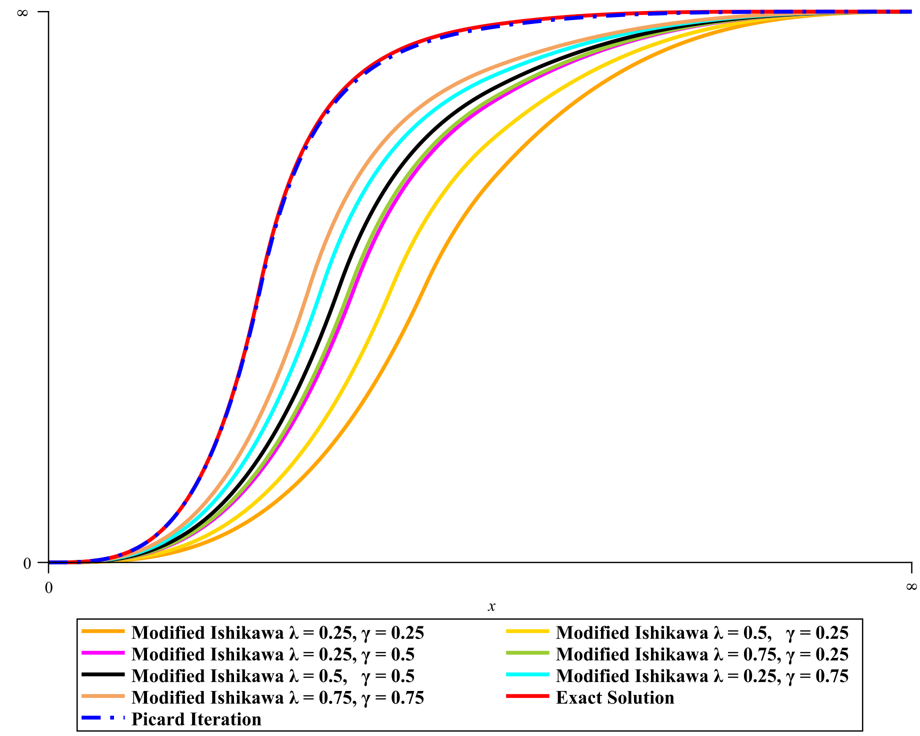

Figure 2 The comparison of exact solution and approximate solution of Example 2.2 for different values of λ and γ.

The best approximation is obtained taking the different values of λ *and* γ *and using the modified Ishikawa iteration method for* $x = 0.2$, $x = 0.4$ *and* $x = 0.5$ *getting* ($\lambda = 0.25$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.25$; $\lambda = 0.25$, $\gamma = 0.5$; $\lambda = 0.75$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.5$; $\lambda = 0.25$, $\gamma = 0.75$; $\lambda = 0.75$, $\gamma = 0.75$) *respectively.*

Similarly, we calculated the solution for $x = 0.6$, $x = 1$ *and* $x = 1.5$ *then the approximation is found more sensitive taking* ($\lambda = 0.25$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.25$; $\lambda = 0.25$, $\gamma = 0.5$; $\lambda = 0.75$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.5$; $\lambda = 0.25$, $\gamma = 0.75$; $\lambda = 0.75$, $\gamma = 0.75$) *respectively.*

Corollary 2.4 *Absolute error of the modified Ishikawa iteration method is computed taking different values of* λ *and* γ ($x = 0.2$, $x = 0.4$ *and* $x = 0.6$), *which is not more effective than Picard, Runge-Kutta and Euler iteration methods.*

Example 2.3 Let us consider the differential equation

$$y' = 2x(y+1)$$

subject to the initial condition

$$y(0) = 0.$$

Using Theorem 1.1 and Corollary 1.2, since $T = \int_{x_0}^{x} F(t, y_n(t))\,dt$, then T has a unique fixed point, which is the unique solution of the differential equation $y' = 2x(y+1)$ with the initial condition $y(0) = 0$.

Table 6 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	λ = 0.5, γ = 0.5	λ = 0.5, γ = 0.25	λ = 0.25, γ = 0.5	λ = 0.25, γ = 0.25
x = 0.2	$y_1 = 0.002666666667$	$y_1 = 0.002666666667$	$y_1 = 0.002666666667$	$y_1 = 0.002666666667$
	$y_2 = 0.0013333$	$y_2 = 0.00066666664$	$y_2 = 0.001333333333$	$y_2 = 0.000666666664$
	$y_3 = 0.002$	$y_3 = 0.00166666528$	$y_3 = 0.001666666664$	$y_3 = 0.001166666666$
	$y_4 = 0.001666664$	$y_4 = 0.00091666432$	$y_4 = 0.001499999992$	$y_4 = 0.000791666664$
	$y_5 = 0.0018333328$	$y_5 = 0.00129166648$	$y_5 = 0.001541666664$	$y_5 = 0.000885416664$
	$y_6 = 0.001749999864$	$y_6 = 0.00101041644$	$y_6 = 0.001520833328$	$y_6 = 0.00081510416$
	$y_7 = 0.001791666568$	$y_7 = 0.001151041456$	$y_7 = 0.001526041656$	$y_7 = 0.00083268228$
	$y_8 = 0.001770833216$	$y_8 = 0.001045572688$	$y_8 = 0.001523437488$	$y_8 = 0.000819498688$
	$y_9 = 0.001781249888$	$y_9 = 0.001098307072$	$y_9 = 0.001522786448$	$y_9 = 0.000822794584$
	$y_{10} = 0.001776041312$	$y_{10} = 0.00105875628$	$y_{10} = 0.001523111968$	$y_{10} = 0.000820322656$
	$y_{11} = 0.00177864572$	$y_{11} = 0.001078531672$	$y_{11} = 0.001523030584$	$y_{11} = 0.00082094064$
x = 0.4	$y_1 = 0.021333333$	$y_1 = 0.021333333$	$y_1 = 0.021333333$	$y_1 = 0.021333333$
	$y_2 = 0.010666666$	$y_2 = 0.00533333312$	$y_2 = 0.010666666$	$y_2 = 0.00533333312$
	$y_3 = 0.016$	$y_3 = 0.013333332$	$y_3 = 0.013333333$	$y_3 = 0.009333333331$
	$y_4 = 0.013333333$	$y_4 = 0.007333331456$	$y_4 = 0.011999999$	$y_4 = 0.006333333312$
	$y_5 = 0.014666662$	$y_5 = 0.010333333$	$y_5 = 0.012333333$	$y_5 = 0.007083333312$
	$y_6 = 0.013999998$	$y_6 = 0.00808333152$	$y_6 = 0.012166666$	$y_6 = 0.00652083328$
	$y_7 = 0.014333332$	$y_7 = 0.009208331648$	$y_7 = 0.012208333$	$y_7 = 0.00666145824$
	$y_8 = 0.014166665$	$y_8 = 0.008364581504$	$y_8 = 0.012187499$	$y_8 = 0.006555989504$
	$y_9 = 0.014249999$	$y_9 = 0.008786456576$	$y_9 = 0.012182291$	$y_9 = 0.006582356672$
	$y_{10} = 0.014208332$	$y_{10} = 0.00847005024$	$y_{10} = 0.012184895$	$y_{10} = 0.006562581248$
	$y_{11} = 0.014229165$	$y_{11} = 0.008628253376$	$y_{11} = 0.012184244$	$y_{11} = 0.00656752512$
x = 0.5	$y_1 = 0.041666666$	$y_1 = 0.041666666$	$y_1 = 0.041666666$	$y_1 = 0.041666666$
	$y_2 = 0.020833333$	$y_2 = 0.010416666$	$y_2 = 0.020833333$	$y_2 = 0.010416666$
	$y_3 = 0.03125$	$y_3 = 0.026041664$	$y_3 = 0.026041666$	$y_3 = 0.018229166$
	$y_4 = 0.026041625$	$y_4 = 0.014322913$	$y_4 = 0.023437499$	$y_4 = 0.012369791$
	$y_5 = 0.028645825$	$y_5 = 0.020182288$	$y_5 = 0.024088541$	$y_5 = 0.013834635$
	$y_6 = 0.027343747$	$y_6 = 0.015787756$	$y_6 = 0.02376302$	$y_6 = 0.012736025$
	$y_7 = 0.02799479$	$y_7 = 0.017985022$	$y_7 = 0.0238444$	$y_7 = 0.01301066$
	$y_8 = 0.027669269$	$y_8 = 0.016337073$	$y_8 = 0.02380371$	$y_8 = 0.012804667$
	$y_9 = 0.027832029$	$y_9 = 0.017161048$	$y_9 = 0.023793538$	$y_9 = 0.012856165$
	$y_{10} = 0.027750649$	$y_{10} = 0.016543066$	$y_{10} = 0.023798624$	$y_{10} = 0.012817541$
	$y_{11} = 0.027791339$	$y_{11} = 0.016852057$	$y_{11} = 0.023797352$	$y_{11} = 0.012827197$

Firstly, we obtained the exact solution of the equation as $y = e^{x^2} - 1$. Then we approach the approximate solution by the Picard iteration method as follows:

$$y_1 = x^2,$$

$$y_2 = x^2 + \frac{x^4}{2},$$

$$y_3 = x^2 + \frac{x^4}{2} + \frac{x^6}{6},$$

$$y_4 = x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}.$$

Now, applying the new modified Ishikawa iteration method to the equation for $\lambda = 0.5$, $\gamma = 0.5$, then

$$y_1 = x^2,$$

$$y_2 = 0.5x^2,$$

$$y_3 = 0.75x^2,$$

Table 7 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	λ = 0.75, γ = 0.25	λ = 0.25, γ = 0.75	λ = 0.75, γ = 0.75
x = 0.2	y_1 = 0.002666666667	y_1 = 0.002666666667	y_1 = 0.002666666667
	y_2 = 0.000666666664	y_2 = 0.002	y_2 = 0.002
	y_3 = 0.002166666664	y_3 = 0.002166666664	y_3 = 0.0025
	y_4 = 0.001041666664	y_4 = 0.002125	y_4 = 0.002375
	y_5 = 0.001885416664	y_5 = 0.002135416664	y_5 = 0.00246875
	y_6 = 0.00125260416	y_6 = 0.002132812496	y_6 = 0.002445312496
	y_7 = 0.001727213536	y_7 = 0.002133463536	y_7 = 0.002462890624
	y_8 = 0.001371256504	y_8 = 0.002133300776	y_8 = 0.002458496088
	y_9 = 0.001638224272	y_9 = 0.002133341464	y_9 = 0.002461791984
	y_{10} = 0.00143799844	y_{10} = 0.002133331288	y_{10} = 0.002406096801
	y_{11} = 0.001588167816	y_{11} = 0.002133333832	y_{11} = 0.002461585992
x = 0.4	y_1 = 0.021333333	y_1 = 0.021333333	y_1 = 0.021333333
	y_2 = 0.005333333312	y_2 = 0.016	y_2 = 0.016
	y_3 = 0.017333333	y_3 = 0.017333333	y_3 = 0.02
	y_4 = 0.008333333312	y_4 = 0.017	y_4 = 0.019
	y_5 = 0.015083333	y_5 = 0.017083333	y_5 = 0.01975
	y_6 = 0.010020833	y_6 = 0.017062499	y_6 = 0.019562499
	y_7 = 0.0138177708	y_7 = 0.017067708	y_7 = 0.019703124
	y_8 = 0.010970052	y_8 = 0.017066406	y_8 = 0.019667968
	y_9 = 0.013105794	y_9 = 0.017066731	y_9 = 0.019694335
	y_{10} = 0.011503987	y_{10} = 0.01706665	y_{10} = 0.019687744
	y_{11} = 0.012705342	y_{11} = 0.01706667	y_{11} = 0.019692687
x = 0.5	y_1 = 0.041666666	y_1 = 0.041666666	y_1 = 0.041666666
	y_2 = 0.010416666	y_2 = 0.03125	y_2 = 0.03125
	y_3 = 0.033854166	y_3 = 0.033854166	y_3 = 0.0390625
	y_4 = 0.016276041	y_4 = 0.033203125	y_4 = 0.037109375
	y_5 = 0.029459635	y_5 = 0.033365885	y_5 = 0.038574218
	y_6 = 0.01957194	y_6 = 0.033325195	y_6 = 0.038208007
	y_7 = 0.026987711	y_7 = 0.033335367	y_7 = 0.038482666
	y_8 = 0.021425882	y_8 = 0.033335367	y_8 = 0.038414001
	y_9 = 0.025597254	y_9 = 0.03333346	y_9 = 0.038465499
	y_{10} = 0.022468725	y_{10} = 0.033333301	y_{10} = 0.038452625
	y_{11} = 0.024815122	y_{11} = 0.033333341	y_{11} = 0.038462281

$$y_4 = 0.625x^2,$$
$$y_5 = 0.6875x^2,$$
$$y_6 = 0.65625x^2,$$
$$y_7 = 0.671875x^2,$$
$$y_8 = 0.6640625x^2,$$
$$y_9 = 0.66796875x^2,$$
$$y_{10} = 0.666015625x^2,$$
$$y_{11} = 0.666992687x^2$$

are found and also for $\lambda = 0.5$, $\gamma = 0.25$,

$$y_1 = x^2,$$
$$y_2 = 0.25x^2,$$
$$y_3 = 0.625x^2,$$

Table 8 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	$\lambda = 0.5, \gamma = 0.5$	$\lambda = 0.5, \gamma = 0.25$	$\lambda = 0.25, \gamma = 0.5$	$\lambda = 0.25, \gamma = 0.25$
$x = 0.6$	$y_1 = 0.072$	$y_1 = 0.072$	$y_1 = 0.072$	$y_1 = 0.072$
	$y_2 = 0.036$	$y_2 = 0.017999992$	$y_2 = 0.036$	$y_2 = 0.017999999$
	$y_3 = 0.054$	$y_3 = 0.044999996$	$y_3 = 0.044999999$	$y_3 = 0.031499999$
	$y_4 = 0.044999928$	$y_4 = 0.024749993$	$y_4 = 0.040499999$	$y_4 = 0.021374999$
	$y_5 = 0.049499985$	$y_5 = 0.034874994$	$y_5 = 0.041624999$	$y_5 = 0.023906249$
	$y_6 = 0.047249996$	$y_6 = 0.027281243$	$y_6 = 0.04062499$	$y_6 = 0.022007812$
	$y_7 = 0.048376997$	$y_7 = 0.031078119$	$y_7 = 0.041203124$	$y_7 = 0.022482421$
	$y_8 = 0.047812496$	$y_8 = 0.028230462$	$y_8 = 0.041132812$	$y_8 = 0.022126464$
	$y_9 = 0.048093746$	$y_9 = 0.02965429$	$y_9 = 0.041115234$	$y_9 = 0.022215453$
	$y_{10} = 0.047953121$	$y_{10} = 0.028586419$	$y_{10} = 0.041124023$	$y_{10} = 0.022148711$
	$y_{11} = 0.048023434$	$y_{11} = 0.029120355$	$y_{11} = 0.041121825$	$y_{11} = 0.022165397$
$x = 1$	$y_1 = 0.333333$	$y_1 = 0.333333$	$y_1 = 0.333333$	$y_1 = 0.333333$
	$y_2 = 0.1666666$	$y_2 = 0.08333333$	$y_2 = 0.1666666$	$y_2 = 0.083333333$
	$y_3 = 0.25$	$y_3 = 0.208333316$	$y_3 = 0.20833333$	$y_3 = 0.1458333333$
	$y_4 = 0.208333$	$y_4 = 0.114583304$	$y_4 = 0.187499999$	$y_4 = 0.098958333$
	$y_5 = 0.2291666$	$y_5 = 0.16145831$	$y_5 = 0.192708333$	$y_5 = 0.110677083$
	$y_6 = 0.218749983$	$y_6 = 0.126302055$	$y_6 = 0.190104166$	$y_6 = 0.10188802$
	$y_7 = 0.223958321$	$y_7 = 0.143880182$	$y_7 = 0.190755207$	$y_7 = 0.104085285$
	$y_8 = 0.221354152$	$y_8 = 0.130696586$	$y_8 = 0.190429686$	$y_8 = 0.102437336$
	$y_9 = 0.222656236$	$y_9 = 0.137288384$	$y_9 = 0.190348306$	$y_9 = 0.102849323$
	$y_{10} = 0.222005194$	$y_{10} = 0.132344535$	$y_{10} = 0.190388996$	$y_{10} = 0.102540332$
	$y_{11} = 0.222330715$	$y_{11} = 0.134816459$	$y_{11} = 0.190378823$	$y_{11} = 0.10261758$
$x = 1.5$	$y_1 = 1.125$	$y_1 = 1.125$	$y_1 = 1.125$	$y_1 = 1.125$
	$y_2 = 0.5625$	$y_2 = 0.2812499888$	$y_2 = 0.5625$	$y_2 = 0.281249998$
	$y_3 = 0.84375$	$y_3 = 0.703124941$	$y_3 = 0.703124998$	$y_3 = 0.492187499$
	$y_4 = 0.703123875$	$y_4 = 0.38654901$	$y_4 = 0.632812496$	$y_4 = 0.333984373$
	$y_5 = 0.773437275$	$y_5 = 0.544921796$	$y_5 = 0.650390623$	$y_5 = 0.373535155$
	$y_6 = 0.738281192$	$y_6 = 0.426269435$	$y_6 = 0.64160156$	$y_6 = 0.343872067$
	$y_7 = 0.755859333$	$y_7 = 0.485595614$	$y_7 = 0.643798823$	$y_7 = 0.351287836$
	$y_8 = 0.747070263$	$y_8 = 0.441100977$	$y_8 = 0.64270019$	$y_8 = 0.345726009$
	$y_9 = 0.751464796$	$y_9 = 0.463348296$	$y_9 = 0.642425532$	$y_9 = 0.347116465$
	$y_{10} = 0.749267529$	$y_{10} = 0.446662805$	$y_{10} = 0.64256274$	$y_{10} = 0.34607362$
	$y_{11} = 0.750366163$	$y_{11} = 0.455005549$	$y_{11} = 0.642528527$	$y_{11} = 0.346334332$

$$y_4 = 0.34375x^2,$$

$$y_5 = 0.484375x^2,$$

$$y_6 = 0.37890625x^2,$$

$$y_7 = 0.431640625x^2,$$

$$y_8 = 0.392089843x^2,$$

$$y_9 = 0.411865234x^2,$$

$$y_{10} = 0.39703369x^2,$$

$$y_{11} = 0.404449462x^2$$

are obtained. On the other hand, for $\lambda = 0.25$, $\gamma = 0.5$,

$$y_1 = x^2,$$

$$y_2 = 0.5x^2,$$

$$y_3 = 0.625x^2,$$

Table 9 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	$\lambda = 0.75, \gamma = 0.25$	$\lambda = 0.25, \gamma = 0.75$	$\lambda = 0.75, \gamma = 0.75$
$x = 0.6$	$y_1 = 0.072$	$y_1 = 0.072$	$y_1 = 0.072$
	$y_2 = 0.017999999$	$y_2 = 0.054$	$y_2 = 0.054$
	$y_3 = 0.058499999$	$y_3 = 0.058499999$	$y_3 = 0.0675$
	$y_4 = 0.028124999$	$y_4 = 0.057375$	$y_4 = 0.064125$
	$y_5 = 0.050906249$	$y_5 = 0.057656249$	$y_5 = 0.06665625$
	$y_6 = 0.033820312$	$y_6 = 0.057585937$	$y_6 = 0.066023437$
	$y_7 = 0.046634765$	$y_7 = 0.057603515$	$y_7 = 0.066498046$
	$y_8 = 0.037023925$	$y_8 = 0.05759912$	$y_8 = 0.066379394$
	$y_9 = 0.044232055$	$y_9 = 0.057600219$	$y_9 = 0.066468383$
	$y_{10} = 0.038825957$	$y_{10} = 0.057599944$	$y_{10} = 0.066446136$
	$y_{11} = 0.042880531$	$y_{11} = 0.057600013$	$y_{11} = 0.066462821$
$x = 1$	$y_1 = 0.333333$	$y_1 = 0.333333$	$y_1 = 0.333333$
	$y_2 = 0.083333333$	$y_2 = 0.25$	$y_2 = 0.25$
	$y_3 = 0.2708333$	$y_3 = 0.270833333$	$y_3 = 0.3125$
	$y_4 = 0.130208333$	$y_4 = 0.265625$	$y_4 = 0.296875$
	$y_5 = 0.235677083$	$y_5 = 0.266927083$	$y_5 = 0.30859375$
	$y_6 = 0.15657552$	$y_6 = 0.266601562$	$y_6 = 0.305664062$
	$y_7 = 0.215901692$	$y_7 = 0.266682942$	$y_7 = 0.307861328$
	$y_8 = 0.171407063$	$y_8 = 0.266662597$	$y_8 = 0.307312011$
	$y_9 = 0.204778034$	$y_9 = 0.266667683$	$y_9 = 0.307723998$
	$y_{10} = 0.179749805$	$y_{10} = 0.266666411$	$y_{10} = 0.307621001$
	$y_{11} = 0.198520977$	$y_{11} = 0.266666729$	$y_{11} = 0.307698249$
$x = 1.5$	$y_1 = 1.175$	$y_1 = 1.125$	$y_1 = 1.125$
	$y_2 = 0.281249998$	$y_2 = 0.84375$	$y_2 = 0.84375$
	$y_3 = 0.914062498$	$y_3 = 0.914062498$	$y_3 = 1.0546875$
	$y_4 = 0.439453123$	$y_4 = 0.896484375$	$y_4 = 1.001953125$
	$y_5 = 0.795410155$	$y_5 = 0.900878905$	$y_5 = 1.041503906$
	$y_6 = 0.52844238$	$y_6 = 0.899780271$	$y_6 = 1.031616209$
	$y_7 = 0.72866821$	$y_7 = 0.900054929$	$y_7 = 1.039031982$
	$y_8 = 0.578498837$	$y_8 = 0.899986264$	$y_8 = 1.037178037$
	$y_9 = 0.691125864$	$y_9 = 0.90000343$	$y_9 = 1.038568493$
	$y_{10} = 0.606655591$	$y_{10} = 0.899999137$	$y_{10} = 1.038220878$
	$y_{11} = 0.670008297$	$y_{11} = 0.90000021$	$y_{11} = 1.03848159$

Table 10 Absolute error of Example 2.2 for different values of λ and γ (x = 0.4 and x = 0.6 respectively)

	$x = 0.2$	$x = 0.4$	$x = 0.6$
$\lambda = 0.5, \gamma = 0.5$	0.0010268706	0.00942023	0.036214166
$\lambda = 0.5, \gamma = 0.25$	0.001726984648	0.015021141	0.055117245
$\lambda = 0.25, \gamma = 0.5$	0.001282485736	0.011465151	0.043115775
$\lambda = 0.25, \gamma = 0.25$	0.00198457568	0.017081869	0.017774779
$\lambda = 0.75, \gamma = 0.25$	0.001217348504	0.010944053	0.041357069
$\lambda = 0.25, \gamma = 0.75$	0.000672182488	0.006582725	0.026637587
$\lambda = 0.75, \gamma = 0.75$	0.000343930328	0.003956708	0.017774779
Picard	0.00000000522	0.000000684	0.000012
Runge-Kutta	0.000998849654	0.001220066	0.001491752
Euler	0.00280551632	0.015649395	0.0426376

$$y_4 = 0.5625x^2,$$

$$y_5 = 0.578125x^2,$$

$$y_6 = 0.5703125x^2,$$

$$y_7 = 0.572265625x^2,$$

$$y_8 = 0.571289062x^2,$$

$$y_9 = 0.571533203x^2,$$
$$y_{10} = 0.571411132x^2,$$
$$y_{11} = 0.57144165x^2$$

are calculated. In the same way, for $\lambda = 0.25$, $\gamma = 0.25$,

$$y_1 = x^2,$$
$$y_2 = 0.25x^2,$$
$$y_3 = 0.4375x^2,$$
$$y_4 = 0.296875x^2,$$
$$y_5 = 0.33203125x^2,$$
$$y_6 = 0.305664062x^2,$$
$$y_7 = 0.312255859x^2,$$
$$y_8 = 0.307312011x^2,$$
$$y_9 = 0.308547973x^2,$$
$$y_{10} = 0.307621001x^2,$$
$$y_{11} = 0.307852994x^2$$

are found and also for $\lambda = 0.75$, $\gamma = 0.25$,

$$y_1 = x^2,$$
$$y_2 = 0.25x^2,$$
$$y_3 = 0.8125x^2,$$
$$y_4 = 0.390625x^2,$$
$$y_5 = 0.70703125x^2,$$
$$y_6 = 0.469726562x^2,$$
$$y_7 = 0.647705078x^2,$$
$$y_8 = 0.514221191x^2,$$
$$y_9 = 0.614334106x^2,$$
$$y_{10} = 0.539249419x^2,$$
$$y_{11} = 0.595562934x^2$$

are obtained. Similarly, for $\lambda = 0.25$, $\gamma = 0.75$,

$$y_1 = x^2,$$
$$y_2 = 0.75x^2,$$
$$y_3 = 0.8125x^2,$$

$$y_4 = 0.796875x^2,$$
$$y_5 = 0.80078125x^2,$$
$$y_6 = 0.799804687x^2,$$
$$y_7 = 0.800048828x^2,$$
$$y_8 = 0.799987792x^2,$$
$$y_9 = 0.800003051x^2,$$
$$y_{10} = 0.799999236x^2,$$
$$y_{11} = 0.80000019x^2$$

are calculated. Finally, for $\lambda = 0.75$, $\gamma = 0.75$,

$$y_1 = x^2,$$
$$y_2 = 0.75x^2,$$
$$y_3 = 0.9375x^2,$$
$$y_4 = 0.890625x^2,$$
$$y_5 = 0.92578125x^2,$$
$$y_6 = 0.916992187x^2,$$
$$y_7 = 0.923583984x^2,$$
$$y_8 = 0.921936035x^2,$$
$$y_9 = 0.923171996x^2,$$
$$y_{10} = 0.922863006x^2,$$
$$y_{11} = 0.923094748x^2$$

are found. Now, we find the approximate solution using by the Euler method. Firstly, we use the formula

$$y_{n+1} = y_n + hF(x_n, y_n)$$

with $F(x,y) = 2x(y+1)$, $h = 0.2$ and $x_0 = 0$, $y_0 = 0$. From the initial condition $y(0) = 0$, we have $F(0,0) = 0$. We now proceed with the calculations as follows:

$$y_1 = y_0 + hf(x_0, y_0) = 0,$$
$$x_1 = x_0 + h = 0.2,$$
$$y_2 = y_1 + hf(x_1, y_1) = 0.08,$$
$$x_2 = x_1 + h = 0.4,$$
$$y_3 = y_2 + hf(x_2, y_2) = 0.2528,$$
$$x_3 = x_2 + h = 0.6.$$

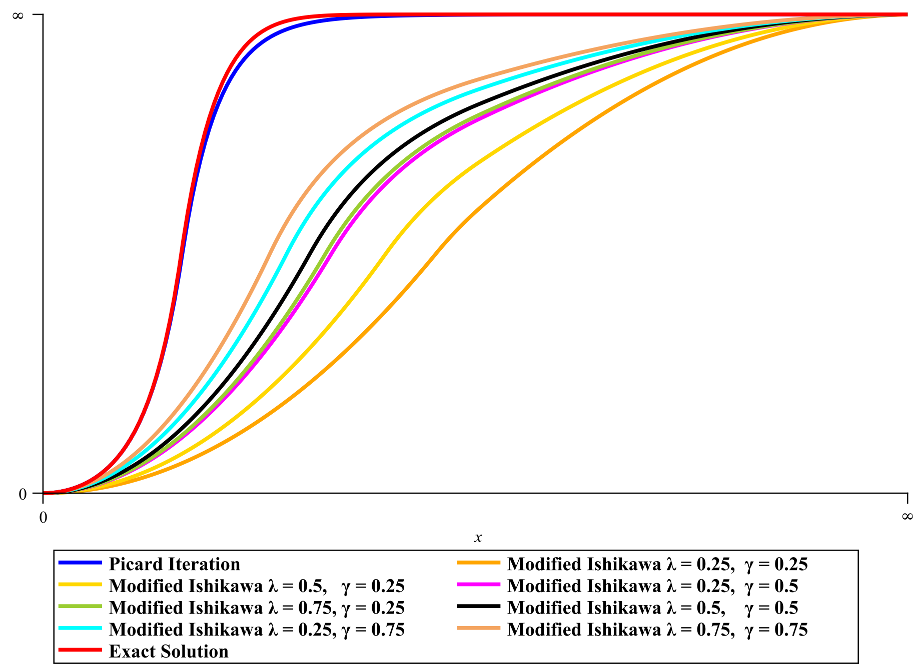

Figure 3 The comparison of exact solution and approximate solution of Example 2.3for different values of λ and γ.

Finally, applying the Runge-Kutta method to the given initial value problem, we carry out the intermediate calculations in each step to give figures after the decimal point and round off the final results at each step to four such places. Here $F(x,y) = 2x(y+1)$, $x_0 = 0$, $y_0 = 0$ and we are to use $h = 0.2$. Using these quantities, we calculated successively k_1, k_2, k_3, k_4 and K_0 defined by

$$k_1 = hg(y_0, x_0),$$

$$k_2 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right),$$

$$k_3 = hg\left(y_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right),$$

$$k_4 = hg(y_0 + h, x_0 + k_3)$$

and $K_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, $y_{n+1} = y_n + K_0$. Thus we find k_1, k_2, k_3, k_4 for $n = 0$ as follows:

$$k_1 = 0.2f(x_0, y_0) = 0,$$

$$k_2 = 0.2f\left(x_0 + 0.1, y_0 + \frac{k_1}{2}\right) = 0.04,$$

$$k_3 = 0.2f\left(x_0 + 0.1, y_0 + \frac{k_2}{2}\right) = 0.0408,$$

$$k_4 = 0.2f(x_0 + 0.2, y_0 + k_3) = 0.083264.$$

Table 11 The solutions obtained by the new modified Ishikawa iteration method for differential values of λ and γ

x	$\lambda = 0.5, \gamma = 0.5$	$\lambda = 0.5, \gamma = 0.25$	$\lambda = 0.25, \gamma = 0.5$	$\lambda = 0.25, \gamma = 0.25$
$x = 0.2$	$y_1 = 0.04$	$y_1 = 0.04$	$y_1 = 0.04$	$y_1 = 0.04$
	$y_2 = 0.02$	$y_2 = 0.01$	$y_2 = 0.02$	$y_2 = 0.01$
	$y_3 = 0.03$	$y_3 = 0.025$	$y_3 = 0.025$	$y_3 = 0.0175$
	$y_4 = 0.025$	$y_4 = 0.01375$	$y_4 = 0.0225$	$y_4 = 0.011875$
	$y_5 = 0.0275$	$y_5 = 0.019375$	$y_5 = 0.023125$	$y_5 = 0.01328125$
	$y_6 = 0.02625$	$y_6 = 0.01515625$	$y_6 = 0.0228125$	$y_6 = 0.012226562$
	$y_7 = 0.026875$	$y_7 = 0.017265625$	$y_7 = 0.022890625$	$y_7 = 0.012490234$
	$y_8 = 0.0265625$	$y_8 = 0.015683593$	$y_8 = 0.022851562$	$y_8 = 0.01229248$
	$y_9 = 0.02671875$	$y_9 = 0.016474609$	$y_9 = 0.022861328$	$y_9 = 0.012341918$
	$y_{10} = 0.026640625$	$y_{10} = 0.015881347$	$y_{10} = 0.022856445$	$y_{10} = 0.01230484$
	$y_{11} = 0.026679707$	$y_{11} = 0.016177978$	$y_{11} = 0.022857666$	$y_{11} = 0.012314119$
$x = 0.4$	$y_1 = 0.16$	$y_1 = 0.16$	$y_1 = 0.16$	$y_1 = 0.16$
	$y_2 = 0.08$	$y_2 = 0.04$	$y_2 = 0.08$	$y_2 = 0.04$
	$y_3 = 0.12$	$y_3 = 0.1$	$y_3 = 0.1$	$y_3 = 0.07$
	$y_4 = 0.1$	$y_4 = 0.055$	$y_4 = 0.09$	$y_4 = 0.0475$
	$y_5 = 0.11$	$y_5 = 0.0775$	$y_5 = 0.0925$	$y_5 = 0.053125$
	$y_6 = 0.105$	$y_6 = 0.060625$	$y_6 = 0.09125$	$y_6 = 0.048906249$
	$y_7 = 0.1075$	$y_7 = 0.0690625$	$y_7 = 0.0915625$	$y_7 = 0.049960937$
	$y_8 = 0.10625$	$y_8 = 0.062734374$	$y_8 = 0.091406249$	$y_8 = 0.049169921$
	$y_9 = 0.106875$	$y_9 = 0.065898437$	$y_9 = 0.091445312$	$y_9 = 0.049367675$
	$y_{10} = 0.1065625$	$y_{10} = 0.06352539$	$y_{10} = 0.091445312$	$y_{10} = 0.04921936$
	$y_{11} = 0.106718829$	$y_{11} = 0.064711913$	$y_{11} = 0.091430664$	$y_{11} = 0.049256479$
$x = 0.5$	$y_1 = 0.25$	$y_1 = 0.25$	$y_1 = 0.25$	$y_1 = 0.25$
	$y_2 = 0.125$	$y_2 = 0.0625$	$y_2 = 0.125$	$y_2 = 0.0625$
	$y_3 = 0.1875$	$y_3 = 0.15625$	$y_3 = 0.15625$	$y_3 = 0.109375$
	$y_4 = 0.15625$	$y_4 = 0.0859375$	$y_4 = 0.140625$	$y_4 = 0.07421875$
	$y_5 = 0.171875$	$y_5 = 0.12109375$	$y_5 = 0.14453125$	$y_5 = 0.083007812$
	$y_6 = 0.1640625$	$y_6 = 0.094726562$	$y_6 = 0.142578125$	$y_6 = 0.076416015$
	$y_7 = 0.16796875$	$y_7 = 0.107910156$	$y_7 = 0.143066406$	$y_7 = 0.078063964$
	$y_8 = 0.166015625$	$y_8 = 0.09802246$	$y_8 = 0.142822265$	$y_8 = 0.076828002$
	$y_9 = 0.166992187$	$y_9 = 0.102966308$	$y_9 = 0.1428833$	$y_9 = 0.077136993$
	$y_{10} = 0.166503906$	$y_{10} = 0.099258422$	$y_{10} = 0.142852783$	$y_{10} = 0.07690525$
	$y_{11} = 0.166748171$	$y_{11} = 0.101112365$	$y_{11} = 0.142860412$	$y_{11} = 0.076963248$

So, $y_1 = 0.040810666$ is obtained for $x_1 = 0.2$. On the other hand, we calculated k_1, k_2, k_3, k_4 for $n = 1$ as follows:

$$k_1 = 0.2f(x_1, y_1) = 0.083264853,$$

$$k_2 = 0.2f\left(x_1 + 0.1, y_1 + \frac{k_1}{2}\right) = 0.129893171,$$

$$k_3 = 0.2f\left(x_1 + 0.1, y_1 + \frac{k_2}{2}\right) = 0.13269087,$$

$$k_4 = 0.2f(x_1 + 0.2, y_1 + k_3) = 0.187760245.$$

Hence $y_2 = 0.173509529$ is calculated for $x_2 = 0.4$. Finally we get k_1, k_2, k_3, k_4 for $n = 2$ as follows:

$$k_1 = 0.2f(x_2, y_2) = 0.187761524,$$

$$k_2 = 0.2f\left(x_2 + 0.1, y_2 + \frac{k_1}{2}\right) = 0.253478058,$$

Table 12 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	λ = 0.75, γ = 0.25	λ = 0.25, γ = 0.75	λ = 0.75, γ = 0.75
x = 0.2	$y_1 = 0.04$	$y_1 = 0.04$	$y_1 = 0.04$
	$y_2 = 0.01$	$y_2 = 0.03$	$y_2 = 0.03$
	$y_3 = 0.0325$	$y_3 = 0.0325$	$y_3 = 0.0375$
	$y_4 = 0.015625$	$y_4 = 0.031875$	$y_4 = 0.035625$
	$y_5 = 0.0282925$	$y_5 = 0.03203125$	$y_5 = 0.03703125$
	$y_6 = 0.018789062$	$y_6 = 0.031992187$	$y_6 = 0.036679687$
	$y_7 = 0.025908203$	$y_7 = 0.032001953$	$y_7 = 0.036943359$
	$y_8 = 0.020568847$	$y_8 = 0.031999511$	$y_8 = 0.036877441$
	$y_9 = 0.024573364$	$y_9 = 0.032000122$	$y_9 = 0.036926879$
	$y_{10} = 0.021569976$	$y_{10} = 0.031999969$	$y_{10} = 0.03691452$
	$y_{11} = 0.023822517$	$y_{11} = 0.032000007$	$y_{11} = 0.036923789$
x = 0.4	$y_1 = 0.16$	$y_1 = 0.16$	$y_1 = 0.16$
	$y_2 = 0.04$	$y_2 = 0.12$	$y_2 = 0.12$
	$y_3 = 0.13$	$y_3 = 0.13$	$y_3 = 0.15$
	$y_4 = 0.0625$	$y_4 = 0.1275$	$y_4 = 0.1425$
	$y_5 = 0.113125$	$y_5 = 0.128125$	$y_5 = 0.148125$
	$y_6 = 0.075156249$	$y_6 = 0.127968747$	$y_6 = 0.146717749$
	$y_7 = 0.103632812$	$y_7 = 0.128007812$	$y_7 = 0.147773437$
	$y_8 = 0.08227539$	$y_8 = 0.127998046$	$y_8 = 0.147509765$
	$y_9 = 0.098293456$	$y_9 = 0.128000488$	$y_9 = 0.147707519$
	$y_{10} = 0.086279907$	$y_{10} = 0.127999877$	$y_{10} = 0.147658081$
	$y_{11} = 0.095290069$	$y_{11} = 0.128000488$	$y_{11} = 0.147695159$
x = 0.5	$y_1 = 0.25$	$y_1 = 0.25$	$y_1 = 0.25$
	$y_2 = 0.0625$	$y_2 = 0.1875$	$y_2 = 0.1875$
	$y_3 = 0.203125$	$y_3 = 0.203125$	$y_3 = 0.234375$
	$y_4 = 0.09765625$	$y_4 = 0.19921875$	$y_4 = 0.22265625$
	$y_5 = 0.176757812$	$y_5 = 0.200195312$	$y_5 = 0.231445312$
	$y_6 = 0.11743164$	$y_6 = 0.199951171$	$y_6 = 0.229248046$
	$y_7 = 0.161926269$	$y_7 = 0.200012207$	$y_7 = 0.230895996$
	$y_8 = 0.128555297$	$y_8 = 0.199996949$	$y_8 = 0.230484008$
	$y_9 = 0.153583526$	$y_9 = 0.200000762$	$y_9 = 0.230792999$
	$y_{10} = 0.134812354$	$y_{10} = 0.199999809$	$y_{10} = 0.230715751$
	$y_{11} = 0.148890733$	$y_{11} = 0.200000047$	$y_{11} = 0.230773687$

$$k_3 = 0.2f\left(x_2 + 0.1, y_2 + \frac{k_2}{2}\right) = 0.260049711,$$

$$k_4 = 0.2f(x_2 + 0.2, y_2 + k_3) = 0.344054217.$$

Thus $y_3 = 0.433321409$ is obtained for $x_3 = 0.6$.

After the necessary calculation which is done above, the comparison is shown schematically in Figure 3.

On the other hand, we may give Table 11, Table 12, Table 13 and Table 14 by the new modified Ishikawa iteration method for different values of λ and γ. Now we may give Table 15 which is expressed that absolute error of Example 2.3 for different values of λ and γ with $x = 0.2$, $x = 0.4$ and $x = 0.6$ respectively.

Corollary 2.5 *If we compare the approximate solution with the different values of λ and γ, then the conclusion may be indicated using by Table* 11, *Table* 12, *Table* 13 *and Table* 14 *as follows.*

The best approximation is obtained taking the different values of λ and γ and using the modified Ishikawa iteration method for $x = 0.2$, $x = 0.4$ *and* $x = 0.5$ *getting* (λ = 0.25, γ =

Table 13 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	$\lambda = 0.5, \gamma = 0.5$	$\lambda = 0.5, \gamma = 0.25$	$\lambda = 0.25, \gamma = 0.5$	$\lambda = 0.25, \gamma = 0.25$
$x = 0.6$	$y_1 = 0.36$	$y_1 = 0.36$	$y_1 = 0.36$	$y_1 = 0.36$
	$y_2 = 0.18$	$y_2 = 0.09$	$y_2 = 0.18$	$y_2 = 0.09$
	$y_3 = 0.27$	$y_3 = 0.225$	$y_3 = 0.225$	$y_3 = 0.1575$
	$y_4 = 0.225$	$y_4 = 0.12375$	$y_4 = 0.2025$	$y_4 = 0.106875$
	$y_5 = 0.2475$	$y_5 = 0.174375$	$y_5 = 0.208125$	$y_5 = 0.11953125$
	$y_6 = 0.23625$	$y_6 = 0.13640625$	$y_6 = 0.2053125$	$y_6 = 0.110039062$
	$y_7 = 0.241875$	$y_7 = 0.155390625$	$y_7 = 0.206015625$	$y_7 = 0.112412109$
	$y_8 = 0.2390625$	$y_8 = 0.141152343$	$y_8 = 0.205664062$	$y_8 = 0.110632324$
	$y_9 = 0.24046875$	$y_9 = 0.148271484$	$y_9 = 0.205751953$	$y_9 = 0.11107727$
	$y_{10} = 0.239765625$	$y_{10} = 0.142932128$	$y_{10} = 0.205708007$	$y_{10} = 0.11074356$
	$y_{11} = 0.240117361$	$y_{11} = 0.145601806$	$y_{11} = 0.205718994$	$y_{11} = 0.110827077$
$x = 1$	$y_1 = 1$	$y_1 = 1$	$y_1 = 1$	$y_1 = 1$
	$y_2 = 0.5$	$y_2 = 0.25$	$y_2 = 0.5$	$y_2 = 0.25$
	$y_3 = 0.75$	$y_3 = 0.625$	$y_3 = 0.625$	$y_3 = 0.4375$
	$y_4 = 0.625$	$y_4 = 0.34375$	$y_4 = 0.5625$	$y_4 = 0.296875$
	$y_5 = 0.6875$	$y_5 = 0.484375$	$y_5 = 0.578125$	$y_5 = 0.33203125$
	$y_6 = 0.65625$	$y_6 = 0.37890625$	$y_6 = 0.5703125$	$y_6 = 0.305664062$
	$y_7 = 0.671875$	$y_7 = 0.431640625$	$y_7 = 0.572265625$	$y_7 = 0.312255859$
	$y_8 = 0.6640625$	$y_8 = 0.392089843$	$y_8 = 0.571289062$	$y_8 = 0.307312011$
	$y_9 = 0.66796875$	$y_9 = 0.411865234$	$y_9 = 0.571533203$	$y_9 = 0.308547973$
	$y_{10} = 0.666015625$	$y_{10} = 0.39703369$	$y_{10} = 0.571411132$	$y_{10} = 0.307621001$
	$y_{11} = 0.666992687$	$y_{11} = 0.404449462$	$y_{11} = 0.57144165$	$y_{11} = 0.307852994$
$x = 1.5$	$y_1 = 2.25$	$y_1 = 2.25$	$y_1 = 2.25$	$y_1 = 2.25$
	$y_2 = 1.125$	$y_2 = 0.5625$	$y_2 = 1.125$	$y_2 = 0.5625$
	$y_3 = 1.6875$	$y_3 = 1.40625$	$y_3 = 1.40625$	$y_3 = 0.984375$
	$y_4 = 1.40625$	$y_4 = 0.7734375$	$y_4 = 1.265625$	$y_4 = 0.66796875$
	$y_5 = 1.546875$	$y_5 = 1.08984375$	$y_5 = 1.30078125$	$y_5 = 0.747070312$
	$y_6 = 1.4765625$	$y_6 = 0.852539062$	$y_6 = 1.283203125$	$y_6 = 0.687744139$
	$y_7 = 1.51171875$	$y_7 = 0.971191406$	$y_7 = 1.287597656$	$y_7 = 0.702575682$
	$y_8 = 1.494140625$	$y_8 = 0.882202146$	$y_8 = 1.28540039$	$y_8 = 0.691452024$
	$y_9 = 1.502929688$	$y_9 = 0.926696776$	$y_9 = 1.285949707$	$y_9 = 0.694232939$
	$y_{10} = 1.498535151$	$y_{10} = 0.893325802$	$y_{10} = 1.285675047$	$y_{10} = 0.692147252$
	$y_{11} = 1.500733546$	$y_{11} = 0.910011289$	$y_{11} = 1.285743713$	$y_{11} = 0.692669236$

0.25; $\lambda = 0.5$, $\gamma = 0.25$; $\lambda = 0.25$, $\gamma = 0.5$; $\lambda = 0.75$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.5$; $\lambda = 0.25$, $\gamma = 0.75$; $\lambda = 0.75$, $\gamma = 0.75$) *respectively.*

Similarly, we calculated the solution for $x = 0.6$, $x = 1$ *and* $x = 1.5$, *then the approximation is found more sensitive taking* ($\lambda = 0.25$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.25$; $\lambda = 0.25$, $\gamma = 0.5$; $\lambda = 0.75$, $\gamma = 0.25$; $\lambda = 0.5$, $\gamma = 0.5$; $\lambda = 0.25$, $\gamma = 0.75$; $\lambda = 0.75$, $\gamma = 0.75$) *respectively.*

Corollary 2.6 *Absolute error of the modified Ishikawa iteration method is computed taking different values of* λ *and* γ ($x = 0.2$, $x = 0.4$ *and* $x = 0.6$), *which is not more effective than Picard, Runge-Kutta and Euler iteration methods.*

3 Conclusion

A new technique, using the new modified Ishikawa iteration method, to numerically solve the different types of differential equations is presented. All the numerical results obtained using the new modified Ishikawa iteration method described earlier show a very good agreement with the exact solution. Comparing the new modified Ishikawa iteration method with several other methods that have been advanced for solving linear and nonlinear differential equations shows that the new technique is reliable, powerful and

Table 14 The solutions obtained by the new modified Ishikawa iteration method for different values of λ and γ

x	$\lambda = 0.75, \gamma = 0.25$	$\lambda = 0.25, \gamma = 0.75$	$\lambda = 0.75, \gamma = 0.75$
$x = 0.6$	$y_1 = 0.36$	$y_1 = 0.36$	$y_1 = 0.36$
	$y_2 = 0.09$	$y_2 = 0.27$	$y_2 = 0.27$
	$y_3 = 0.2925$	$y_3 = 0.2925$	$y_3 = 0.3375$
	$y_4 = 0.140625$	$y_4 = 0.286875$	$y_4 = 0.320625$
	$y_5 = 0.25453125$	$y_5 = 0.28828125$	$y_5 = 0.33328125$
	$y_6 = 0.169101562$	$y_6 = 0.287929687$	$y_6 = 0.330117187$
	$y_7 = 0.233173828$	$y_7 = 0.288017578$	$y_7 = 0.332490234$
	$y_8 = 0.185119628$	$y_8 = 0.287995605$	$y_8 = 0.331896972$
	$y_9 = 0.221160278$	$y_9 = 0.288001083$	$y_9 = 0.332341918$
	$y_{10} = 0.19412979$	$y_{10} = 0.287999725$	$y_{10} = 0.332230682$
	$y_{11} = 0.214402656$	$y_{11} = 0.288000068$	$y_{11} = 0.332314109$
$x = 1$	$y_1 = 1$	$y_1 = 1$	$y_1 = 1$
	$y_2 = 0.25$	$y_2 = 0.75$	$y_2 = 0.75$
	$y_3 = 0.8125$	$y_3 = 0.8125$	$y_3 = 0.9375$
	$y_4 = 0.390625$	$y_4 = 0.796875$	$y_4 = 0.890625$
	$y_5 = 0.70703125$	$y_5 = 0.80078125$	$y_5 = 0.92578125$
	$y_6 = 0.469726562$	$y_6 = 0.799804687$	$y_6 = 0.916992187$
	$y_7 = 0.647705078$	$y_7 = 0.800048828$	$y_7 = 0.923583984$
	$y_8 = 0.514221191$	$y_8 = 0.799987792$	$y_8 = 0.921936035$
	$y_9 = 0.614334106$	$y_9 = 0.800003051$	$y_9 = 0.923171996$
	$y_{10} = 0.539249419$	$y_{10} = 0.799999236$	$y_{10} = 0.922863006$
	$y_{11} = 0.595562934$	$y_{11} = 0.80000019$	$y_{11} = 0.923094748$
$x = 1.5$	$y_1 = 2.25$	$y_1 = 2.25$	$y_1 = 2.25$
	$y_2 = 0.5625$	$y_2 = 1.6875$	$y_2 = 1.6875$
	$y_3 = 1.828125$	$y_3 = 1.828125$	$y_3 = 2.109375$
	$y_4 = 0.87890625$	$y_4 = 1.79296875$	$y_4 = 2.00390625$
	$y_5 = 1.590820313$	$y_5 = 1.801757813$	$y_5 = 2.083007813$
	$y_6 = 1.056884765$	$y_6 = 1.799560546$	$y_6 = 2.063232421$
	$y_7 = 1.457336426$	$y_7 = 1.800109863$	$y_7 = 2.078063964$
	$y_8 = 1.15699768$	$y_8 = 1.799972532$	$y_8 = 2.074356079$
	$y_9 = 1.382251739$	$y_9 = 1.800006865$	$y_9 = 2.077136991$
	$y_{10} = 1.213311193$	$y_{10} = 1.799998281$	$y_{10} = 2.076441764$
	$y_{11} = 1.340016602$	$y_{11} = 1.800000428$	$y_{11} = 2.076963183$

Table 15 Absolute error of Example 2.3 for different values of λ and γ ($x = 0.4$ and $x = 0.6$ respectively)

	$x = 0.2$	$x = 0.4$	$x = 0.6$
$\lambda = 0.5, \gamma = 0.5$	0.014131067	0.066792042	0.193212053
$\lambda = 0.5, \gamma = 0.25$	0.024632796	0.108798958	0.287727608
$\lambda = 0.25, \gamma = 0.5$	0.017953108	0.082080207	0.22761042
$\lambda = 0.25, \gamma = 0.25$	0.028496655	0.124254392	0.322502337
$\lambda = 0.75, \gamma = 0.25$	0.016988257	0.078220802	0.218926758
$\lambda = 0.25, \gamma = 0.75$	0.008810767	0.045510383	0.145329346
$\lambda = 0.75, \gamma = 0.75$	0.003886985	0.025815712	0.101015305
Picard	0.000000002	0.000000899	0.000053574
Runge-Kutta	0.000000108	0.000001342	0.000008005
Euler	0.040810774	0.093510871	0.180529414

promising. We believe that the efficiency of the new modified Ishikawa iteration method gives it much wider applicability which should be explored further.

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