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# The combination of the set of solutions of equilibrium problem for convergence theorem of the set of fixed points of strictly pseudo-contractive mappings and variational inequalities problem

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**Abstract**

For the purpose of this article, we introduce a new problem using the concept of equilibrium problem and prove the strong convergence theorem for finding a common element of the set of fixed points of an infinite family of $\kappa_i$-strictly pseudo contractive mappings and of a finite family of the set of solutions of equilibrium problem and variational inequalities problem. Furthermore, we utilize our main theorem for the numerical example.

**Keywords:** strictly pseudo-contractive mapping; *S*-mapping; variational inequality; the combination of equilibrium problem

## 1 Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let $A$ be a strongly positive linear bounded operator on $H$ if there is a constant $\bar{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

We now recall some well-known concepts and results as follows.

**Definition 1.1** Let $B : C \to H$ be a mapping. Then $B$ is called

(i) monotone if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) $\upsilon$-strongly monotone if there exists a positive real number $\upsilon$ such that

$$\langle Bx - By, x - y\rangle \geq \upsilon\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) $\xi$*-inverse-strongly monotone* if there exists a positive real number $\xi$ such that

$$\langle x-y, Bx-By\rangle \geq \xi\|Bx-By\|^2, \quad \forall x,y\in C.$$

**Definition 1.2** Let $T:C\to C$ be a mapping. Then

(i) an element $x\in C$ is said to be a fixed point of $T$ if $Tx=x$ and $\operatorname{Fix}(T)=\{x\in C: Tx=x\}$ denotes the set of fixed points of $T$;

(ii) a mapping $T$ is called nonexpansive if

$$\|Tx-Ty\| \leq \|x-y\|, \quad \forall x,y\in C;$$

(iii) $T$ is said to be $\kappa$*-strictly pseudo-contractive* if there exists a constant $\kappa\in[0,1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + \kappa\big\|(I-T)x-(I-T)y\big\|^2, \quad \forall x,y\in C. \tag{1.1}$$

Note that the class of $\kappa$-strict pseudo-contractions strictly includes the class of nonexpansive mappings.

Fixed point problems arise in many areas such as the vibration of masses attached to strings or nets (see the book by Cheng [1]) and a network bandwidth allocation problem [2] which is one of the central issues in modern communication networks. In applications to neural networks, fixed point theorems can be used to design a dynamic neural network in order to solve steady state solutions [3]. For general information on neural networks, see the books by Robert [4] or by Haykin [5].

Let $G:C\to H$. The *variational inequality problem* is to find a point $u\in C$ such that

$$\langle Gu, v-u\rangle \geq 0 \tag{1.2}$$

for all $v\in C$. The set of solutions of the variational inequality is denoted by $VI(C,G)$.

Variational inequalities were introduced and investigated by Stampacchia [6] in 1964. It is now well known that variational inequalities cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance; see [7–9].

Let $F:C\times C\to\mathbb{R}$ be a bifunction. The equilibrium problem for $F$ is to determine its equilibrium point, *i.e.*, the set

$$EP(F)=\big\{x\in C: F(x,y)\geq 0, \forall y\in C\big\}. \tag{1.3}$$

Equilibrium problems, which were introduced by [10] in 1994, have had a great impact and influence on the development of several branches of pure and applied sciences. Numerous problems in physics, optimization and economics are related to seeking some elements of $EP(F)$; see [10, 11]. Many authors have studied an iterative scheme for the equilibrium problem; see, for example, [11–14].

In 2005, Combettes and Hirstoaga [11] introduced some iterative schemes for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved the strong convergence theorem.

In 2007, Takahashi and Takahashi [14] proved the following theorem.

**Theorem 1.1** *Let $C$ be a nonempty closed convex subset of $H$. Let $F$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying*

(A1) *$F(x,x) = 0$ for all $x \in C$;*

(A2) *$F$ is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;*

(A3) *For each $x,y,z \in C$,*

$$\lim_{t\to 0^+} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) *For each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous;*

*and let $S$ be a nonexpansive mapping of $C$ into $H$ such that $F(S) \cap EP(F) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x\rangle \geq 0, \quad \forall y \in C,$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [0,1]$ satisfy some control conditions. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.*

For $i = 1,2,\ldots,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0,1)$ with $\sum_{i=1}^N a_i = 1$. Define the mapping $\sum_{i=1}^N a_i F_i : C \times C \to \mathbb{R}$. *The combination of equilibrium problem* is to find $x \in C$ such that

$$\left(\sum_{i=1}^N a_i F_i\right)(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions (1.4) is denoted by

$$EP\left(\sum_{i=1}^N a_i F_i\right) = \left\{x \in C : \left(\sum_{i=1}^N a_i F_i\right)(x,y) \geq 0, \forall y \in C\right\}.$$

If $F_i = F$, $\forall i = 1,2,\ldots,N$, then (1.4) reduces to (1.3).

Motivated by Theorem 1.1 and (1.4), we prove the strong convergence theorem for finding a common element of the set of fixed points of an infinite family of $\kappa_i$-strictly pseudo-contractive mappings and a finite family of the set of solutions of equilibrium problem and variational inequalities problem.

## 2 Preliminaries

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. We denote weak convergence and strong convergence by '$\rightharpoonup$' and '$\rightarrow$', respectively. In a real Hilbert space $H$, it is well known that

$$\bigl\|\alpha x - (1-\alpha)y\bigr\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$$

for all $x,y \in H$ and $\alpha \in [0,1]$.

Recall that the (nearest point) projection $P_C$ from $H$ onto $C$ assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

The following lemmas are needed to prove the main theorem.

**Lemma 2.1** [15] *For given $z \in H$ and $u \in C$,*

$$u = P_C z \quad \Leftrightarrow \quad \langle u - z, v - u \rangle \geq 0, \quad \forall v \in C.$$

*It is well known that $P_C$ is a firmly nonexpansive mapping of $H$ onto $C$ and satisfies*

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H.$$

**Lemma 2.2** [16] *Each Hilbert space $H$ satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|$$

*holds for every $y \in H$ with $y \neq x$.*

**Lemma 2.3** [17] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \delta_n, \quad \forall n \geq 0,$$

*where $\alpha_n$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that*
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(2) $\limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
*Then* $\lim_{n \to \infty} s_n = 0$.

**Lemma 2.4** *Let $H$ be a real Hilbert space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle,$$

*for all $x, y \in H$.*

**Lemma 2.5** [15] *Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$, and let $A$ be a mapping of $C$ into $H$. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A) u \quad \Leftrightarrow \quad u \in VI(C, A),$$

*where $P_C$ is the metric projection of $H$ onto $C$.*

**Lemma 2.6** [18] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $S : C \to C$ be a self-mapping of $C$. If $S$ is a $\kappa$-strictly pseudo-contractive mapping, then*

*S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \quad \forall x, y \in C.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that $F$ and $C$ satisfy the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) $F$ is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) For each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) For each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

By using the concept of equilibrium problem, we have Lemma 2.7.

**Lemma 2.7** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying* (A1)-(A4) *with $\bigcap_{i=1}^N EP(F_i) \neq \emptyset$. Then*

$$EP\left(\sum_{i=1}^N a_i F_i\right) = \bigcap_{i=1}^N EP(F_i),$$

*where $a_i \in (0,1)$ for every $i = 1, 2, \ldots, N$ and $\sum_{i=1}^N a_i = 1$.*

*Proof* It is easy to show that $\bigcap_{i=1}^N EP(F_i) \subseteq EP(\sum_{i=1}^N a_i F_i)$.

Let $x_0 \in EP(\sum_{i=1}^N a_i F_i)$ and $x^* \in \bigcap_{i=1}^N EP(F_i)$. Then we have

$$\left(\sum_{i=1}^N a_i F_i\right)(x_0, y) \geq 0, \quad \forall y \in C \tag{2.1}$$

and

$$F_k\bigl(x^*, y\bigr) \geq 0, \quad \text{for all } k = 1, 2, \ldots, N, \text{ and } y \in C. \tag{2.2}$$

From (2.2) and $x_0 \in C$, we have

$$F_k\bigl(x^*, x_0\bigr) \geq 0, \tag{2.3}$$

for all $k = 1, 2, \ldots, N$. From (2.3) and (A2), we obtain

$$F_k\bigl(x_0, x^*\bigr) \leq F_k\bigl(x^*, x_0\bigr) + F_k\bigl(x_0, x^*\bigr) \leq 0. \tag{2.4}$$

Since $x^* \in C$, it follows from (2.1) that

$$\sum_{i=1}^N a_i F_i\bigl(x_0, x^*\bigr) \geq 0. \tag{2.5}$$

Applying (2.5), for each $k = 1, 2, \dots, N$, we obtain

$$\begin{aligned} 0 &\le \sum_{i=1}^{N} a_i F_i(x_0, x^*) \\ &= \sum_{i=1}^{k-1} a_i F_i(x_0, x^*) + a_k F_k(x_0, x^*) + \sum_{i=k+1}^{N} a_i F_i(x_0, x^*). \end{aligned} \tag{2.6}$$

From (2.3), (2.6) and (A2), it follows that

$$\begin{aligned} a_k F_k(x_0, x^*) &\ge -\sum_{i=1}^{k-1} a_i F_i(x_0, x^*) - \sum_{i=k+1}^{N} a_i F_i(x_0, x^*) \\ &\ge \sum_{i=1}^{k-1} a_i F_i(x^*, x_0) + \sum_{i=k+1}^{N} a_i F_i(x^*, x_0) \ge 0. \end{aligned} \tag{2.7}$$

Inequalities (2.7) and (2.4) guarantee that

$$F_k(x_0, x^*) = 0 \quad \text{for every } k = 1, 2, \dots, N. \tag{2.8}$$

By using (2.8) and (A1), deduce that

$$x_0 = x^*.$$

It implies that

$$x_0 \in \bigcap_{i=1}^{N} EP(F_i).$$

Therefore,

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) \subseteq \bigcap_{i=1}^{N} EP(F_i).$$

Hence, we have

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i).$$

□

**Example 2.8** Let $\mathbb{R}$ be the set of real numbers, and let bifunctions $F_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, be defined by

$$\begin{aligned} F_1(x, y) &= -x^2 + y^2, \\ F_2(x, y) &= -2x^2 + xy + y^2, \\ F_3(x, y) &= -\frac{x^2}{2} - 2xy + \frac{5y^2}{2}, \quad \forall x, y \in \mathbb{R}. \end{aligned}$$

It is easy to check that $F_i(x,y)$ satisfy (A1)-(A4) for every $i = 1,2,3$ and

$$\bigcap_{i=1}^{3} EP(F_i) = \{0\}. \tag{2.9}$$

By choosing $a_1 = \frac{1}{12}$, $a_2 = \frac{2}{3}$ and $a_3 = \frac{1}{4}$, we obtain

$$\sum_{i=1}^{3} a_i F_i(x,y) = \frac{1}{24}\left(-37x^2 + 4xy + 33y^2\right). \tag{2.10}$$

From (2.10), we have

$$EP\left(\sum_{i=1}^{3} a_i F_i\right) = \{0\}. \tag{2.11}$$

From (2.9) and (2.11), we obtain

$$EP\left(\sum_{i=1}^{3} a_i F_i\right) = \bigcap_{i=1}^{3} EP(F_i) = \{0\}.$$

**Remark 2.9** By using Lemma 2.7, we can guarantee the result of Example 2.8.

**Lemma 2.10** [10] *Let $C$ be a nonempty closed convex subset of $H$, and let $F$ be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

**Lemma 2.11** [11] *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

*for all $x \in H$. Then the following hold*:

(i) *$T_r$ is single-valued*;

(ii) *$T_r$ is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\left\|T_r(x) - T_r(y)\right\|^2 \leq \left\langle T_r(x) - T_r(y), x - y\right\rangle;$$

(iii) $\mathrm{Fix}(T_r) = EP(F)$;

(iv) *$EP(F)$ is closed and convex.*

**Remark 2.12** From Lemma 2.7, it is easy to see that $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4). By using Lemma 2.11, we obtain

$$\mathrm{Fix}(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$, for each $i = 1,2,\ldots,N$, and $\sum_{i=1}^{N} a_i = 1$.

**Definition 2.1** [19] Let $C$ be a nonempty convex subset of a real Hilbert space. Let $T_i$, $i = 1,2,\dots$, be mappings of $C$ into itself. For each $j = 1,2,\dots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$, we define the mapping $S_n : C \to C$ as follows:

$$
\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I,\\
&U_{n,n-1} = \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I,\\
&\vdots\\
&U_{n,k+1} = \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I,\\
&U_{n,k} = \alpha_1^k T_k U_{n,k+1} + \alpha_2^k U_{n,k+1} + \alpha_3^k I,\\
&\vdots\\
&U_{n,2} = \alpha_1^2 T_2 U_{n,3} + \alpha_2^2 U_{n,3} + \alpha_3^2 I,\\
&S_n = U_{n,1} = \alpha_1^1 T_1 U_{n,2} + \alpha_2^1 U_{n,2} + \alpha_3^1 I.
\end{aligned}
$$

This mapping is called *S-mapping* generated by $T_n, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$.

**Lemma 2.13** [19] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_i\}_{i=1}^\infty$ be $\kappa_i$-strictly pseudo-contractive mappings of $C$ into itself with $\bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$ and $\kappa = \sup_{i\in\mathbb{N}} \kappa_i$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\dots$. For every $n \in \mathbb{N}$, let $S_n$ be an* $S$-mapping *generated by $T_n, T_{n-1}, \dots, T_1$ and $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k}x$ exists.*

For every $k \in \mathbb{N}$ and $x \in C$, Kangtunyakarn [19] defined the mapping $U_{\infty,k}$ and $S : C \to C$ as follows:

$$\lim_{n\to\infty} U_{n,k}x = U_{\infty,k}x$$

and

$$\lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x = Sx.$$

Such a mapping $S$ is called $S$-mapping generated by $T_n, T_{n-1}, \dots$ and $\alpha_n, \alpha_{n-1}, \dots$.

**Remark 2.14** [19] For every $n \in \mathbb{N}$, $S_n$ is nonexpansive and $\lim_{n\to\infty} \sup_{x\in D} \|S_n x - Sx\| = 0$ for every bounded subset $D$ of $C$.

**Lemma 2.15** [19] *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_i\}_{i=1}^\infty$ be $\kappa_i$-strictly pseudo-contractive mappings of $C$ into itself with $\bigcap_{i=1}^\infty \mathrm{Fix}(T_i) \neq \emptyset$ and $\kappa = \sup_{i\in\mathbb{N}} \kappa_i$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1,2,\dots$. For every $n \in \mathbb{N}$, let $S_n$ and $S$ be* $S$-mappings *generated by* $T_n, T_{n-1}, \dots, T_1$ *and* $\alpha_n, \alpha_{n-1}, \dots, \alpha_1$ *and* $T_n, T_{n-1}, \dots$ *and* $\alpha_n, \alpha_{n-1}, \dots$, *respectively. Then* $\operatorname{Fix}(S) = \bigcap_{i=1}^{\infty} \operatorname{Fix}(T_i)$.

**Lemma 2.16** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For every $i = 1, 2, \dots, N$, let $A_i$ be a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\dots,N} \gamma_i$. Let $\{a_i\}_{i=1}^N \subseteq (0,1)$ with $\sum_{i=1}^N a_i = 1$. Then the following properties hold*:

(i) $\|I - \rho \sum_{i=1}^N a_i A_i\| \leq 1 - \rho\bar{\gamma}$ *and* $I - \rho \sum_{i=1}^N a_i A_i$ *is a nonexpansive mapping for every* $0 < \rho < \|A_i\|^{-1}$ $(i = 1, 2, \dots, N)$.

(ii) $VI(C, \sum_{i=1}^N a_i A_i) = \bigcap_{i=1}^N VI(C, A_i)$.

*Proof* To show (i), it is obvious that $I - \rho \sum_{i=1}^N a_i A_i$ is a positive linear bounded operator on $H$, which yields that

$$\left\| I - \rho \sum_{i=1}^N a_i A_i \right\| = \sup\left\{ \left| \left\langle I - \rho \sum_{i=1}^N a_i A_i x, x \right\rangle \right| : x \in H, \|x\| = 1 \right\}. \tag{2.12}$$

Since $A_i$ is a strongly positive operator for all $i = 1, 2, \dots, N$, we get

$$\begin{aligned} \left\langle \sum_{i=1}^N a_i A_i x, x \right\rangle &= \sum_{i=1}^N a_i \langle A_i x, x\rangle \\ &\geq \sum_{i=1}^N a_i \gamma_i \|x\|^2 \\ &\geq \sum_{i=1}^N a_i \bar{\gamma} \|x\|^2 \\ &= \bar{\gamma} \|x\|^2, \end{aligned} \tag{2.13}$$

which implies that $\sum_{i=1}^N a_i A_i$ is a $\bar{\gamma}$-strongly positive operator.

Let $\|x\| = 1$. Then, by using (2.13), we obtain

$$\begin{aligned} \left\langle \left( I - \rho \sum_{i=1}^N a_i A_i \right) x, x \right\rangle &= \left\langle x - \rho \sum_{i=1}^N a_i A_i x, x \right\rangle \\ &= \|x\|^2 - \rho \left\langle \sum_{i=1}^N a_i A_i x, x \right\rangle \\ &\leq (1 - \rho\bar{\gamma}) \|x\|^2 \\ &= 1 - \rho\bar{\gamma}. \end{aligned} \tag{2.14}$$

From (2.12) and (2.14), we have

$$\left\| I - \rho \sum_{i=1}^N a_i A_i \right\| \leq 1 - \rho\bar{\gamma}. \tag{2.15}$$

Next, we show that $I-\rho\sum_{i=1}^N a_iA_i$ is a nonexpansive mapping. Let $x,y\in C$. Then, using (2.15), we obtain

$$\begin{aligned}\left\|\left(I-\rho\sum_{i=1}^N a_iA_i\right)x-\left(I-\rho\sum_{i=1}^N a_iA_i\right)y\right\| &= \left\|\left(I-\rho\sum_{i=1}^N a_iA_i\right)(x-y)\right\| \\ &\leq \left\|\left(I-\rho\sum_{i=1}^N a_iA_i\right)\right\|\|x-y\| \\ &\leq (1-\rho\bar{\gamma})\|x-y\| \\ &\leq \|x-y\|.\end{aligned}$$

Hence, $I-\rho\sum_{i=1}^N a_iA_i$ is a nonexpansive mapping.

To prove (ii), it is easy to see that

$$\bigcap_{i=1}^N VI(C,A_i)\subseteq VI\left(C,\sum_{i=1}^N a_iA_i\right). \tag{2.16}$$

Let $x_0\in VI(C,\sum_{i=1}^N a_iA_i)$ and $x^*\in\bigcap_{i=1}^N VI(C,A_i)$. Then we have

$$\left\langle y-x_0,\sum_{i=1}^N a_iA_ix_0\right\rangle\geq 0,\quad \forall y\in C. \tag{2.17}$$

From (2.16), we have $x^*\in VI(C,\sum_{i=1}^N a_iA_i)$. It implies that

$$\left\langle y-x^*,\sum_{i=1}^N a_iA_ix^*\right\rangle\geq 0,\quad \forall y\in C. \tag{2.18}$$

From (2.17), (2.18) and $x^*,x_0\in C$, we obtain

$$\left\langle x^*-x_0,\sum_{i=1}^N a_iA_ix_0\right\rangle\geq 0 \tag{2.19}$$

and

$$\left\langle x_0-x^*,\sum_{i=1}^N a_iA_ix^*\right\rangle\geq 0. \tag{2.20}$$

By summing up (2.19) and (2.20), we have

$$\begin{aligned}0 &\leq \left\langle x_0-x^*,\sum_{i=1}^N a_iA_ix^*-\sum_{i=1}^N a_iA_ix_0\right\rangle \\ &= \sum_{i=1}^N a_i\langle x_0-x^*,A_ix^*-A_ix_0\rangle \\ &\leq -\sum_{i=1}^N a_i\gamma_i\|x_0-x^*\|^2\end{aligned}$$

$$\leq -\sum_{i=1}^{N} a_i \bar{\gamma} \|x_0 - x^*\|^2$$
$$= -\bar{\gamma} \|x_0 - x^*\|^2.$$

It implies that $x_0 = x^*$.

Then we can conclude that $x_0 \in \bigcap_{i=1}^{N} VI(C, A_i)$. Therefore

$$VI\left(C, \sum_{i=1}^{N} a_i A_i\right) \subseteq \bigcap_{i=1}^{N} VI(C, A_i).$$

Hence, we have

$$VI\left(C, \sum_{i=1}^{N} a_i A_i\right) = \bigcap_{i=1}^{N} VI(C, A_i).$$

□

## 3 Main result

**Theorem 3.1** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4), *and let $A_i : C \to H$ be a strongly positive linear bounded operator on $H$ with coefficient $\gamma_i > 0$ and $\bar{\gamma} = \min_{i=1,2,\ldots,N} \gamma_i$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of $\kappa_i$-strictly pseudo-contractive mappings of $C$ into itself, and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p, q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $S_n$ and $S$ be the S-mappings generated by $T_n, T_{n-1}, \ldots, T_1$ and $\sigma_n, \sigma_{n-1}, \ldots, \sigma_1$ and $T_n, T_{n-1}, \ldots$ and $\sigma_n, \sigma_{n-1}, \ldots$, respectively. Assume that $\mathbb{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(C, A_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n) S_n x_n) + (1-\beta_n) P_C(I - \rho_n \sum_{i=1}^{N} b_i A_i) u_n, & \forall n \geq 1, \end{cases} \tag{3.1}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ and $0 \leq a_i, b_i \leq 1$, for every $i = 1, 2, \ldots, N$, satisfy the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $0 < a \leq \beta_n \leq b < 1$, $\forall n \in \mathbb{N}$,
(iii) $\lim_{n\to\infty} \rho_n = 0$,
(iv) $0 < c \leq r_n \leq d < 1$, $\forall n \in \mathbb{N}$,
(v) $\sum_{i=1}^{N} a_i = \sum_{i=1}^{N} b_i = 1$,
(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_1^n < \infty$.

*Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathbb{F}} u$.*

*Proof* Since $\rho_n \to 0$ as $n \to \infty$, without loss of generality, we may assume that $\rho_n < \frac{1}{\|A_i\|}$, $\forall n \in \mathbb{N}$ and $i = 1, 2, \ldots, N$.

The proof will be divided into five steps.

Step 1. We will show that $\{x_n\}$ is bounded.

Since $\sum_{i=1}^N a_iF_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^N a_iF_i(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y\in C,$$

by Lemma 2.11 and Remark 2.12, we have $u_n = T_{r_n}x_n$ and $\operatorname{Fix}(T_{r_n}) = \bigcap_{i=1}^N EP(F_i)$.

Let $z\in\mathbb{F}$. Since $z\in\bigcap_{i=1}^N VI(C,A_i)$, by Lemma 2.5 and Lemma 2.16, we have

$$VI\left(C,\sum_{i=1}^N b_iA_i\right) = F\left(P_C\left(I-\rho_n\sum_{i=1}^N b_iA_i\right)\right).$$

From Lemma 2.16 and nonexpansiveness of $T_{r_n}$, we have

$$\begin{aligned}
&\|x_{n+1}-z\| \\
&\quad= \left\|\beta_n\big(\alpha_nu+(1-\alpha_n)S_nx_n\big)+(1-\beta_n)P_C\left(I-\rho_n\sum_{i=1}^N b_iA_i\right)u_n - z\right\| \\
&\quad= \left\|\beta_n\big(\alpha_nu+(1-\alpha_n)S_nx_n - z\big)+(1-\beta_n)\left(P_C\left(I-\rho_n\sum_{i=1}^N b_iA_i\right)u_n - z\right)\right\| \\
&\quad\leq \beta_n\big\|\alpha_n(u-z)+(1-\alpha_n)(S_nx_n-z)\big\| + (1-\beta_n)\left\|P_C\left(I-\rho_n\sum_{i=1}^N b_iA_i\right)u_n - z\right\| \\
&\quad\leq \beta_n\big(\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|\big)+(1-\beta_n)\|u_n-z\| \\
&\quad= \beta_n\big(\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|\big)+(1-\beta_n)\|T_{r_n}x_n-z\| \\
&\quad\leq \beta_n\big(\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|\big)+(1-\beta_n)\|x_n-z\| \\
&\quad\leq \max\big\{\|u-z\|,\|x_1-z\|\big\}.
\end{aligned}$$

By induction on $n$, for some $M>0$, we have $\|x_n-z\|\leq M$, $\forall n\in\mathbb{N}$. It implies that $\{x_n\}$ is bounded and so $\{u_n\}$ is a bounded sequence.

Step 2. We will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Putting $D=\sum_{i=1}^N b_iA_i$, from the definition of $x_n$, we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad= \big\|\beta_n\big(\alpha_nu+(1-\alpha_n)S_nx_n\big)+(1-\beta_n)P_C(I-\rho_nD)u_n \\
&\qquad - \big(\beta_{n-1}\big(\alpha_{n-1}u+(1-\alpha_{n-1})S_{n-1}x_{n-1}\big)+(1-\beta_{n-1})P_C(I-\rho_{n-1}D)u_{n-1}\big)\big\| \\
&\quad= \big\|\beta_n\big(\alpha_nu+(1-\alpha_n)S_nx_n\big)-\beta_n\big(\alpha_{n-1}u+(1-\alpha_{n-1})S_{n-1}x_{n-1}\big) \\
&\qquad + \beta_n\big(\alpha_{n-1}u+(1-\alpha_{n-1})S_{n-1}x_{n-1}\big)-\beta_{n-1}\big(\alpha_{n-1}u+(1-\alpha_{n-1})S_{n-1}x_{n-1}\big) \\
&\qquad + (1-\beta_n)P_C(I-\rho_nD)u_n-(1-\beta_n)P_C(I-\rho_nD)u_{n-1} \\
&\qquad + (1-\beta_n)P_C(I-\rho_nD)u_{n-1}-(1-\beta_n)P_C(I-\rho_{n-1}D)u_{n-1} \\
&\qquad + (1-\beta_n)P_C(I-\rho_{n-1}D)u_{n-1}-(1-\beta_{n-1})P_C(I-\rho_{n-1}D)u_{n-1}\big\| \\
&\quad\leq \beta_n\big[|\alpha_n-\alpha_{n-1}|\|u\| + \big\|(1-\alpha_n)S_nx_n-(1-\alpha_n)S_nx_{n-1}+(1-\alpha_n)S_nx_{n-1}
\end{aligned}$$

$$\begin{aligned}
&- (1-\alpha_n)S_{n-1}x_{n-1} + (1-\alpha_n)S_{n-1}x_{n-1} - (1-\alpha_{n-1})S_{n-1}x_{n-1}\big\|\big] \\
&+ |\beta_n - \beta_{n-1}|\big[\alpha_{n-1}\|u\| + (1-\alpha_{n-1})\|S_{n-1}x_{n-1}\|\big] \\
&+ (1-\beta_n)\big\|P_C(I-\rho_n D)u_n - P_C(I-\rho_n D)u_{n-1}\big\| \\
&+ (1-\beta_n)\big\|P_C(I-\rho_n D)u_{n-1} - P_C(I-\rho_{n-1} D)u_{n-1}\big\| \\
&+ |\beta_n - \beta_{n-1}|\big\|P_C(I-\rho_{n-1}D)u_{n-1}\big\| \\
\le{}& \beta_n\big[|\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|S_n x_n - S_n x_{n-1}\| \\
&+ (1-\alpha_n)\|S_n x_{n-1} - S_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_{n-1}x_{n-1}\|\big] \\
&+ |\beta_n - \beta_{n-1}|\big[\alpha_{n-1}\|u\| + (1-\alpha_{n-1})\|S_{n-1}x_{n-1}\|\big] \\
&+ (1-\beta_n)\|u_n - u_{n-1}\| + (1-\beta_n)\big\|(I-\rho_n D)u_{n-1} - (I-\rho_{n-1}D)u_{n-1}\big\| \\
&+ |\beta_n - \beta_{n-1}|\big\|P_C(I-\rho_{n-1}D)u_{n-1}\big\| \\
\le{}& \beta_n\big[|\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\
&+ (1-\alpha_n)\|S_n x_{n-1} - S_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|S_{n-1}x_{n-1}\|\big] \\
&+ |\beta_n - \beta_{n-1}|\big[\alpha_{n-1}\|u\| + (1-\alpha_{n-1})\|S_{n-1}x_{n-1}\|\big] \\
&+ (1-\beta_n)\|u_n - u_{n-1}\| + (1-\beta_n)|\rho_n - \rho_{n-1}|\|Du_{n-1}\| \\
&+ |\beta_n - \beta_{n-1}|\big\|P_C(I-\rho_{n-1}D)u_{n-1}\big\|.
\end{aligned} \tag{3.2}$$

By using the same method as in step 2 of Theorem 3.1 in [20], we have

$$\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \le \alpha_1^n \frac{2}{1-\kappa}\|x_{n-1} - z\|. \tag{3.3}$$

Since $u_n = T_{r_n}x_n$, by utilizing the definition of $T_{r_n}$, we obtain

$$\sum_{i=1}^N a_i F_i(T_{r_n}x_n, y) + \frac{1}{r_n}\langle y - T_{r_n}x_n, T_{r_n}x_n - x_n\rangle \ge 0, \quad \forall y \in C, \tag{3.4}$$

and

$$\sum_{i=1}^N a_i F_i(T_{r_{n+1}}x_{n+1}, y) + \frac{1}{r_{n+1}}\langle y - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ge 0, \quad \forall y \in C. \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\sum_{i=1}^N a_i F_i(T_{r_n}x_n, T_{r_{n+1}}x_{n+1}) + \frac{1}{r_n}\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\rangle \ge 0, \tag{3.6}$$

and

$$\sum_{i=1}^N a_i F_i(T_{r_{n+1}}x_{n+1}, T_{r_n}x_n) + \frac{1}{r_{n+1}}\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \ge 0. \tag{3.7}$$

From (3.6) and (3.7) and the fact that $\sum_{i=1}^{N} a_i F_i$ satisfies (A2), we have

$$\frac{1}{r_n}\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\rangle + \frac{1}{r_{n+1}}\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\rangle \geq 0,$$

which implies that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \geq 0.$$

It follows that

$$\begin{aligned}&\Big\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n \\ &\quad - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1} - x_{n+1})\Big\rangle \geq 0. \end{aligned} \tag{3.8}$$

From (3.8), we obtain

$$\begin{aligned}&\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|^2 \\ &\quad \leq \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle \\ &\quad = \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\rangle \\ &\quad \leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left\|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(T_{r_{n+1}}x_{n+1} - x_{n+1})\right\| \\ &\quad \leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}x_{n+1} - x_{n+1}\|\right] \\ &\quad = \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|T_{r_{n+1}}x_{n+1} - x_{n+1}\|\right] \\ &\quad \leq \|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\|\left[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n|\|T_{r_{n+1}}x_{n+1} - x_{n+1}\|\right],\end{aligned}$$

which yields that

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\|. \tag{3.9}$$

From (3.9), we have

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}|\|u_n - x_n\|. \tag{3.10}$$

By substituting (3.3) and (3.10) into (3.2), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| \leq \beta_n\Big[&|\alpha_n - \alpha_{n-1}|\|u\| + (1-\alpha_n)\|x_n - x_{n-1}\| \\ &+ (1-\alpha_n)\alpha_1^n\frac{2}{1-\kappa}\|x_{n-1} - z\| + |\alpha_n - \alpha_{n-1}|\|S_{n-1}x_{n-1}\|\Big]\end{aligned}$$

$$
\begin{aligned}
&\quad + |\beta_n - \beta_{n-1}|\big[\alpha_{n-1}\|u\| + (1-\alpha_{n-1})\|S_{n-1}x_{n-1}\|\big] \\
&\quad + (1-\beta_n)\bigg[\|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}|\|u_n - x_n\|\bigg] \\
&\quad + (1-\beta_n)|\rho_n - \rho_{n-1}|\|Du_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\big\|P_C(I-\rho_{n-1}D)u_{n-1}\big\| \\
&\leq \beta_n(1-\alpha_n)\|x_n - x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|u\| + \alpha_1^n\frac{2}{1-\kappa}\|x_{n-1} - z\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\|S_{n-1}x_{n-1}\| + |\beta_n - \beta_{n-1}|\big[\|u\| + \|S_{n-1}x_{n-1}\|\big] \\
&\quad + \frac{1}{d}|r_n - r_{n-1}|\|u_n - x_n\| + |\rho_n - \rho_{n-1}|\|Du_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}|\big\|P_C(I-\rho_{n-1}D)u_{n-1}\big\| \\
&\leq (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|K + \alpha_1^n\frac{2}{1-\kappa}K \\
&\quad + |\alpha_n - \alpha_{n-1}|K + |\beta_n - \beta_{n-1}|2K + \frac{1}{d}|r_n - r_{n-1}|K \\
&\quad + |\rho_n - \rho_{n-1}|K + |\beta_n - \beta_{n-1}|K \\
&= (1-\alpha_n\beta_n)\|x_n - x_{n-1}\| + 2K|\alpha_n - \alpha_{n-1}| + \frac{2K}{1-\kappa}\alpha_1^n \\
&\quad + 3K|\beta_n - \beta_{n-1}| + \frac{K}{d}|r_n - r_{n-1}| + K|\rho_n - \rho_{n-1}|, \qquad (3.11)
\end{aligned}
$$

where $K = \max_{n\in\mathbb{N}}\{\|u\|, \|x_n - z\|, \|S_nx_n\|, \|u_n - x_n\|, \|Du_n\|, \|P_C(I-\rho_nD)u_n\|\}$. From (3.11), conditions (i), (ii), (vi) and Lemma 2.3, we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.12)$$

Step 3. We will show that $\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|P_C(I-\rho_nD)x_n - x_n\| = \lim_{n\to\infty}\|S_nx_n - x_n\| = 0$, where $D = \sum_{n=1}^N b_iA_i$.

To show this, let $z \in \mathbb{F}$. Since $u_n = T_{r_n}x_n$ and $T_{r_n}$ is a firmly nonexpansive mapping, then we obtain

$$
\begin{aligned}
\|z - T_{r_n}x_n\|^2 &= \|T_{r_n}z - T_{r_n}x_n\|^2 \leq \langle T_{r_n}z - T_{r_n}x_n, z - x_n\rangle \\
&= \frac{1}{2}\big(\|T_{r_n}x_n - z\|^2 + \|x_n - z\|^2 - \|T_{r_n}x_n - x_n\|^2\big),
\end{aligned}
$$

which yields that

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \qquad (3.13)$$

By nonexpansiveness of $P_C(I-\rho_nD)$, (3.13) and the definition of $x_n$, we have

$$
\begin{aligned}
&\|x_{n+1} - z\|^2 \\
&\quad = \big\|\beta_n\big(\alpha_nu + (1-\alpha_n)S_nx_n\big) + (1-\beta_n)P_C(I-\rho_nD)u_n - z\big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \beta_n\big(\alpha_n(u-z) + (1-\alpha_n)(S_nx_n - z)\big) + (1-\beta_n)\big(P_C(I-\rho_nD)u_n - z\big)\right\|^2 \\
&\le \beta_n \left\| \alpha_n(u-z) + (1-\alpha_n)(S_nx_n - z)\right\|^2 + (1-\beta_n)\left\|P_C(I-\rho_nD)u_n - z\right\|^2 \\
&\le \beta_n\big(\alpha_n\|u-z\|^2 + (1-\alpha_n)\|S_nx_n - z\|^2\big) + (1-\beta_n)\|u_n - z\|^2 \\
&\le \beta_n\big(\alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2\big) + (1-\beta_n)\big(\|x_n - z\|^2 - \|u_n - x_n\|^2\big) \\
&\le \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - (1-\beta_n)\|u_n - x_n\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(1-\beta_n)\|u_n - x_n\|^2 &\le \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\le \alpha_n\|u-z\|^2 + \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\|. \qquad (3.14)
\end{aligned}
$$

By (3.12), (3.14), conditions (i) and (ii), we have

$$
\lim_{n\to\infty} \|u_n - x_n\| = 0. \qquad (3.15)
$$

Put $w_n = \alpha_n u + (1-\alpha_n)S_nx_n$. By the definition of $x_n$ and $z \in \mathbb{F}$, we obtain

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\beta_n w_n + (1-\beta_n)P_C(I-\rho_nD)u_n - z\right\|^2 \\
&= \left\|\beta_n(w_n - z) + (1-\beta_n)\big(P_C(I-\rho_nD)u_n - z\big)\right\|^2 \\
&= \beta_n\|w_n - z\|^2 + (1-\beta_n)\left\|P_C(I-\rho_nD)u_n - z\right\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|w_n - P_C(I-\rho_nD)u_n\right\|^2 \\
&\le \beta_n\left\|\alpha_n u + (1-\alpha_n)S_nx_n - z\right\|^2 + (1-\beta_n)\|u_n - z\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|w_n - P_C(I-\rho_nD)u_n\right\|^2 \\
&\le \beta_n\big(\alpha_n\|u-z\|^2 + (1-\alpha_n)\|S_nx_n - z\|^2\big) + (1-\beta_n)\|x_n - z\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|w_n - P_C(I-\rho_nD)u_n\right\|^2 \\
&\le \beta_n\big(\alpha_n\|u-z\|^2 + (1-\alpha_n)\|x_n - z\|^2\big) + (1-\beta_n)\|x_n - z\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|w_n - P_C(I-\rho_nD)u_n\right\|^2 \\
&\le \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \beta_n(1-\beta_n)\left\|w_n - P_C(I-\rho_nD)u_n\right\|^2,
\end{aligned}
$$

which yields that

$$
\begin{aligned}
&\beta_n(1-\beta_n)\left\|w_n - P_C(I-\rho_nD)u_n\right\|^2 \\
&\quad \le \alpha_n\|u-z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\quad \le \alpha_n\|u-z\|^2 + \big(\|x_n - z\| + \|x_{n+1} - z\|\big)\|x_{n+1} - x_n\|. \qquad (3.16)
\end{aligned}
$$

By (3.12) and conditions (i) and (ii), we have

$$
\lim_{n\to\infty} \left\|w_n - P_C(I-\rho_nD)u_n\right\| = 0. \qquad (3.17)
$$

By the definition of $x_n$, we obtain

$$x_{n+1} - P_C(I - \rho_n D)u_n = \beta_n\bigl(w_n - P_C(I - \rho_n D)u_n\bigr). \tag{3.18}$$

By (3.18) and the definition of $x_n$, we have

$$\begin{aligned}\bigl\|x_n - P_C(I - \rho_n D)x_n\bigr\| &\le \|x_n - x_{n+1}\| + \bigl\|x_{n+1} - P_C(I - \rho_n D)u_n\bigr\| \\ &\quad + \bigl\|P_C(I - \rho_n D)u_n - P_C(I - \rho_n D)x_n\bigr\| \\ &\le \|x_n - x_{n+1}\| + \beta_n\bigl\|w_n - P_C(I - \rho_n D)u_n\bigr\| \\ &\quad + \|u_n - x_n\|.\end{aligned}$$

From (3.12), (3.15) and (3.17), we have

$$\lim_{n\to\infty}\bigl\|x_n - P_C(I - \rho_n D)x_n\bigr\| = 0. \tag{3.19}$$

Since

$$\begin{aligned}&\bigl\|x_n - P_C(I - \rho_n D)u_n\bigr\| \\ &\quad \le \bigl\|x_n - P_C(I - \rho_n D)x_n\bigr\| + \bigl\|P_C(I - \rho_n D)x_n - P_C(I - \rho_n D)u_n\bigr\| \\ &\quad \le \bigl\|x_n - P_C(I - \rho_n D)x_n\bigr\| + \|x_n - u_n\|,\end{aligned}$$

by using (3.15) and (3.19), we have

$$\lim_{n\to\infty}\bigl\|x_n - P_C(I - \rho_n D)u_n\bigr\| = 0. \tag{3.20}$$

By the definition of $x_n$, we obtain

$$\begin{aligned}&x_{n+1} - x_n \\ &\quad = \beta_n\bigl(\alpha_n u + (1 - \alpha_n)S_n x_n\bigr) + (1 - \beta_n)P_C(I - \rho_n D)u_n - x_n \\ &\quad = \beta_n\bigl(\alpha_n u + (1 - \alpha_n)S_n x_n - x_n\bigr) + (1 - \beta_n)\bigl(P_C(I - \rho_n D)u_n - x_n\bigr) \\ &\quad = \beta_n\bigl(\alpha_n(u - x_n) + (1 - \alpha_n)(S_n x_n - x_n)\bigr) + (1 - \beta_n)\bigl(P_C(I - \rho_n D)u_n - x_n\bigr) \\ &\quad = \alpha_n\beta_n(u - x_n) + \beta_n(1 - \alpha_n)(S_n x_n - x_n) + (1 - \beta_n)\bigl(P_C(I - \rho_n D)u_n - x_n\bigr),\end{aligned}$$

from which it follows that

$$\begin{aligned}&\beta_n(1 - \alpha_n)\|S_n x_n - x_n\| \\ &\quad \le \alpha_n\beta_n\|u - x_n\| + \|x_{n+1} - x_n\| + (1 - \beta_n)\bigl\|P_C(I - \rho_n D)u_n - x_n\bigr\|.\end{aligned}$$

From (3.12), (3.20), conditions (i) and (ii), we have

$$\lim_{n\to\infty}\|S_n x_n - x_n\| = 0. \tag{3.21}$$

Step 4. We will show that $\limsup_{n\to\infty}\langle u-z, x_n-z\rangle \le 0$, where $z = P_{\mathbb{F}}u$.

To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z, x_n-z\rangle = \lim_{k\to\infty}\langle u-z, x_{n_k}-z\rangle. \tag{3.22}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k\to\infty$, where $\omega\in C$. From (3.15), we obtain $u_{n_k}\rightharpoonup\omega$ as $k\to\infty$.

Assume that $\omega \notin \bigcap_{i=1}^N VI(C,A_i)$. Since $\bigcap_{i=1}^N VI(C,A_i) = F(P_C(I-\rho_{n_k}D))$, we have $\omega \neq P_C(I-\rho_{n_k}D)\omega$, where $D = \sum_{i=1}^N b_iA_i$.

By nonexpansiveness of $P_C(I-\rho_{n_k}D)$, (3.19) and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\|x_{n_k}-\omega\| &< \liminf_{k\to\infty}\bigl\|x_{n_k}-P_C(I-\rho_{n_k}D)\omega\bigr\| \\
&\le \liminf_{k\to\infty}\bigl\|x_{n_k}-P_C(I-\rho_{n_k}D)x_{n_k}\bigr\| \\
&\quad + \liminf_{k\to\infty}\bigl\|P_C(I-\rho_{n_k}D)x_{n_k}-P_C(I-\rho_{n_k}D)\omega\bigr\| \\
&\le \liminf_{k\to\infty}\|x_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in\bigcap_{i=1}^N VI(C,A_i). \tag{3.23}$$

Next, we will show that $\omega\in\bigcap_{i=1}^\infty \mathrm{Fix}(T_i)$.

By Lemma 2.15, we have $\mathrm{Fix}(S) = \bigcap_{i=1}^\infty \mathrm{Fix}(T_i)$. Assume that $\omega\neq S\omega$. Using Opial's condition, (3.21) and Remark 2.14, we obtain

$$\begin{aligned}
&\liminf_{k\to\infty}\|x_{n_k}-\omega\| \\
&\quad < \liminf_{k\to\infty}\|x_{n_k}-S\omega\| \\
&\quad \le \liminf_{k\to\infty}\bigl(\|x_{n_k}-S_{n_k}x_{n_k}\| + \|S_{n_k}x_{n_k}-S_{n_k}\omega\| + \|S_{n_k}\omega-S\omega\|\bigr) \\
&\quad \le \liminf_{k\to\infty}\|x_{n_k}-\omega\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega\in\mathrm{Fix}(S) = \bigcap_{i=1}^\infty \mathrm{Fix}(T_i). \tag{3.24}$$

Next, we will show that $\omega\in\bigcap_{i=1}^N EP(F_i)$.

Since

$$\sum_{i=1}^N a_iF_i(u_n,y) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \ge 0, \quad \forall y\in C,$$

and $\sum_{i=1}^{N} a_i F_i$ satisfies conditions (A1)-(A4), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \sum_{i=1}^{N} a_i F_i(y, u_n), \quad \forall y \in C.$$

In particular, it follows that

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \geq \sum_{i=1}^{N} a_i F_i(y, u_{n_k}), \quad \forall y \in C. \tag{3.25}$$

From (3.15), (3.25) and (A4), we have

$$\sum_{i=1}^{N} a_i F_i(y, \omega) \leq 0, \quad \forall y \in C. \tag{3.26}$$

Put $y_t := ty + (1-t)\omega$, $t \in (0,1]$, we have $y_t \in C$. By using (A1), (A4) and (3.26), we have

$$\begin{aligned}
0 &= \sum_{i=1}^{N} a_i F_i(y_t, y_t) \\
&= \sum_{i=1}^{N} a_i F_i\bigl(y_t, ty + (1-t)\omega\bigr) \\
&\leq t\sum_{i=1}^{N} a_i F_i(y_t, y) + (1-t)\sum_{i=1}^{N} a_i F_i(y_t, \omega) \\
&\leq t\sum_{i=1}^{N} a_i F_i(y_t, y).
\end{aligned}$$

It implies that

$$\sum_{i=1}^{N} a_i F_i\bigl(ty + (1-t)\omega, y\bigr) \geq 0, \quad \forall t \in (0,1] \text{ and } \forall y \in C. \tag{3.27}$$

From (3.27), taking $t \to 0^+$ and using (A3), we can conclude that

$$0 \leq \sum_{i=1}^{N} a_i F_i(\omega, y), \quad \forall y \in C.$$

Therefore, $\omega \in EP(\sum_{i=1}^{N} a_i F_i)$. By Lemma 2.7, we obtain $EP(\sum_{i=1}^{N} a_i F_i) = \bigcap_{i=1}^{N} EP(F_i)$. It follows that

$$\omega \in \bigcap_{i=1}^{N} EP(F_i). \tag{3.28}$$

From (3.23), (3.24) and (3.28), we can deduce that $\omega \in \mathbb{F}$.

Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \mathbb{F}$, then, by Lemma 2.1, we can conclude that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, x_n - z\rangle &= \lim_{k\to\infty}\langle u - z, x_{n_k} - z\rangle \\ &= \langle u - z, \omega - z\rangle \\ &\leq 0. \end{aligned} \tag{3.29}$$

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathbb{F}}u$.
By nonexpansiveness of $S_n$ and $P_C(I - \rho_n D)$, we have

$$\begin{aligned}&\|x_{n+1} - z\|^2 \\ &\quad = \big\|\beta_n\big(\alpha_n u + (1-\alpha_n)S_n x_n\big) + (1-\beta_n)P_C(I-\rho_n D)u_n - z\big\|^2 \\ &\quad = \big\|\beta_n\big(\alpha_n(u - z) + (1-\alpha_n)(S_n x_n - z)\big) + (1-\beta_n)\big(P_C(I-\rho_n D)u_n - z\big)\big\|^2 \\ &\quad = \big\|\alpha_n\beta_n(u-z) + \beta_n(1-\alpha_n)(S_n x_n - z) + (1-\beta_n)\big(P_C(I-\rho_n D)u_n - z\big)\big\|^2 \\ &\quad \leq \big\|\beta_n(1-\alpha_n)(S_n x_n - z) + (1-\beta_n)\big(P_C(I-\rho_n D)u_n - z\big)\big\|^2 \\ &\qquad + 2\alpha_n\beta_n\langle u - z, x_{n+1} - z\rangle \\ &\quad \leq \big(\beta_n(1-\alpha_n)\|x_n - z\| + (1-\beta_n)\|u_n - z\|\big)^2 + 2\alpha_n\beta_n\langle u - z, x_{n+1} - z\rangle \\ &\quad \leq \big(\beta_n(1-\alpha_n)\|x_n - z\| + (1-\beta_n)\|x_n - z\|\big)^2 + 2\alpha_n\beta_n\langle u - z, x_{n+1} - z\rangle \\ &\quad = (1-\alpha_n\beta_n)^2\|x_n - z\|^2 + 2\alpha_n\beta_n\langle u - z, x_{n+1} - z\rangle \\ &\quad \leq (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n\langle u - z, x_{n+1} - z\rangle.\end{aligned}$$

From (3.29), conditions (i), (ii) and Lemma 2.3, we can conclude that $\{x_n\}$ converges strongly to $z = P_{\mathbb{F}}u$. By (3.15), we have $\{u_n\}$ converges strongly to $z = P_{\mathbb{F}}u$. This completes the proof. □

## 4 Application

In this section, we apply our main theorem to prove strong convergence theorems involving equilibrium problem, variational inequality problem and fixed point problem.

**Theorem 4.1** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. For $i = 1, 2, \ldots, N$, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4), *and let $A : C \to H$ be a strongly positive linear bounded operator on $H$ with coefficient $\gamma > 0$. Let $\{T_i\}_{i=1}^{\infty}$ be an infinite family of $\kappa_i$-strictly pseudo-contractive mappings of $C$ into itself, and let $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p,q] \subset (\kappa, 1)$ for all $j \in \mathbb{N}$. For every $n \in \mathbb{N}$, let $S_n$ and $S$ be the S-mappings generated by $T_n, T_{n-1}, \ldots, T_1$ and $\sigma_n, \sigma_{n-1}, \ldots, \sigma_1$ and $T_n, T_{n-1}, \ldots$ and $\sigma_n, \sigma_{n-1}, \ldots$, respectively. Assume that $\mathbb{F} = EP(F) \cap VI(C,A) \cap \bigcap_{i=1}^{\infty}\operatorname{Fix}(T_i) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)S_n x_n) + (1-\beta_n)P_C(I - \rho_n A)u_n, & \forall n \geq 1, \end{cases} \tag{4.1}$$

*where* $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ *and* $0 \leq a_i, b_i \leq 1$, *for every* $i = 1, 2, \ldots, N$, *satisfy the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $0 < a \leq \beta_n \leq b < 1$, $\forall n \in \mathbb{N}$,
(iii) $\lim_{n\to\infty} \rho_n = 0$,
(iv) $0 < c \leq r_n \leq d < 1$, $\forall n \in \mathbb{N}$,
(v) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_1^n < \infty$.

*Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z_0 = P_{\mathbb{F}} u$.

*Proof* Take $F = F_i$ and $A = A_i$, $\forall i = 1, 2, \ldots, N$. By Theorem 3.1, we obtain the desired conclusion. □

**Theorem 4.2** *Let* $C$ *be a nonempty closed convex subset of a real Hilbert space* $H$. *For every* $i = 1, 2, \ldots, N$, *let* $A_i : C \to H$ *be a strongly positive linear bounded operator on* $H$ *with coefficient* $\gamma_i > 0$ *and* $\bar{\gamma} = \min \gamma_i$. *Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinite family of* $\kappa_i$*-strictly pseudo-contractive mappings of* $C$ *into itself, and let* $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0,1]$ *and* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ *and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p,q] \subset (\kappa, 1)$ *for all* $j \in \mathbb{N}$. *For every* $n \in \mathbb{N}$, *let* $S_n$ *and* $S$ *be the S-mappings generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\sigma_n, \sigma_{n-1}, \ldots, \sigma_1$ *and* $T_n, T_{n-1}, \ldots$ *and* $\sigma_n, \sigma_{n-1}, \ldots$, *respectively. Assume that* $\mathbb{F} = \bigcap_{i=1}^{N} VI(C, A_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. *Let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $x_1, u \in C$ *and*

$$x_{n+1} = \beta_n \big(\alpha_n u + (1 - \alpha_n) S_n x_n\big) + (1 - \beta_n) P_C \left(I - \rho_n \sum_{i=1}^{N} b_i A_i\right) x_n, \quad \forall n \geq 1, \tag{4.2}$$

*where* $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ *satisfy the following conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $0 < a \leq \beta_n \leq b < 1$, $\forall n \in \mathbb{N}$,
(iii) $\lim_{n\to\infty} \rho_n = 0$,
(iv) $0 < c \leq r_n \leq d < 1$, $\forall n \in \mathbb{N}$,
(v) $\sum_{i=1}^{N} b_i = 1$,
(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\rho_{n+1} - \rho_n| < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_1^n < \infty$.

*Then the sequences* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z_0 = P_{\mathbb{F}} u$.

*Proof* Put $F_i = 0$, $\forall i = 1, 2, \ldots, N$. Then we have $u_n = P_C x_n = x_n$, $\forall n \in \mathbb{N}$. Therefore the conclusion of Theorem 4.2 can be obtained from Theorem 3.1. □

**Theorem 4.3** *Let* $C$ *be a nonempty closed convex subset of a real Hilbert space* $H$. *For* $i = 1, 2, \ldots, N$, *let* $F_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4). *Let* $\{T_i\}_{i=1}^{\infty}$ *be an infinite family of nonexpansive mappings of* $C$ *into itself, and let* $\sigma_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0,1]$ *and* $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq \eta < 1$ *and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [p,q] \subset (0,1)$ *for all* $j \in \mathbb{N}$. *For every* $n \in \mathbb{N}$, *let* $S_n$ *and* $S$ *be the S-mappings generated by* $T_n, T_{n-1}, \ldots, T_1$ *and* $\sigma_n, \sigma_{n-1}, \ldots, \sigma_1$ *and* $T_n, T_{n-1}, \ldots$ *and* $\sigma_n, \sigma_{n-1}, \ldots$, *respectively. Assume that* $\mathbb{F} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. *Let the sequences* $\{x_n\}$ *and* $\{u_n\}$ *be generated by*

*$x_1, u \in C$ and*

$$\begin{cases} \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)S_n x_n) + (1-\beta_n)P_C u_n, & \forall n \geq 1, \end{cases} \tag{4.3}$$

*where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ and $0 \leq a_i \leq 1$, for every $i = 1, 2, \ldots, N$, satisfy the following conditions:*

(i) $\sum_{n=1}^\infty \alpha_n = \infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $0 < a \leq \beta_n \leq b < 1$, $\forall n \in \mathbb{N}$,

(iii) $\lim_{n\to\infty} \rho_n = 0$,

(iv) $0 < c \leq r_n \leq d < 1$, $\forall n \in \mathbb{N}$,

(v) $\sum_{i=1}^N a_i = 1$,

(vi) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^\infty |\rho_{n+1} - \rho_n| < \infty$ *and* $\sum_{n=1}^\infty \alpha_1^n < \infty$.

*Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_{\mathbb{F}}u$.*

*Proof* Put $A_i = 0$, $\forall i = 1, 2, \ldots, N$. Let $\kappa_i = 0$, then $T_i$ is a nonexpansive mapping for every $i = 1, 2, \ldots$. The result of Theorem 4.3 can be obtained by Theorem 3.1. □

## 5 Example and numerical results

In this section, an example is given to support Theorem 3.1.

**Example 5.1** Let $\mathbb{R}$ be the set of real numbers, and let the mapping $A_i : \mathbb{R} \to \mathbb{R}$ be defined by $A_i x = \frac{ix}{2}$, $\forall x \in \mathbb{R}$ and $b_i = \frac{7}{8^i} + \frac{1}{N8^N}$ for every $i = 1, 2, \ldots, N$. For $n \in \mathbb{N}$, let the mapping $T_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$T_n x = \frac{7n}{7n+1}x, \quad \forall x \in \mathbb{R},$$

and let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F_i(x, y) = i\bigl(-3x^2 + xy + 2y^2\bigr), \quad \forall x, y \in \mathbb{R}.$$

Furthermore, let $a_i = \frac{2}{3^i} + \frac{1}{N3^N}$ for every $i = 1, 2, \ldots, N$. Then we have

$$\sum_{i=1}^N a_i F_i(x, y) = \sum_{i=1}^N \left(\frac{2}{3^i} + \frac{1}{N3^N}\right) i\bigl(-3x^2 + xy + 2y^2\bigr) = E\bigl(-3x^2 + xy + 2y^2\bigr),$$

where $E = \sum_{i=1}^N (\frac{2}{3^i} + \frac{1}{N3^N})i$. It is easy to check that $\sum_{i=1}^N a_i F_i$ satisfies all the conditions of Theorem 3.1 and $EP(\sum_{i=1}^N a_i F_i) = \bigcap_{i=1}^N EP(F_i) = \{0\}$. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by (3.1). By the definition of $F$, we have

$$\begin{aligned} 0 &\leq \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \\ &= E\bigl(-3u_n^2 + u_n y + 2y^2\bigr) + \frac{1}{r_n}(y - u_n)(u_n - x_n) \\ &= E\bigl(-3u_n^2 + u_n y + 2y^2\bigr) + \frac{1}{r_n}\bigl(yu_n - yx_n - u_n^2 + u_n x_n\bigr) \end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \\
&0 \leq Er_n\big(-3u_n^2 + u_n y + 2y^2\big) + \big(yu_n - yx_n - u_n^2 + u_n x_n\big) \\
&\quad = -3Eu_n^2 r_n + Eu_n r_n y + 2Ey^2 r_n + yu_n - yx_n - u_n^2 + u_n x_n \\
&\quad = 2Er_n y^2 + (Eu_n r_n + u_n - x_n)y + u_n x_n - u_n^2 - 3Eu_n^2 r_n.
\end{aligned}$$

Let $G(y) = 2Er_n y^2 + (u_n(Er_n + 1) - x_n)y + u_n x_n - u_n^2 - 3Eu_n^2 r_n$. $G(y)$ is a quadratic function of $y$ with coefficient $a = 2Er_n$, $b = u_n(Er_n + 1) - x_n$, and $c = u_n x_n - u_n^2 - 3Eu_n^2 r_n$. Determine the discriminant $\Delta$ of $G$ as follows:

$$\begin{aligned}
\Delta &= b^2 - 4ac \\
&= \big(u_n(Er_n + 1) - x_n\big)^2 - 4(2Er_n)\big(u_n x_n - u_n^2 - 3Eu_n^2 r_n\big) \\
&= u_n^2(Er_n + 1)^2 - 2x_n u_n(Er_n + 1) + x_n^2 - 8Er_n u_n x_n + 24E^2 u_n^2 r_n^2 + 8Eu_n^2 r_n \\
&= E^2 u_n^2 r_n^2 + 2Er_n u_n^2 + u_n^2 - 2Ex_n u_n r_n - 2x_n u_n + x_n^2 - 8Er_n u_n x_n \\
&\quad + 24E^2 u_n^2 r_n^2 + 8Eu_n^2 r_n \\
&= u_n^2 + 10Er_n u_n^2 + 25E^2 u_n^2 r_n^2 - 2x_n u_n - 10Ex_n u_n r_n + x_n^2 \\
&= (u_n + 5Eu_n r_n)^2 - 2x_n(u_n + 5Eu_n r_n) + x_n^2 \\
&= (u_n + 5Eu_n r_n - x_n)^2.
\end{aligned}$$

We know that $G(y) \geq 0$, $\forall y \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so we obtain

$$u_n = \frac{x_n}{1 + 5\sum_{i=1}^{N}(\frac{2}{3^i} + \frac{1}{N3^N})ir_n}. \tag{5.1}$$

For every $n \in \mathbb{N}$, $x, y \in \mathbb{R}$ and $T_n x = \frac{7n}{7n+1}x$, then we have $T_n$ is a nonexpansive mapping. It implies that $T$ is 0-strictly pseudo-contractive for every $n \in \mathbb{N}$. For every $j = 1, 2, \ldots$, let $\alpha_1^j = \frac{3}{5j^2+3}$, $\alpha_2^j = \frac{3j^2}{5j^2+3}$, $\alpha_3^j = \frac{2j^2}{5j^2+3}$. Then $\sigma_j = (\frac{3}{5j^2+3}, \frac{3j^2}{5j^2+3}, \frac{2j^2}{5j^2+3})$ for all $j = 1, 2, \ldots$. Since $S_n$ is an $S$-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\sigma_n, \sigma_{n-1}, \ldots, \sigma_1$, we obtain

$$\begin{aligned}
U_{n,n+1}x_n &= x_n, \\
U_{n,n}x_n &= \left(\frac{3}{5n^2+3}\left(\frac{7n}{7n+1}\right)U_{n,n+1} + \left(\frac{3n^2}{5n^2+3}\right)U_{n,n+1} + \frac{2n^2}{5n^2+3}\right)x_n, \\
U_{n,n-1}x_n &= \left(\frac{3}{5(n-1)^2+3}\left(\frac{7(n-1)}{7(n-1)+1}\right)U_{n,n} + \left(\frac{3(n-1)^2}{5(n-1)^2+3}\right)U_{n,n}\right. \\
&\quad \left. + \frac{2(n-1)^2}{5(n-1)^2+3}\right)x_n, \\
&\vdots \\
U_{n,k+1}x_n &= \left(\frac{3}{5(k+1)^2+3}\left(\frac{7(k+1)}{7(k+1)+1}\right)U_{n,k+2} + \left(\frac{3(k+1)^2}{5(k+1)^2+3}\right)U_{n,k+1}\right. \\
&\quad \left. + \frac{2(k+1)^2}{5(k+1)^2+3}\right)x_n,
\end{aligned}$$

$$U_{n,k}x_n = \left(\frac{3}{5k^2+3}\left(\frac{7k}{7k+1}\right)U_{n,k+1} + \left(\frac{3k^2}{5k^2+3}\right)U_{n,k} + \frac{2k^2}{5k^2+3}\right)x_n,$$

$$\vdots$$

$$U_{n,2}x_n = \left(\frac{3}{23}\left(\frac{14}{15}\right)U_{n,3} + \left(\frac{12}{23}\right)U_{n,2} + \frac{8}{23}\right)x_n,$$

$$S_nx_n = U_{n,1}x_n = \left(\frac{3}{8}\left(\frac{7}{8}\right)U_{n,2} + \left(\frac{3}{8}\right)U_{n,2} + \frac{2}{8}\right)x_n.$$

From the definition of $T_n$, we obtain

$$\{0\} = \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i). \tag{5.2}$$

From (5.2) and the definitions of $A_i$ and $F_i$, we have

$$\{0\} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(C,A_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i).$$

Put $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n+1}{7n+2}$, $r_n = \frac{n}{2n+1}$, $\rho_n = \frac{1}{n^3}$, $\forall n \in \mathbb{N}$. For every $n \in \mathbb{N}$, from (5.1) we rewrite (3.1) as follows:

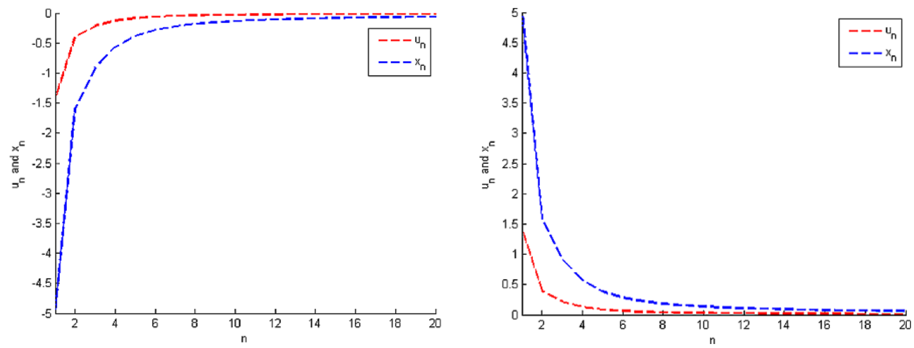
$$\begin{aligned} x_{n+1} = {} & \frac{n+1}{7n+2}\left(\frac{1}{n}u + \left(1-\frac{1}{n}\right)S_nx_n\right) + \left(1 - \frac{n+1}{7n+2}\right) \\ & \times \left(I - \frac{1}{n^3}\sum_{i=1}^{N}\left(\frac{7}{8^i} + \frac{1}{N8^N}\right)A_i\right)\frac{x_n}{1+5\sum_{i=1}^{N}(\frac{2}{3^i} + \frac{1}{N3^N})ir_n}, \quad \forall n \geq 1. \end{aligned} \tag{5.3}$$

It is clear that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ and $\{\rho_n\}$ satisfy all the conditions of Theorem 4.1. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Table 1 shows the values of sequences $\{x_n\}$ and $\{u_n\}$, where $u = x_1 = -5$ and $u = x_1 = 5$ and $n = N = 20$.

**Table 1 The values of $\{u_n\}$ and $\{x_n\}$ with $n = N = 20$**

| $n$ | $u = x_1 = -5$ | | $u = x_1 = 5$ | |
|---|---|---|---|---|
| | $u_n$ | $x_n$ | $u_n$ | $x_n$ |
| 1 | −1.428571 | −5.000000 | 1.428571 | 5.000000 |
| 2 | −0.396825 | −1.587302 | 0.396825 | 1.587302 |
| 3 | −0.215646 | −0.908795 | 0.215646 | 0.908795 |
| 4 | −0.130112 | −0.563820 | 0.130112 | 0.563820 |
| 5 | −0.086732 | −0.382407 | 0.086732 | 0.382407 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | −0.029311 | −0.133993 | 0.029311 | 0.133993 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 16 | −0.016296 | −0.075553 | 0.016296 | 0.075553 |
| 17 | −0.015173 | −0.070448 | 0.015173 | 0.070448 |
| 18 | −0.014196 | −0.065991 | 0.014196 | 0.065991 |
| 19 | −0.013336 | −0.062064 | 0.013336 | 0.062064 |
| 20 | −0.012575 | −0.058579 | 0.012575 | 0.058579 |




(a) $u = x_1 = -5$.  (b) $u = x_1 = 5$.

**Figure 1 The convergence comparison with different initial values *u* and $x_1$.**

## Conclusion

1. Table 1 and Figure 1 show that the sequences $\{x_n\}$ and $\{u_n\}$ converge to 0, where $\{0\} = \bigcap_{i=1}^{N} EP(F_i) \cap \bigcap_{i=1}^{N} VI(C, A_i) \cap \bigcap_{i=1}^{\infty} \mathrm{Fix}(T_i)$.
2. Theorem 3.1 guarantees the convergence of $\{x_n\}$ and $\{u_n\}$ in Example 5.1.

### Competing interests
The authors declare that they have no competing interests.

### Authors' contributions
Both authors contributed equally and significantly to this research article. Both authors read and approved the final manuscript.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