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Brouwer fixed point theorem in $(L^0)^d$

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Abstract

The classical Brouwer fixed point theorem states that in $\mathbb{R}^d$ every continuous function from a convex, compact set on itself has a fixed point. For an arbitrary probability space, let $L^0 = L^0(\Omega, \mathcal{A}, P)$ be the set of random variables. We consider $(L^0)^d$ as an L^0-module and show that local, sequentially continuous functions on L^0-convex, closed and bounded subsets have a fixed point which is measurable by construction.

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Introduction

The Brouwer fixed point theorem states that a continuous function from a compact and convex set in $\mathbb{R}^d$ to itself has a fixed point. This result and its extensions play a central role in analysis, optimization and economic theory among others. To show the result, one approach is to consider functions on simplexes first and use Sperner's lemma.

Recently, Cheridito *et al.* [1], inspired by the theory developed by Filipović *et al.* [2] and Guo [3], studied $(L^0)^d$ as an L^0-module, discussing concepts like linear independence, σ-stability, locality and L^0-convexity. Based on this, we define affine independence and conditional simplexes in $(L^0)^d$. Showing first a result similar to Sperner's lemma, we obtain a fixed point for local, sequentially continuous functions on conditional simplexes. From the measurable structure of the problem, it turns out that we have to work with local, measurable labeling functions. To cope with this difficulty and to maintain some uniform properties, we subdivide the conditional simplex barycentrically. We then prove the existence of a measurable completely labeled conditional simplex, contained in the original one, which turns out to be a suitable σ-combination of elements of the barycentric subdivision along a partition of Ω. Thus, we can construct a sequence of conditional simplexes converging to a point. By applying always the same rule of labeling using the locality of the function, we show that this point is a fixed point. Due to the measurability of the labeling function, the fixed point is measurable by construction. Hence, even though we follow the constructions and methods used in the proof of the classical result in $\mathbb{R}^d$ (*cf.* [4]), we do not need any measurable selection argument.

In probabilistic analysis theory, the problem of finding random fixed points of random operators is an important issue. Given $\mathcal{C}$, a compact convex set of a Banach space, a continuous random operator is a function $R : \Omega \times \mathcal{C} \to \mathcal{C}$ satisfying

(i) $R(\cdot, x) : \Omega \to \mathcal{C}$ is a random variable for any fixed $x \in \mathcal{C}$,

(ii) $R(\omega, \cdot) : \mathcal{C} \to \mathcal{C}$ is a continuous function for any fixed $\omega \in \Omega$.

For R there exists a random fixed point which is a random variable $\xi : \Omega \to \mathcal{C}$ such that $\xi(\omega) = R(\omega, \xi(\omega))$ for any ω (*cf.* [5–7]). In contrast to this ω-wise consideration, our approach is completely within the theory of L^0. All objects and properties are therefore defined in that language and proofs are done with L^0-methods. Moreover, the connection between continuous random operators on $\mathbb{R}^d$ and sequentially continuous functions on $(L^0)^d$ is not entirely clear.

An application, though not studied in this paper, is for instance possible in economic theory or optimization in the context of [8]. Therein the methods from convex analysis are used to obtain equilibrium results for translation invariant utility functionals on $(L^0)^d$. Without translation invariance, these methods fail and will be replaced by fixed point arguments in an ongoing work. Thus, our result is helpful to develop the theory of non-translation invariant preference functionals mapping to L^0.

The present paper is organized as follows. In the first section, we present the basic concepts concerning $(L^0)^d$ as an L^0-module. We define conditional simplexes and examine their basic properties. In the second section, we define measurable labeling functions and show the Brouwer fixed point theorem for conditional simplexes via a construction in the spirit of Sperner's lemma. In the third section, we show a fixed point result for L^0-convex, bounded and sequentially closed sets in $(L^0)^d$. With this result at hand, we present the topological implications known from the real-valued case. On the one hand, we show the impossibility of contracting a ball to a sphere in $(L^0)^d$ and, on the other hand, an intermediate value theorem in L^0.

1 Conditional simplex

For a probability space $(\Omega, \mathcal{A}, P)$, let $L^0 = L^0(\Omega, \mathcal{A}, P)$ be the space of all $\mathcal{A}$-measurable random variables, where P-almost surely equal random variables are identified. In particular, for $X, Y \in L^0$, the relations $X \geq Y$ and $X > Y$ have to be understood P-almost surely. The set L^0 with the P-almost everywhere order is a lattice ordered ring, and for a non-empty subset $\mathcal{C} \subseteq L^0$, we denote the least upper bound by $\operatorname{ess\,sup} \mathcal{C}$ and the greatest lower bound by $\operatorname{ess\,inf} \mathcal{C}$, respectively (*cf.* [1]). For $m \in \mathbb{R}$, we denote the constant random variable $m1_\Omega$ by m. Further, we define the sets $L^0_+ = \{X \in L^0 : X \geq 0\}$, $L^0_{++} = \{X \in L^0 : X > 0\}$ and $\mathcal{A}_+ = \{A \in \mathcal{A} : P(A) > 0\}$. The set of random variables with values in a set $M \subseteq \mathbb{R}$ is denoted by $M(\mathcal{A})$. For example, $\{1, \dots, r\}(\mathcal{A})$ is the set of $\mathcal{A}$-measurable functions with values in $\{1, \dots, r\} \subseteq \mathbb{N}$, $[0,1](\mathcal{A}) = \{Z \in L^0 : 0 \leq Z \leq 1\}$ and $(0,1)(\mathcal{A}) = \{Z \in L^0 : 0 < Z < 1\}$.

The *convex hull* of $X_1, \dots, X_N \in (L^0)^d$, $N \in \mathbb{N}$, is defined as

$$\operatorname{conv}(X_1, \dots, X_N) = \left\{ \sum_{i=1}^N \lambda_i X_i : \lambda_i \in L^0_+, \sum_{i=1}^N \lambda_i = 1 \right\}.$$

An element $Y = \sum_{i=1}^N \lambda_i X_i$ such that $\lambda_i > 0$ for all $i \in I \subseteq \{1, \dots, N\}$ is called a *strict convex combination* of $\{X_i : i \in I\}$. Moreover, a set $\mathcal{C} \subseteq (L^0)^d$ is said to be L^0-*convex* if for any $X, Y \in \mathcal{C}$ and $\lambda \in [0,1](\mathcal{A})$, it holds that $\lambda X + (1 - \lambda) Y \in \mathcal{C}$.

The σ-*stable hull* of a set $\mathcal{C} \subseteq (L^0)^d$ is defined as

$$\sigma(\mathcal{C}) = \left\{ \sum_{i \in \mathbb{N}} 1_{A_i} X_i : X_i \in \mathcal{C}, (A_i)_{i \in \mathbb{N}} \text{ is a } \textit{partition} \right\},$$

where a partition is a countable family $(A_i)_{i \in \mathbb{N}} \subseteq \mathcal{A}$ such that $P(A_i \cap A_j) = 0$ for $i \neq j$ and $P(\bigcup_{i \in \mathbb{N}} A_i) = 1$. We call a non-empty set $\mathcal{C}$ σ-*stable* if it is equal to $\sigma(\mathcal{C})$. For a σ-stable set $\mathcal{C} \subseteq (L^0)^d$, a function $f : \mathcal{C} \to (L^0)^d$ is called *local* if $f(\sum_{i\in\mathbb{N}} 1_{A_i} X_i) = \sum_{i\in\mathbb{N}} 1_{A_i} f(X_i)$ for every partition $(A_i)_{i\in\mathbb{N}}$ and $X_i \in \mathcal{C}$, $i \in \mathbb{N}$. For $\mathcal{X}, \mathcal{Y} \subseteq (L^0)^d$, we call a function $f : \mathcal{X} \to \mathcal{Y}$ *sequentially continuous* if for every sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{X}$ converging to $X \in \mathcal{X}$ P-almost-surely, it holds that $f(X_n)$ converges to $f(X)$ P-almost surely. Further, the L^0*-scalar product* and L^0*-norm* on $(L^0)^d$ are defined as

$$\langle X, Y\rangle = \sum_{i=1}^{d} X_i Y_i \quad \text{and} \quad \|X\| = \langle X, X\rangle^{\frac{1}{2}}.$$

We call $\mathcal{C} \subseteq (L^0)^d$ *bounded* if $\operatorname{ess\,sup}_{X\in\mathcal{C}} \|X\| \in L^0$ and *sequentially closed* if it contains all P-almost sure limits of sequences in $\mathcal{C}$. Further, the diameter of $\mathcal{C} \subseteq (L^0)^d$ is defined as $\operatorname{diam}(\mathcal{C}) = \operatorname{ess\,sup}_{X,Y\in\mathcal{C}} \|X - Y\|$.

Definition 1.1 Elements $X_1, \ldots, X_N$ of $(L^0)^d$, $N \in \mathbb{N}$, are said to be *affinely independent*, if either $N = 1$ or $N > 1$ and $\{X_i - X_N\}_{i=1}^{N-1}$ are *linearly independent*, that is,

$$\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0 \quad \text{implies} \quad \lambda_1 = \cdots = \lambda_{N-1} = 0, \tag{1.1}$$

where $\lambda_1, \ldots, \lambda_{N-1} \in L^0$.

The definition of affine independence is equivalent to

$$\sum_{i=1}^{N} \lambda_i X_i = 0 \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i = 0 \quad \text{implies} \quad \lambda_1 = \cdots = \lambda_N = 0. \tag{1.2}$$

Indeed, first we show that (1.1) implies (1.2). Let $\sum_{i=1}^{N} \lambda_i X_i = 0$ and $\sum_{i=1}^{N} \lambda_i = 0$. Then $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = 0$. By assumption (1.1), it holds that $\lambda_1 = \cdots = \lambda_{N-1} = 0$, thus also $\lambda_N = 0$. To see that (1.2) implies (1.1), let $\sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. With $\lambda_N = -\sum_{i=1}^{N-1} \lambda_i$, it holds that $\sum_{i=1}^{N} \lambda_i X_i = \lambda_N X_N + \sum_{i=1}^{N-1} \lambda_i X_i = \sum_{i=1}^{N-1} \lambda_i (X_i - X_N) = 0$. By assumption (1.2), $\lambda_1 = \cdots = \lambda_N = 0$.

Remark 1.2 We observe that if $(X_i)_{i=1}^{N} \subseteq (L^0)^d$ are affinely independent, then $(\lambda X_i)_{i=1}^{N}$ for $\lambda \in L^0_{++}$ and $(X_i + Y)_{i=1}^{N}$ for $Y \in (L^0)^d$ are affinely independent. Moreover, if a family $X_1, \ldots, X_N$ is affinely independent, then also $1_B X_1, \ldots, 1_B X_N$ are affinely independent on $B \in \mathcal{A}_+$, which means from $\sum_{i=1}^{N} 1_B \lambda_i X_i = 0$ and $\sum_{i=1}^{N} 1_B \lambda_i = 0$ it always follows that $1_B \lambda_i = 0$ for all $i = 1, \ldots, N$.

Definition 1.3 A *conditional simplex* in $(L^0)^d$ is a set of the form

$$\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$$

such that $X_1, \ldots, X_N \in (L^0)^d$ are affinely independent. We call $N \in \mathbb{N}$ the dimension of $\mathcal{S}$.

Remark 1.4 In a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \ldots, X_N)$, the coefficients of convex combinations are unique in the sense that if $\sum_{i=1}^{N} \lambda_i = \sum_{i=1}^{N} \mu_i = 1$, then

$$\sum_{i=1}^{N} \lambda_i X_i = \sum_{i=1}^{N} \mu_i X_i \quad \text{implies} \quad \lambda_i = \mu_i \quad \text{for all } i = 1, \ldots, N. \tag{1.3}$$

Indeed, since $\sum_{i=1}^N(\lambda_i-\mu_i)X_i=0$ and $\sum_{i=1}^N(\lambda_i-\mu_i)=0$, it follows from (1.2) that $\lambda_i-\mu_i=0$ for all $i=1,\dots,N$.

Remark 1.5 Note that the present setting - L^0-modules and the sequential P-almost sure convergence - is of local nature. This is, for instance, not the case for subsets of L^p or the convergence in the L^p-norm for $1\le p<\infty$. First, L^p is not closed under multiplication and hence neither a ring nor a module over itself, so that we cannot even speak about affine independence. Second, it is in general not a σ-stable subspace of L^0. However, for a conditional simplex $\mathcal{S}=\operatorname{conv}(X_1,\dots,X_N)$ in $(L^0)^d$ such that any X_k is in $(L^p)^d$, it holds that $\mathcal{S}$ is uniformly bounded by $N\sup_{k=1,\dots,N}\|X_k\|\in L^p$. This uniform boundedness yields that any P-almost sure converging sequence in $\mathcal{S}$ is also converging in the L^p-norm for $1\le p<\infty$ due to the dominated convergence theorem. This shows how one can translate results from L^0 to L^p.

Since a conditional simplex is a convex hull, it is in particular σ-stable. In contrast to a simplex in $\mathbb{R}^d$, the representation of $\mathcal{S}$ as a convex hull of affinely independent elements is unique but up to σ-stability.

Proposition 1.6 *Let $(X_i)_{i=1}^N$ and $(Y_i)_{i=1}^N$ be families in $(L^0)^d$ with $\sigma(X_1,\dots,X_N)=\sigma(Y_1,\dots,Y_N)$. Then $\operatorname{conv}(X_1,\dots,X_N)=\operatorname{conv}(Y_1,\dots,Y_N)$. Moreover, $(X_i)_{i=1}^N$ are affinely independent if and only if $(Y_i)_{i=1}^N$ are affinely independent.*

If $\mathcal{S}$ is a conditional simplex such that $\mathcal{S}=\operatorname{conv}(X_1,\dots,X_N)=\operatorname{conv}(Y_1,\dots,Y_N)$, then it holds that $\sigma(X_1,\dots,X_N)=\sigma(Y_1,\dots,Y_N)$.

Proof Suppose $\sigma(X_1,\dots,X_N)=\sigma(Y_1,\dots,Y_N)$. For $i=1,\dots,N$, it holds that

$$X_i\in\sigma(X_1,\dots,X_N)=\sigma(Y_1,\dots,Y_N)\subseteq\operatorname{conv}(Y_1,\dots,Y_N).$$

Therefore, $\operatorname{conv}(X_1,\dots,X_N)\subseteq\operatorname{conv}(Y_1,\dots,Y_N)$ and the reverse inclusion holds analogously.

Now, let $(X_i)_{i=1}^N$ be affinely independent and $\sigma(X_1,\dots,X_N)=\sigma(Y_1,\dots,Y_N)$. We want to show that $(Y_i)_{i=1}^N$ are affinely independent. To that end, we define the affine hull

$$\operatorname{aff}(X_1,\dots,X_N)=\left\{\sum_{i=1}^N\lambda_iX_i:\lambda_i\in L^0,\sum_{i=1}^N\lambda_i=1\right\}.$$

First, let $Z_1,\dots,Z_M\in(L^0)^d$, $M\in\mathbb{N}$, such that $\sigma(X_1,\dots,X_N)=\sigma(Z_1,\dots,Z_M)$. We show that if $1_A\operatorname{aff}(X_1,\dots,X_N)\subseteq 1_A\operatorname{aff}(Z_1,\dots,Z_M)$ for $A\in\mathcal{A}_+$ and $X_1,\dots,X_N$ are affinely independent, then $M\ge N$. Since $X_i\in\sigma(X_1,\dots,X_N)=\sigma(Z_1,\dots,Z_M)\subseteq\operatorname{aff}(Z_1,\dots,Z_M)$, we have $\operatorname{aff}(X_1,\dots,X_N)\subseteq\operatorname{aff}(Z_1,\dots,Z_M)$. Further, it holds that $X_1=\sum_{i=1}^M 1_{B_i^1}Z_i$ for a partition $(B_i^1)_{i=1}^M$ and hence there exists at least one $B_{k_1}^1$ such that $A_{k_1}^1:=B_{k_1}^1\cap A\in\mathcal{A}_+$, and $1_{A_{k_1}^1}X_1=1_{A_{k_1}^1}Z_{k_1}$. Therefore,

$$\begin{aligned}1_{A_{k_1}^1}\operatorname{aff}(X_1,\dots,X_N)&\subseteq 1_{A_{k_1}^1}\operatorname{aff}(Z_1,\dots,Z_M)\\&=1_{A_{k_1}^1}\operatorname{aff}\big(\{X_1,Z_1,\dots,Z_M\}\setminus\{Z_{k_1}\}\big).\end{aligned}$$

For $X_2 = \sum_{i=1}^{M} 1_{A_i^2} Z_i$, we find a set A_k^2 such that $A_{k_2}^2 = A_k^2 \cap A_{k_1}^1 \in \mathcal{A}_+$, $1_{A_{k_2}^2} X_2 = 1_{A_{k_2}^2} Z_{k_2}$ and $k_1 \neq k_2$. Assume to the contrary $k_2 = k_1$, then there exists a set $B \in \mathcal{A}_+$ such that $1_B X_1 = 1_B X_2$, which is a contradiction to the affine independence of $(X_i)_{i=1}^N$. Hence, we can again substitute Z_{k_2} by X_2 on $A_{k_2}^2$. Inductively, we find $k_1, \dots, k_N$ such that

$$1_{A_{k_N}} \operatorname{aff}(X_1, \dots, X_N) \subseteq 1_{A_{k_N}} \operatorname{aff}\bigl(\{X_1, \dots, X_N, Z_1, \dots, Z_M\} \setminus \{Z_{k_1}, \dots, Z_{k_N}\}\bigr),$$

which shows $M \geq N$. Now suppose that $Y_1, \dots, Y_N$ are not affinely independent. This means that there exist $(\lambda_i)_{i=1}^N$ such that $\sum_{i=1}^N \lambda_i Y_i = \sum_{i=1}^N \lambda_i = 0$ but not all coefficients λ_i are zero, without loss of generality, $\lambda_1 > 0$ on $A \in \mathcal{A}_+$. Thus, $1_A Y_1 = -1_A \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i$ and it holds that $1_A \operatorname{aff}(Y_1, \dots, Y_N) = 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. To see this, consider $1_A Z = 1_A \sum_{i=1}^N \mu_i Y_i$ in $1_A \operatorname{aff}(Y_1, \dots, Y_N)$, which means $1_A \sum_{i=1}^N \mu_i = 1_A$. Thus, inserting for Y_1,

$$1_A Z = 1_A \left[\sum_{i=2}^N \mu_i Y_i - \mu_1 \sum_{i=2}^N \frac{\lambda_i}{\lambda_1} Y_i \right] = 1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1} \right) Y_i \right].$$

Moreover,

$$\begin{aligned} 1_A \left[\sum_{i=2}^N \left(\mu_i - \mu_1 \frac{\lambda_i}{\lambda_1} \right) \right] &= 1_A \left[\sum_{i=2}^N \mu_i \right] + 1_A \left[-\frac{\mu_1}{\lambda_1} \sum_{i=2}^N \lambda_i \right] \\ &= 1_A (1 - \mu_1) + 1_A \frac{\mu_1}{\lambda_1} \lambda_1 = 1_A. \end{aligned}$$

Hence, it holds that $1_A Z \in 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. It follows that $1_A \operatorname{aff}(X_1, \dots, X_N) = 1_A \operatorname{aff}(Y_1, \dots, Y_N) = 1_A \operatorname{aff}(Y_2, \dots, Y_N)$. This is a contradiction to the former part of the proof (because $N - 1 \ngeq N$).

Next, we show that in a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ it holds that X is in $\sigma(X_1, \dots, X_N)$ if and only if there do not exist Y and Z in $\mathcal{S} \setminus \{X\}$ and $\lambda \in (0,1)(\mathcal{A})$ such that $\lambda Y + (1-\lambda) Z = X$. Consider $X \in \sigma(X_1, \dots, X_N)$ which is $X = \sum_{k=1}^N 1_{A_k} X_k$ for a partition $(A_k)_{k=1,\dots,N}$. Now assume to the contrary that we find $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ in $\mathcal{S} \setminus \{X\}$ such that $X = \lambda Y + (1-\lambda) Z$. This means that $X = \sum_{k=1}^N (\lambda \lambda_k + (1-\lambda)\mu_k) X_k$. Due to uniqueness of the coefficients (*cf.* (1.3)) in a conditional simplex, we have $\lambda \lambda_k + (1-\lambda)\mu_k = 1_{A_k}$ for all $k = 1, \dots, N$. By means of $0 < \lambda < 1$, it holds that $\lambda \lambda_k + (1-\lambda)\mu_k = 1_{A_k}$ if and only if $\lambda_k = \mu_k = 1_{A_k}$. Since the last equality holds for all k, it follows that $Y = Z = X$. Therefore, we cannot find Y and Z in $\mathcal{S} \setminus \{X\}$ such that X is a strict convex combination of them. On the other hand, consider $X \in \mathcal{S}$ such that $X \notin \sigma(X_1, \dots, X_N)$. This means $X = \sum_{k=1}^N \nu_k X_k$ such that there exist ν_{k_1} and ν_{k_2} and $B \in \mathcal{A}_+$ with $0 < \nu_{k_1} < 1$ on B and $0 < \nu_{k_2} < 1$ on B. Define $\varepsilon := \operatorname{ess\,inf}\{\nu_{k_1}, \nu_{k_2}, 1 - \nu_{k_1}, 1 - \nu_{k_2}\}$. Then define $\mu_k = \lambda_k = \nu_k$ if $k_1 \neq k \neq k_2$ and $\lambda_{k_1} = \nu_{k_1} - \varepsilon$, $\lambda_{k_2} = \nu_{k_2} + \varepsilon$, $\mu_{k_1} = \nu_{k_1} + \varepsilon$ and $\mu_{k_2} = \nu_{k_2} - \varepsilon$. Thus, $Y = \sum_{k=1}^N \lambda_k X_k$ and $Z = \sum_{k=1}^N \mu_k X_k$ fulfill $0.5Y + 0.5Z = X$ but both are not equal to X by construction. Hence, X can be written as a strict convex combination of elements in $\mathcal{S} \setminus \{X\}$. To conclude, consider $X \in \sigma(X_1, \dots, X_N) \subseteq \mathcal{S} = \operatorname{conv}(X_1, \dots, X_N) = \operatorname{conv}(Y_1, \dots, Y_N)$. Since $X \in \sigma(X_1, \dots, X_N)$, it is not a strict convex combination of elements in $\mathcal{S} \setminus \{X\}$, in particular, of elements in $\operatorname{conv}(Y_1, \dots, Y_N) \setminus \{X\}$. Therefore, X is also in $\sigma(Y_1, \dots, Y_N)$. Hence, $\sigma(X_1, \dots, X_N) \subseteq \sigma(Y_1, \dots, Y_N)$. With the same argumentation, the other inclusion follows. $\square$

As an example, let us consider $[0,1](\mathcal{A})$. For an arbitrary $A \in \mathcal{A}$, it holds that 1_A and 1_{A^c} are affinely independent and $\operatorname{conv}(1_A, 1_{A^c}) = \{\lambda 1_A + (1-\lambda)1_{A^c} : 0 \leq \lambda \leq 1\} = [0,1](\mathcal{A})$. Thus, the conditional simplex $[0,1](\mathcal{A})$ can be written as a convex combination of different affinely independent elements of L^0. This is due to the fact that $\sigma(0,1) = \{1_B : B \in \mathcal{A}\} = \sigma(1_A, 1_{A^c})$ for all $A \in \mathcal{A}$.

Remark 1.7 In $(L^0)^d$, let e_i be the random variable which is 1 in the ith component and 0 in any other. Then the family $0, e_1, \dots, e_d$ is affinely independent and $(L^0)^d = \operatorname{aff}(0, e_1, \dots, e_d)$. Hence, the maximal number of affinely independent elements in $(L^0)^d$ is $d+1$.

The characterization of $X \in \sigma(X_1, \dots, X_N)$ leads to the following definition.

Definition 1.8 Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex. We define the set of *extremal points* $\operatorname{ext}(\mathcal{S}) = \sigma(X_1, \dots, X_N)$. For an index set I and a collection $\mathscr{S} = (\mathcal{S}_i)_{i \in I}$ of conditional simplexes, we denote $\operatorname{ext}(\mathscr{S}) = \sigma(\bigcup_{i \in I} \operatorname{ext}(\mathcal{S}_i))$.

Remark 1.9 Let $\mathcal{S}^j = \operatorname{conv}(X_1^j, \dots, X_N^j)$, $j \in \mathbb{N}$, be conditional simplexes of the same dimension N and $(A_j)_{j \in \mathbb{N}}$ a partition. Then $\sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j$ is again a conditional simplex. To that end, we define $Y_k = \sum_{j \in \mathbb{N}} 1_{A_j} X_k^j$ and recognize $\sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j = \operatorname{conv}(Y_1, \dots, Y_N)$. Indeed,

$$\sum_{k=1}^{N} \lambda_k Y_k = \sum_{k=1}^{N} \lambda_k \sum_{j \in \mathbb{N}} 1_{A_j} X_k^j = \sum_{j \in \mathbb{N}} 1_{A_j} \sum_{k=1}^{N} \lambda_k X_k^j \in \sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j, \tag{1.4}$$

shows $\operatorname{conv}(Y_1, \dots, Y_N) \subseteq \sum_{j \in \mathbb{N}} 1_{A_j} \mathcal{S}^j$. The other inclusion follows by considering $\sum_{k=1}^{N} \lambda_k^j \times X_k^j \in \mathcal{S}^j$ and defining $\lambda_k = \sum_{j \in \mathbb{N}} 1_{A_j} \lambda_k^j$. To show that $Y_1, \dots, Y_N$ are affinely independent, we consider $\sum_{k=1}^{N} \lambda_k Y_k = 0 = \sum_{k=1}^{N} \lambda_k$. Then by (1.4) it holds that $1_{A_j} \sum_{k=1}^{N} \lambda_k X_k^j = 0$ and since $\mathcal{S}^j$ is a conditional simplex, $1_{A_j} \lambda_k = 0$ for all $j \in \mathbb{N}$ and $k = 1, \dots, N$. From the fact that $(A_j)_{j \in \mathbb{N}}$ is a partition, it follows that $\lambda_k = 0$ for all $k = 1, \dots, N$.

We will prove the Brouwer fixed point theorem in the present setting using an L^0-module version of Sperner's lemma. As in the unconditional case, we have to subdivide a conditional simplex into smaller ones. For our argumentation, we cannot use arbitrary subdivisions and need very special properties of the conditional simplexes in which we subdivide. This leads to the following definition.

Definition 1.10 Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex and S_N be the group of permutations of $\{1, \dots, N\}$. Then, for $\pi \in \mathrm{S}_N$, we define

$$Y_k^\pi = \frac{1}{k} \sum_{i=1}^{k} X_{\pi(i)}, \quad k = 1, \dots, N,$$

$$\mathcal{C}_\pi = \operatorname{conv}\left(Y_1^\pi, \dots, Y_N^\pi\right).$$

We call $(\mathcal{C}_\pi)_{\pi \in \mathrm{S}_N}$ the *barycentric subdivision* of $\mathcal{S}$.

Lemma 1.11 *Let $X_1, \dots, X_N \in (L^0)^d$ be affinely independent. The barycentric subdivision of $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ is a collection of finitely many conditional simplexes satisfying the following properties*:

(i) $\sigma(\bigcup_{\pi\in S_N}\mathcal{C}_\pi)=\mathcal{S}$.
(ii) $\mathcal{C}_\pi$ *has dimension* N, $\pi\in S_N$.
(iii) $\mathcal{C}_\pi\cap\mathcal{C}_{\overline{\pi}}$ *is a conditional simplex of dimension* $r\in\mathbb{N}$ *and* $r<N$ *for* $\pi,\overline{\pi}\in S_N$, $\pi\neq\overline{\pi}$.
(iv) *For* $s=1,\dots,N-1$, *let* $\mathcal{B}_s:=\operatorname{conv}(X_1,\dots,X_s)$. *All conditional simplexes* $\mathcal{C}_\pi\cap\mathcal{B}_s$, $\pi\in S_N$, *of dimension* s *subdivide* $\mathcal{B}_s$ *barycentrically.*

Proof We show the affine independence of $Y_1^\pi,\dots,Y_N^\pi$ in $\mathcal{C}_\pi$. It holds that

$$\lambda_{\pi(1)}X_{\pi(1)}+\lambda_{\pi(2)}\frac{X_{\pi(1)}+X_{\pi(2)}}{2}+\cdots+\lambda_{\pi(N)}\frac{\sum_{k=1}^N X_{\pi(k)}}{N}=\sum_{i=1}^N\mu_iX_i,$$

with $\mu_i=\sum_{k=\pi^{-1}(i)}^N\frac{\lambda_{\pi(k)}}{k}$. Since $\sum_{i=1}^N\mu_i=\sum_{i=1}^N\lambda_i$, the affine independence of $Y_1^\pi,\dots,Y_N^\pi$ is obtained by the affine independence of $X_1,\dots,X_N$. Therefore all $\mathcal{C}_\pi$ are conditional simplexes.

As for Condition (i), it clearly holds that $\sigma(\bigcup_{\pi\in S_N}\mathcal{C}_\pi)\subseteq\mathcal{S}$. On the other hand, let $X=\sum_{i=1}^N\lambda_iX_i\in\mathcal{S}$. Then we find a partition $(A_n)_{n=1,\dots,M}$, for some $M\in\mathbb{N}$, such that on every A_n the indexes are completely ordered, which is $\lambda_{i_1^n}\geq\lambda_{i_2^n}\geq\cdots\geq\lambda_{i_N^n}$ on A_n.[a] This means that $X\in 1_{A_n}\mathcal{C}_{\pi^n}$ with $\pi^n(j)=i_j^n$. Indeed, we can rewrite X on A_n as

$$X=(\lambda_{i_1^n}-\lambda_{i_2^n})X_{i_1^n}+\cdots+(N-1)(\lambda_{i_{N-1}^n}-\lambda_{i_N^n})\frac{\sum_{k=1}^{N-1}X_{i_k^n}}{N-1}+N\lambda_{i_N^n}\frac{\sum_{k=1}^N X_{i_k^n}}{N},$$

which shows that $X\in\mathcal{C}_{\pi^n}$ on A_n. Condition (ii) is fulfilled by construction.

The intersection of two conditional simplexes $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ can be expressed in the following manner. Let $J=\{j:\{\pi(1),\dots,\pi(j)\}=\{\overline{\pi}(1),\dots,\overline{\pi}(j)\}\}$ be the set of indexes up to which both π and $\overline{\pi}$ have the same set of images. Then

$$\mathcal{C}_\pi\cap\mathcal{C}_{\overline{\pi}}=\operatorname{conv}\bigl(Y_j^\pi:j\in J\bigr).\tag{1.5}$$

To show $\supseteq$, let $j\in J$. It holds that Y_j^π is in both $\mathcal{C}_\pi$ and $\mathcal{C}_{\overline{\pi}}$ since $\{\pi(1),\dots,\pi(j)\}=\{\overline{\pi}(1),\dots,\overline{\pi}(j)\}$. Since the intersection of L^0-convex sets is L^0-convex, we get this inclusion. As for the reverse inclusion, consider $X\in\mathcal{C}_\pi\cap\mathcal{C}_{\overline{\pi}}$. From $X\in\mathcal{C}_\pi\cap\mathcal{C}_{\overline{\pi}}$, it follows that $X=\sum_{i=1}^N\lambda_i(\sum_{k=1}^i\frac{X_{\pi(k)}}{i})=\sum_{i=1}^N\mu_i(\sum_{k=1}^i\frac{X_{\overline{\pi}(k)}}{i})$. Consider $j\notin J$. By definition of J, there exist $p,q\leq j$ with $\overline{\pi}^{-1}(\pi(p))$, $\pi^{-1}(\overline{\pi}(q))\notin\{1,\dots,j\}$. By (1.3), the coefficients of $X_{\pi(p)}$ are equal: $\sum_{i=p}^N\frac{\lambda_i}{i}=\sum_{i=\overline{\pi}^{-1}(\pi(p))}^N\frac{\mu_i}{i}$. The same holds for $X_{\pi(q)}$: $\sum_{i=q}^N\frac{\mu_i}{i}=\sum_{i=\pi^{-1}(\overline{\pi}(q))}^N\frac{\lambda_i}{i}$. Put together

$$\sum_{i=j+1}^N\frac{\mu_i}{i}\leq\sum_{i=q}^N\frac{\mu_i}{i}=\sum_{i=\pi^{-1}(\overline{\pi}(q))}^N\frac{\lambda_i}{i}\leq\sum_{i=j+1}^N\frac{\lambda_i}{i}\leq\sum_{i=p}^N\frac{\lambda_i}{i}=\sum_{i=\overline{\pi}^{-1}(\pi(p))}^N\frac{\mu_i}{i}\leq\sum_{i=j+1}^N\frac{\mu_i}{i},$$

which is only possible if $\mu_j=\lambda_j=0$ since $p,q\leq j$. Furthermore, if $\mathcal{C}_\pi\cap\mathcal{C}_{\overline{\pi}}$ is of dimension N, by (1.5) it follows that $\pi=\overline{\pi}$. This shows (iii).

Further, for $\mathcal{B}_s=\operatorname{conv}(X_1,\dots,X_s)$, the elements $\mathcal{C}_{\pi'}\cap\mathcal{B}_s$ of dimension s are exactly the ones with $\{\pi'(i):i=1,\dots,s\}=\{1,\dots,s\}$. To this end, let $C_{\pi'}\cap\mathcal{B}_s$ be of dimension s. This means there exists an element Y in this intersection such that $Y=\sum_{i=1}^N\lambda_iX_i$ with $\lambda_i>0$ for all $i=1,\dots,s$ and $\lambda_i=0$ for $i>s$. As an element of $C_{\pi'}$, this Y has a representation of the form $Y=\sum_{j=1}^N(\sum_{k=j}^N\frac{\mu_k}{k})X_{\pi'(j)}$ for $\sum_{k=1}^N\mu_k=1$ and $\mu_k\in L_+^0$ for every $k=1,\dots,N$. Suppose now that there exists some $j_0 \leq s$ with $\pi'(j_0) > s$. Then due to $\lambda_{\pi'(j_0)} = 0$ and the uniqueness of the coefficients (*cf.* (1.3)) in a conditional simplex, it holds that $\sum_{k=j_0}^{N} \frac{\mu_k}{k} = 0$ and within $\sum_{k=j}^{N} \frac{\mu_k}{k} = 0$ for all $j \geq j_0$. This means $Y = \sum_{j=1}^{j_0-1}(\sum_{k=j}^{N} \frac{\mu_k}{k})X_{\pi'(j)}$ and hence Y is the convex combination of $j_0 - 1$ elements with $j_0 - 1 < s$. This contradicts the property that $\lambda_i > 0$ for s elements. Therefore, $(\mathcal{C}_{\pi'} \cap \mathcal{B}_s)_{\pi'}$ is exactly the barycentric subdivision of $\mathcal{B}_s$, which has been shown to fulfill the properties (i)-(iii). $\square$

Subdividing a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ barycentrically, we obtain $(\mathcal{C}_\pi)_{\pi \in S_N}$. Dividing every $\mathcal{C}_\pi$ barycentrically results in a new collection of conditional simplexes and we call this the two-fold barycentric subdivision of $\mathcal{S}$. Inductively, we can subdivide every conditional simplex of the $(m-1)$th step barycentrically and call the resulting collection of conditional simplexes the m-fold barycentric subdivision of $\mathcal{S}$ and denote it by $\mathscr{S}^m$. Further, we define $\operatorname{ext}(\mathscr{S}^m) = \sigma(\{\operatorname{ext}(\mathcal{C}) : \mathcal{C} \in \mathscr{S}^m\})$ to be the σ-stable hull of all extremal points of the conditional simplexes of the m-fold barycentric subdivision of $\mathcal{S}$. Notice that this is the σ-stable hull of only finitely many elements, since there are only finitely many simplexes in the subdivision, each of which is the convex hull of N elements.

Remark 1.12 Consider an arbitrary $\mathcal{C}_\pi = \operatorname{conv}(Y_1^\pi, \dots, Y_N^\pi)$, $\pi \in \mathrm{S}_N$ in the barycentric subdivision of a conditional simplex $\mathcal{S}$. Then it holds that

$$\operatorname{diam}(\mathcal{C}_\pi) = \operatorname*{ess\,sup}_{i,j=1,\dots,N} \| Y_i^\pi - Y_j^\pi \| \leq \frac{N-1}{N} \operatorname{diam}(\mathcal{S}).$$

Since this holds for any $\pi \in S_N$, it follows that the diameter of $\mathcal{S}^m$, which is an arbitrary conditional simplex of the m-fold barycentric subdivision of $\mathcal{S}$, fulfills $\operatorname{diam}(\mathcal{S}^m) \leq (\frac{N-1}{N})^m \operatorname{diam}(\mathcal{S})$. Since $\operatorname{diam}(\mathcal{S}) < \infty$ and $(\frac{N-1}{N})^m \to 0$, for $m \to \infty$, it follows that $\operatorname{diam}(\mathcal{S}^m) \to 0$ for $m \to \infty$ for every sequence $(\mathcal{S}^m)_{m \in \mathbb{N}}$.

2 Brouwer fixed point theorem for conditional simplexes

Definition 2.1 Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex, m-fold barycentrically subdivided in $\mathscr{S}^m$. A local function $\phi : \operatorname{ext}(\mathscr{S}^m) \to \{1, \dots, N\}(\mathcal{A})$ is called a *labeling function* of $\mathcal{S}$. For fixed $X_1, \dots, X_N \in \operatorname{ext}(\mathcal{S})$ with $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, the labeling function is called *proper* if for any $Y \in \operatorname{ext}(\mathscr{S}^m)$ it holds that

$$P\big(\{\omega : \phi(Y)(\omega) = i, \lambda_i(\omega) = 0\}\big) = 0$$

for $i = 1, \dots, N$, where $Y = \sum_{i=1}^{N} \lambda_i X_i$. A conditional simplex $\mathcal{C} = \operatorname{conv}(Y_1, \dots, Y_N)$, with $\mathcal{C} \subseteq \mathcal{S}$, with $Y_j \in \operatorname{ext}(\mathscr{S}^m)$, $j = 1, \dots, N$, is said to be *completely labeled* by ϕ if ϕ is a proper labeling function of $\mathcal{S}$ and

$$P\big(\{\omega : \text{there exists } j \in \{1, \dots, N\}, \phi(Y_j)(\omega) = i\}\big) = 1$$

for all $i \in \{1, \dots, N\}$.

Lemma 2.2 *Let $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ be a conditional simplex and $f : \mathcal{S} \to \mathcal{S}$ be a local function. Let $\phi : \operatorname{ext}(\mathscr{S}^m) \to \{0, \dots, N\}(\mathcal{A})$ be a local function such that for every $X \in \operatorname{ext}(\mathscr{S}^m)$ it holds that*

(i) $P(\{\omega : \phi(X)(\omega) = i; \lambda_i(\omega) = 0 \text{ or } \mu_i(\omega) > \lambda_i(\omega)\}) = 0$ *for all* $i = 1, \dots, N$,

(ii) $P(\{\omega : \phi(X)(\omega) = 0, \exists i \in \{1, \dots, N\}, \lambda_i(\omega) > 0, \lambda_i(\omega) \geq \mu_i(\omega)\}) = 0$,

where $(\lambda_i)_{i=1,\dots,N}$ *and* $(\mu_i)_{i=1,\dots,N}$ *are determined by* $X = \sum_{i=1}^N \lambda_i X_i$ *and* $f(X) = \sum_{i=1}^N \mu_i X_i$. *Then* ϕ *is a proper labeling function.*

Moreover, the set of functions fulfilling these properties is non-empty.

Proof First we show that ϕ is a labeling function. Since ϕ is local, we just have to prove that ϕ actually maps into $\{1, \dots, N\}$. Due to (ii), we have to show that

$$P\big(\{\omega : \text{there exists } i \in \{1, \dots, N\}, \lambda_i(\omega) \geq \mu_i(\omega), \lambda_i(\omega) > 0\}\big) = 1.$$

Assume, to the contrary, that $\mu_i > \lambda_i$ on $A \in \mathcal{A}_+$ for all λ_i with $\lambda_i > 0$ on A. Then it holds that $1 = \sum_{i=1}^N \lambda_i 1_{\{\lambda_i > 0\}} < \sum_{i=1}^N \mu_i 1_{\{\mu_i > 0\}} = 1$ on A, which yields a contradiction. Thus, ϕ is a labeling function. Moreover, due to (i), it holds in particular that $P(\{\omega : \phi(X)(\omega) = i, \lambda_i(\omega) = 0\}) = 0$, which shows that ϕ is proper.

To prove the existence for $X \in \operatorname{ext}(\mathscr{S}^m)$ with $X = \sum_{i=1}^N \lambda_i X_i, f(X) = \sum_{i=1}^N \mu_i$, let $B_i := \{\omega : \lambda_i(\omega) > 0\} \cap \{\omega : \lambda_i(\omega) \geq \mu_i(\omega)\}$, $i = 1, \dots, N$. Then we define the function ϕ at X as $\{\omega : \phi(X)(\omega) = i\} = B_i \setminus (\bigcup_{k=1}^{i-1} B_k)$, $i = 1, \dots, N$. It has been shown that ϕ maps to $\{1, \dots, N\}(\mathcal{A})$ and is proper. It remains to show that ϕ is local. To this end, consider $X = \sum_{j \in \mathbb{N}} 1_{A_j} X^j$, where $X^j = \sum_{i=1}^N \lambda_i^j X_i$ and $f(X^j) = \sum_{i=1}^N \mu_i^j X_i$. Due to uniqueness of the coefficients in a conditional simplex, it holds that $\lambda_i = \sum_{j \in \mathbb{N}} 1_{A_j} \lambda_i^j$, and due to locality of f, it follows that $\mu_i = \sum_{j \in \mathbb{N}} 1_{A_j} \mu_i^j$. Therefore it holds that $B_i = \bigcup_{j \in \mathbb{N}} (\{\omega : \lambda_i^j(\omega) > 0\} \cap \{\omega : \lambda_i^j(\omega) \geq \mu_i^j(\omega)\} \cap A_j) = \bigcup_{j \in \mathbb{N}} (B_i^j \cap A_j)$. Hence, $\phi(X) = i$ on $B_i \setminus (\bigcup_{k=1}^{i-1} B_k) = [\bigcup_{j \in \mathbb{N}} (B_i^j \cap A_j)] \setminus [\bigcup_{k=1}^{i-1} (\bigcup_{j \in \mathbb{N}} B_k^j \cap A_j)] = \bigcup_{j \in \mathbb{N}} [(B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j) \cap A_j]$. On the other hand, we see that $\sum_{j \in \mathbb{N}} 1_{A_j} \phi(X^j)$ is i on any $A_j \cap \{\omega : \phi(X^j)(\omega) = i\}$, hence it is i on $\bigcup_{j \in \mathbb{N}} (B_i^j \setminus \bigcup_{k=1}^{i-1} B_k^j) \cap A_j$. Thus, $\sum_{j \in \mathbb{N}} 1_{A_j} \phi(X^j) = \phi(\sum_{j \in \mathbb{N}} 1_{A_j} X^j)$, which shows that ϕ is local. □

The reason to demand locality of a labeling function is exactly because we want to label by the function ϕ mentioned in the existence proof of Lemma 2.2 and hence keep local information with it. For example, consider a conditional simplex $\mathcal{S} = \operatorname{conv}(X_1, X_2, X_3, X_4)$ and $\Omega = \{\omega_1, \omega_2\}$. Let $Y \in \operatorname{ext}(\mathscr{S})$ be given by $Y = \frac{1}{3} \sum_{i=1}^3 X_i$. Now consider a function f on $\mathcal{S}$ such that

$$f(Y)(\omega_1) = \frac{1}{4} X_1(\omega_1) + \frac{3}{4} X_3(\omega_1); \qquad f(Y)(\omega_2) = \frac{2}{5} X_1(\omega_2) + \frac{2}{5} X_2(\omega_2) + \frac{1}{5} X_4(\omega_2).$$

If we label Y by the rule explained in Lemma 2.2, ϕ takes the values $\phi(Y)(\omega_1) \in \{1, 2\}$ and $\phi(Y)(\omega_2) = 3$. Therefore, we can really distinguish on which sets $\lambda_i \geq \mu_i$. Yet, using a deterministic labeling of Y, we would lose this information.

Theorem 2.3 *Let* $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$ *be a conditional simplex in* $(L^0)^d$. *Let further* $f : \mathcal{S} \to \mathcal{S}$ *be a local, sequentially continuous function. Then there exists* $Y \in \mathcal{S}$ *such that* $f(Y) = Y$.

Proof We consider the barycentric subdivision $(\mathcal{C}_\pi)_{\pi \in S_N}$ of $\mathcal{S}$ and a proper labeling function ϕ on $\operatorname{ext}(\mathscr{S})$. First, we show that we can find a completely labeled conditional simplex in $\mathcal{S}$. By induction on the dimension of $\mathcal{S} = \operatorname{conv}(X_1, \dots, X_N)$, we show that there exists a partition $(A_k)_{k=1,\ldots,K}$ such that on any A_k there is an odd number of completely labeled $\mathcal{C}_\pi$. The case $N = 1$ is clear since a point can be labeled with the constant index 1 only.

Suppose that the case $N - 1$ is proven. Since the number of Y_i^π of the barycentric subdivision is finite and ϕ can only take finitely many values, it holds for all $V \in (Y_i^\pi)_{i=1,\ldots,N,\pi\in S_N}$ that there exists a partition $(A_k^V)_{k=1,\ldots,K}$, $K < \infty$, where $\phi(V)$ is constant on any A_k^V. Therefore, we find a partition $(A_k)_{k=1,\ldots,K}$ such that $\phi(V)$ on A_k is constant for all V and A_k. Fix A_k now.

In the following, we denote by $\mathcal{C}_{\pi^b}$ those conditional simplexes for which $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1}$ are $N-1$-dimensional (*cf.* Lemma 1.11(iv)), therefore $\pi^b(N) = N$. Further we denote by $\mathcal{C}_{\pi^c}$ these conditional simplexes which are not of the type $\mathcal{C}_{\pi^b}$, that is, $\pi^c(N) \neq N$. If we use $\mathcal{C}_\pi$, we mean a conditional simplex of arbitrary type. We define:

- $\mathscr{C} \subseteq (\mathcal{C}_\pi)_{\pi\in S_N}$ to be the set of $\mathcal{C}_\pi$ which are completely labeled on A_k.
- $\mathscr{A} \subseteq (\mathcal{C}_\pi)_{\pi\in S_N}$ to be the set of P-almost completely labeled $\mathcal{C}_\pi$, that is,

$$\{\phi(Y_k^\pi), k \in \{1,\ldots,N\}\} = \{1,\ldots,N-1\} \quad \text{on } A_k.$$

- $\mathscr{E}_\pi$ to be the set of intersections $(\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l})_{\pi_l\in S_N}$ which are $N-1$-dimensional and completely labeled on A_k.[b]
- $\mathscr{B}_\pi$ to be the set of intersections $\mathcal{C}_\pi \cap \mathcal{B}_{N-1}$ which are completely labeled on A_k.

It holds that $\mathscr{E}_\pi \cap \mathscr{B}_\pi = \emptyset$ and hence $|\mathscr{E}_\pi \cup \mathscr{B}_\pi| = |\mathscr{E}_\pi| + |\mathscr{B}_\pi|$. Since $\mathcal{C}_{\pi^c} \cap \mathcal{B}_{N-1}$ is at most $N-2$-dimensional, it holds that $\mathscr{B}_{\pi^c} = \emptyset$ and hence $|\mathscr{B}_{\pi^c}| = 0$. Moreover, we know that $\mathcal{C}_\pi \cap \mathcal{C}_{\pi_l}$ is $N-1$-dimensional on A_k if and only if this holds on the whole Ω (*cf.* Lemma 1.11(iii)) and $\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1} \neq \emptyset$ on A_k if and only if this also holds on the whole Ω (*cf.* Lemma 1.11(iv)). So, it does not play any role if we look at these sets which are intersections on A_k or on Ω since they are exactly the same sets.

If $\mathcal{C}_{\pi^c} \in \mathscr{C}$, then $|\mathscr{E}_{\pi^c}| = 1$ and if $\mathcal{C}_{\pi^b} \in \mathscr{C}$, then $|\mathscr{E}_{\pi^b} \cup \mathscr{B}_{\pi^b}| = 1$. If $\mathcal{C}_{\pi^c} \in \mathscr{A}$, then $|\mathscr{E}_{\pi^c}| = 2$ and if $\mathcal{C}_{\pi^b} \in \mathscr{A}$, then $|\mathscr{E}_{\pi^b} \cup \mathscr{B}_{\pi^b}| = 2$. Therefore it holds that $\sum_{\pi\in S_N} |\mathscr{E}_\pi \cup \mathscr{B}_\pi| = |\mathscr{C}| + 2|\mathscr{A}|$.

If we pick $E_\pi \in \mathscr{E}_\pi$, we know that there always exists exactly one other π_l such that $E_\pi \in \mathscr{E}_{\pi_l}$ (Lemma 1.11(iii)). Therefore $\sum_{\pi\in S_N} |\mathscr{E}_\pi|$ is even. Moreover, $(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})_{\pi^b}$ subdivides $\mathcal{B}_{N-1}$ barycentrically, and hence we can apply the inductive hypothesis (on $\mathrm{ext}(\mathcal{C}_{\pi^b} \cap \mathcal{B}_{N-1})$). Indeed, the set $\mathcal{B}_{N-1}$ is a σ-stable set, so if it is partitioned by the labeling function into $(A_k)_{k=1,\ldots,K}$, we know that $\mathcal{B}_{N-1}(\mathcal{S}) = \sum_{k=1}^K 1_{A_k}\mathcal{B}_{N-1}(1_{A_k}\mathcal{S})$ and by Lemma 1.11(iv) we can apply the induction hypothesis also to every A_k, $k = 1,\ldots,K$. Thus, the number of completely labeled conditional simplexes is odd on a partition of Ω, but since ϕ is constant on A_k, it also has to be odd there. This means that $\sum_{\pi^b} |\mathscr{B}_{\pi^b}|$ has to be odd. Hence, we also have that $\sum_\pi |\mathscr{E}_\pi \cup \mathscr{B}_\pi|$ is the sum of an even and an odd number and thus odd. So, we conclude $|\mathscr{C}| + 2|\mathscr{A}|$ is odd and hence also $|\mathscr{C}|$. Thus, we find for any A_k a completely labeled $\mathcal{C}_{\pi_k}$.

We define $\mathcal{S}^1 = \sum_{k=1}^K 1_{A_k}\mathcal{C}_{\pi_k}$ which by Remark 1.9 is indeed a conditional simplex. Due to σ-stability of $\mathcal{S}$, it holds that $\mathcal{S}^1 \subseteq \mathcal{S}$. By Remark 1.12, $\mathcal{S}^1$ has a diameter which is less than $\frac{N-1}{N}\,\mathrm{diam}(\mathcal{S})$ and since ϕ is local, $\mathcal{S}^1$ is completely labeled on the whole Ω.

The same argumentation holds for every m-fold barycentric subdivision $\mathscr{S}^m$ of $\mathcal{S}$, $m \in \mathbb{N}$, that is, there exists a completely labeled conditional simplex in every m-fold barycentrically subdivided conditional simplex which is properly labeled. Henceforth, subdividing $\mathcal{S}$ m-fold barycentrically and labeling it by $\phi^m : \mathrm{ext}(\mathscr{S}^m) \to \{1,\ldots,N\}(\mathcal{A})$, which is a labeling function as in Lemma 2.2, we always obtain a completely labeled conditional simplex

$\mathcal{S}^{m+1} \subseteq \mathcal{S}$ for $m \in \mathbb{N}$. Moreover, since $\mathcal{S}^1$ is completely labeled, it holds $\mathcal{S}^1 = \sum_{k=1}^K 1_{A_k}\mathcal{C}_{\pi_k}$ as above, where $\mathcal{C}_{\pi_k}$ is completely labeled on A_k. This means $\mathcal{C}_{\pi_k} = \operatorname{conv}(Y_1^k, \dots, Y_N^k)$ with $\phi(Y_j^k) = j$ on A_k for every $j = 1, \dots, N$. Defining $V_j^1 = \sum_{k=1}^K 1_{A_k} Y_j^k$ for every $j = 1, \dots, N$ yields $P(\{\omega : \phi(V_j^1)(\omega) = j\}) = 1$ for every $j = 1, \dots, N$ and $\mathcal{S}^1 = \operatorname{conv}(V_1^1, \dots, V_N^1)$. The same holds for any $m \in \mathbb{N}$ and so that we can write $\mathcal{S}^m = \operatorname{conv}(V_1^m, \dots, V_N^m)$ with $P(\{\omega : \phi^{m-1}(V_j^m)(\omega) = j\}) = 1$ for every $j = 1, \dots, N$.

Now, $(V_1^m)_{m\in\mathbb{N}}$ is a sequence in the sequentially closed, L^0-bounded set $\mathcal{S}$, so that by [1, Corollary 3.9], there exists $Y \in \mathcal{S}$ and a sequence $(M_m)_{m\in\mathbb{N}}$ in $\mathbb{N}(\mathcal{A})$ such that $M_{m+1} > M_m$ for all $m \in \mathbb{N}$ and $\lim_{m\to\infty} V_1^{M_m} = Y$ P-almost surely. For $M_m \in \mathbb{N}(\mathcal{A})$, $V_1^{M_m}$ is defined as $\sum_{n\in\mathbb{N}} 1_{\{M_m=n\}} V_1^n$. This means an element with index M_m, for some $m \in \mathbb{N}$, equals V_1^n on A_n, $n \in \mathbb{N}$, where the sets A_n are determined by M_m via $A_n = \{\omega : M_m(\omega) = n\}$, $n \in \mathbb{N}$. Furthermore, as m goes to ∞, $\operatorname{diam}(S^m)$ is converging to zero P-almost surely, and therefore it also follows that $\lim_{m\to\infty} V_k^{M_m} = Y$ P-almost surely for every $k = 1, \dots, N$. Indeed, it holds that $|V_k^m - Y| \le \operatorname{diam}(\mathcal{S}^m) + |V_1^m - Y|$ for every $k = 1, \dots, N$ and $m \in \mathbb{N}$, so we can use the sequence $(M_m)_{m\in\mathbb{N}}$ for every $k = 1, \dots, N$.

Let $Y = \sum_{l=1}^N \alpha_l X_l$ and $f(Y) = \sum_{l=1}^N \beta_l X_l$ as well as $V_k^m = \sum_{l=1}^N \lambda_l^{m,k} X_l$ and $f(V_k^m) = \sum_{l=1}^N \mu_l^{m,k} X_l$ for $m \in \mathbb{N}$. As f is local, it holds that $f(V_1^{M_m}) = \sum_{n\in\mathbb{N}} 1_{\{M_m=n\}} f(V_1^n)$. By sequential continuity of f, it follows that $\lim_{m\to\infty} f(V_k^{M_n}) = f(Y)$ P-almost surely for every $k = 1, \dots, N$. In particular, $\lim_{m\to\infty} \lambda_l^{M_m,l} = \alpha_l$ and $\lim_{m\to\infty} \mu_l^{M_m,l} = \beta_l$ P-almost surely for every $l = 1, \dots, N$. However, by construction, $\phi^{m-1}(V_l^m) = l$ for every $l = 1, \dots, N$, and from the choice of ϕ^{m-1}, it follows that $\lambda_l^{m,l} \ge \mu_l^{m,l}$ P-almost surely for every $l = 1, \dots, N$ and $m \in \mathbb{N}$. Hence, $\alpha_l = \lim_{m\to\infty} \lambda_l^{M_m,l} \ge \lim_{m\to\infty} \mu_l^{M_m,l} = \beta_l$ P-almost surely for every $l = 1, \dots, N$. This is possible only if $\alpha_l = \beta_l$ P-almost surely for every $l = 1, \dots, N$, showing that $f(Y) = Y$. □

3 Applications

3.1 Fixed point theorem for sequentially closed and bounded sets in $(L^0)^d$

Proposition 3.1 *Let $\mathcal{K}$ be an L^0-convex, sequentially closed and bounded subset of $(L^0)^d$, and let $f : \mathcal{K} \to \mathcal{K}$ be a local, sequentially continuous function. Then f has a fixed point.*

Proof Since $\mathcal{K}$ is bounded, there exists a conditional simplex $\mathcal{S}$ such that $\mathcal{K} \subseteq \mathcal{S}$. Now define the function $h : \mathcal{S} \to \mathcal{K}$ by

$$h(X) = \begin{cases} X, & \text{if } X \in \mathcal{K}, \\ \arg\min\{\|X - Y\| : Y \in \mathcal{K}\}, & \text{else.} \end{cases}$$

This means, that h is the identity function on $\mathcal{K}$ and the projection on $\mathcal{K}$ for the elements in $\mathcal{S} \setminus \mathcal{K}$. Due to [1, Corollary 4.5] this minimum exists and is unique. Therefore h is well defined.

We can characterize h by

$$Y = h(X) \quad \Leftrightarrow \quad \langle X - Y, Z - Y\rangle \le 0 \quad \text{for all } Z \in \mathcal{K}. \tag{3.1}$$

Indeed, let $\langle X - Y, Z - Y\rangle \le 0$ for all $Z \in \mathcal{K}$. Then

$$\begin{aligned} \|X - Z\|^2 &= \big\|(X - Y) + (Y - Z)\big\| \\ &= \|X - Y\|^2 + 2\langle X - Y, Y - Z\rangle + \|Y - Z\|^2 \ge \|X - Y\|^2, \end{aligned}$$

which shows the minimizing property of h. On the other hand, let $Y = h(X)$. Since $\mathcal{K}$ is L^0-convex, $\lambda Z + (1-\lambda)Y \in \mathcal{K}$ for any $\lambda \in (0,1](\mathcal{A})$ and $Z \in \mathcal{K}$. By a standard calculation,

$$\left\|X - \bigl(\lambda Z + (1-\lambda)Y\bigr)\right\|^2 \geq \|X - Y\|^2$$

yields $0 \geq -2\lambda\langle X, -Y\rangle + (2\lambda - \lambda^2)\langle Y, Y\rangle + 2\lambda\langle X, Z\rangle - \lambda^2\|Z\|^2 - 2\lambda(1-\lambda)\langle Z, Y\rangle$. Dividing by $\lambda > 0$ and letting $\lambda \downarrow 0$ afterwards yields

$$0 \geq -2\langle X, -Y\rangle + 2\langle Y, Y\rangle + 2\langle X, Z\rangle - 2\langle Z, Y\rangle = 2\langle X - Y, Z - Y\rangle,$$

which is the desired claim.

Furthermore, for any $X, Y \in \mathcal{S}$, it holds that

$$\left\|h(X) - h(Y)\right\| \leq \|X - Y\|.$$

Indeed,

$$X - Y = \bigl(h(X) - h(Y)\bigr) + X - h(X) + h(Y) - Y =: \bigl(h(X) - h(Y)\bigr) + c,$$

which means

$$\|X - Y\|^2 = \left\|h(X) - h(Y)\right\|^2 + \|c\|^2 + 2\bigl\langle c, h(X) - h(Y)\bigr\rangle. \tag{3.2}$$

Since

$$\bigl\langle c, h(X) - h(Y)\bigr\rangle = -\bigl\langle X - h(X), h(Y) - h(X)\bigr\rangle - \bigl\langle Y - h(Y), h(X) - h(Y)\bigr\rangle,$$

by (3.1), it follows that $\langle c, h(X) - h(Y)\rangle \geq 0$. Therefore, $\|X - Y\|^2 \geq \|h(X) - h(Y)\|^2$ by (3.2). This shows that h is sequentially continuous.

The function $f \circ h$ is a sequentially continuous function mapping from $\mathcal{S}$ to $\mathcal{K} \subseteq \mathcal{S}$. Hence, there exists a fixed point $f \circ h(Z) = Z$. Since $f \circ h$ maps into $\mathcal{K}$, this Z has to be in $\mathcal{K}$. But then we know $h(Z) = Z$ and therefore $f(Z) = Z$, which ends the proof. □

Remark 3.2 In Drapeau *et al.* [9] the concept of conditional compactness is introduced and it is shown that there is an equivalence between conditional compactness and conditional closed- and boundedness in $(L^0)^d$. In that context we can formulate the conditional Brouwer fixed point theorem as follows. A sequentially continuous function $f : \mathcal{K} \to \mathcal{K}$ such that $\mathcal{K}$ is a conditionally compact and L^0-convex subset of $(L^0)^d$ has a fixed point.

3.2 Applications in conditional analysis on $(L^0)^d$

Working in $\mathbb{R}^d$, the Brouwer fixed point theorem can be used to prove several topological properties and is even equivalent to some of them. In the theory of $(L^0)^d$, we will show that the conditional Brouwer fixed point theorem has several implications as well.

Define the *unit ball* in $(L^0)^d$ by $\mathcal{B}(d) = \{X \in (L^0)^d : \|X\| \leq 1\}$. Then, by the former theorem, any local, sequentially continuous function $f : \mathcal{B}(d) \to \mathcal{B}(d)$ has a fixed point. The *unit sphere* $\mathcal{S}(d-1)$ is defined as $\mathcal{S}(d-1) = \{X \in (L^0)^d : \|X\| = 1\}$.

Definition 3.3 Let $\mathcal{X}$ and $\mathcal{Y}$ be subsets of $(L^0)^d$. An L^0*-homotopy* of two local, sequentially continuous functions $f,g:\mathcal{X}\to\mathcal{Y}$ is a jointly local, sequentially continuous function $H:\mathcal{X}\times[0,1](\mathcal{A})\to\mathcal{Y}$ such that $H(X,0)=f(X)$ and $H(X,1)=g(X)$. Jointly local means $H(\sum_{j\in\mathbb{N}}1_{A_j}X_j,\sum_{j\in\mathbb{N}}1_{A_j}t_j)=\sum_{j\in\mathbb{N}}1_{A_j}H(X_j,t_j)$ for any partition $(A_j)_{j\in\mathbb{N}}$, $(X_j)_{j\in\mathbb{N}}$ in $\mathcal{X}$ and $(t_j)_{j\in\mathbb{N}}$ in $[0,1](\mathcal{A})$. Sequential continuity of H is therefore $H(X_n,t_n)\to H(X,t)$ whenever $X_n\to X$ and $t_n\to t$ both P-almost surely for $X_n,X\in\mathcal{X}$ and $t_n,t\in[0,1](\mathcal{A})$.

Lemma 3.4 *The identity function of the sphere is not L^0-homotopic to a constant function.*

The proof is a consequence of the following lemma.

Lemma 3.5 *There does not exist a local, sequentially continuous function $f:\mathcal{B}(d)\to\mathcal{S}(d-1)$ which is the identity on $\mathcal{S}(d-1)$.*

Proof Suppose that there is this local, sequentially continuous function f. Define the function $g:\mathcal{S}(d-1)\to\mathcal{S}(d-1)$ by $g(X)=-X$. Then the composition $g\circ f:\mathcal{B}(d)\to\mathcal{B}(d)$, which actually maps to $\mathcal{S}(d-1)$, is local and sequentially continuous. Therefore, this has a fixed point Y which has to be in $\mathcal{S}(d-1)$ since this is the image of $g\circ f$. But we know $f(Y)=Y$ and $g(Y)=-Y$ and hence $g\circ f(Y)=-Y$. Therefore, Y cannot be a fixed point (since $0\notin\mathcal{S}(d-1)$), which is a contradiction. □

It directly follows that the identity on the sphere is not L^0-homotopic to a constant function. In the case $d=1$, we get the following result which is the L^0-module version of an intermediate value theorem.

Lemma 3.6 *Let $X,\overline{X}\in L^0$ with $X\le\overline{X}$ and $[X,\overline{X}]:=\{Z\in L^0:X\le Z\le\overline{X}\}$. Let further $f:[X,\overline{X}]\to L^0$ be a local, sequentially continuous function and $A:=\{\omega:f(X)(\omega)\le f(\overline{X})(\omega)\}$. If Y is in $[1_Af(X)+1_{A^c}f(\overline{X}),1_Af(\overline{X})+1_{A^c}f(X)]$, then there exists $\overline{Y}\in[X,\overline{X}]$ with $f(\overline{Y})=Y$.*

Proof Since f is local, it is sufficient to prove the case for $f(X)\le f(\overline{X})$ which is $A=\Omega$. For the general case, we would consider A and A^c separately, obtain $1_Af(\overline{Y}_1)=1_AY$, $1_{A^c}f(\overline{Y}_2)=1_{A^c}Y$ and by locality we have $f(1_A\overline{Y}_1+1_{A^c}\overline{Y}_2)=Y$. So, suppose that Y is in $[f(X),f(\overline{X})]$ in the rest of the proof.

Let first $f(X)<Y<f(\overline{X})$. Define the function $g:[X,\overline{X}]\to[X,\overline{X}]$ by

$$g(V):=p\bigl(V-f(V)+Y\bigr)\quad\text{with}\quad p(Z)=1_{\{Z\le X\}}X+1_{\{X\le Z\le\overline{X}\}}Z+1_{\{\overline{X}\le Z\}}\overline{X}.$$

Notice that as a sum, product, and composition of local, sequentially continuous functions, g is so as well. Hence, g has a fixed point $\overline{Y}$. If $\overline{Y}=X$, it must hold that $X-f(X)+Y\le X$, which means $Y\le f(X)$, which is a contradiction. If $\overline{Y}=\overline{X}$, it follows that $f(\overline{X})\le Y$, which is also a contradiction. Hence, $\overline{Y}=\overline{Y}-f(\overline{Y})+Y$, which means $f(\overline{Y})=Y$.

If $Y=f(X)$ on B and $Y=f(\overline{X})$ on C, then $f(X)<Y<f(\overline{X})$ on $D:=(B\cup C)^c$. Then we find $\overline{Y}$ such that $f(\overline{Y})=Y$ on D. In total $f(1_BX+1_{C\setminus B}\overline{X}+1_D\overline{Y})=1_Bf(X)+1_{C\setminus B}f(\overline{X})+1_Df(\overline{Y})=Y$. This shows the claim for general Y in $[f(X),f(\overline{X})]$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed in equal parts to the paper. All authors read and approved the final manuscript.

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Endnotes

[a] Let $B_\pi := \{\omega : \lambda_{\pi(1)}(\omega) \geq \lambda_{\pi(2)}(\omega) \geq \cdots \geq \lambda_{\pi(N)}(\omega)\}$, $\pi \in S_N$. This finite collection of measurable sets fulfills $P(\bigcup_{\pi \in S_N} B_\pi) = 1$. We can construct a partition $(A_n)_{n=1,\dots,M}$ such that $A_n \subseteq B_{\pi_n}$ for some $\pi_n \in S_N$ and for all $n = 1, \dots, M$. Such a partition fulfills the required property.

[b] That is bearing exactly the label $1, \dots, N-1$ on A_k.

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References

1. Cheridito, P, Kupper, M, Vogelpoth, N: Conditional analysis on $\mathbb{R}^d$. Preprint (2012)
2. Filipović, D, Kupper, M, Vogelpoth, N: Separation and duality in locally L^0-convex modules. J. Funct. Anal. **256**, 3996-4029 (2009)
3. Guo, T: Relations between some basic results derived from two kinds of topologies for a random locally convex module. J. Funct. Anal. **258**(9), 3024-3047 (2010)
4. Border, KC: Fixed Point Theorems with Applications to Economics and Game Theory. Cambridge University Press, Cambridge (1985)
5. Bharucha-Reid, AT: Fixed point theorems in probabilistic analysis. Bull. Am. Math. Soc. **82**(5), 641-657 (1976)
6. Shahzad, N: Random fixed points of set-valued maps. Nonlinear Anal. **45**, 689-692 (2001)
7. Fierro, R, Martinez, C, Morales, CH: Fixed point theorems for random lower semi-continuous mappings. Fixed Point Theory Appl. **2009**, 584178 (2009)
8. Cheridito, P, Horst, U, Kupper, M, Pirvu, T: Equilibrium pricing in incomplete markets under translation invariant preferences (2011, submitted)
9. Drapeau, S, Jamneshan, A, Karliczek, M, Kupper, M: The algebra of conditional sets and the concepts of conditional topology and compactness. Preprint (2013)

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