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A note on common fixed points for (ψ, α, β)-weakly contractive mappings in generalized metric spaces

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Abstract

Very recently, Isik and Turkoglu (Fixed Point Theory Appl. 2013:131, 2013) proved a common fixed point theorem in a rectangular metric space by using three auxiliary distance functions. In this paper, we note that this result can be derived from the recent paper of Lakzian and Samet (Appl. Math. Lett. 25:902-906, 2012).

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1 Introduction and preliminaries

In 2012, Lakzian and Samet [1] proved a fixed point theorem of a self-mapping with certain conditions in the context of a rectangular metric space via two auxiliary functions. Very recently, as a generalization of the main result of [1], Isik and Turkoglu [2] reported a common fixed point result of two self-mappings in the setting of a rectangular metric space by using three auxiliary functions. In this paper, unexpectedly, we conclude that the main result of Isik and Turkoglu [2] is a consequence of the main results of [1]. The obtained results are inspired by the techniques and ideas of, *e.g.*, [3–11].

Throughout the paper, we follow the notations used in [2]. For the sake of completeness, we recall some basic definitions, notations and results.

Definition 1.1 Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$, each of which is different from x and y:

(RM1) $d(x, y) = 0$ if and only if $x = y$,

(RM2) $d(x, y) = d(y, x)$,

(RM3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

Then the map d is called a rectangular metric and the pair (X, d) is called a rectangular metric space (or, for short, *RMS*).

We note that a rectangular metric space is also known as a generalized metric space (g.m.s.) in some sources.

We first recall the definitions of the following auxiliary functions: Let $\mathcal{F}$ be the set of functions $\xi : [0, \infty) \to [0, \infty)$ satisfying the condition $\xi(t) = 0$ if and only if $t = 0$. We denote by Ψ the set of functions $\psi \in \mathcal{F}$ such that ψ is continuous and nondecreasing. We

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reserve Φ for the set of functions $\alpha \in \mathcal{F}$ such that α is continuous. Finally, by Γ we denote the set of functions $\beta \in \mathcal{F}$ satisfying the following condition: β is lower semi-continuous.

Lakzian and Samet [1] proved the following fixed point theorem.

Theorem 1.1 [1] *Let (X,d) be a Hausdorff and complete RMS, and let $T : X \to X$ be a self-map satisfying*

$$\psi\big(d(Tx,Ty)\big) \leq \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \tag{1}$$

for all $x,y \in X$, where $\psi \in \Psi$ and $\phi \in \Phi$. Then T has a unique fixed point in X.

Lemma 1.1 [3] *Let X be a nonempty set and $T : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $T(E) = T(X)$ and $T : E \to X$ is one-to-one.*

Definition 1.2 Let X be a nonempty set, and let $T,F : X \to X$ be self-mappings. The mappings are said to be weakly compatible if they commute at their coincidence points, that is, if $Tx = Fx$ for some $x \in X$ implies that $TFx = FTx$.

Theorem 1.2 [2] *Let (X,d) be a Hausdorff and complete RMS, and let $T,F : X \to X$ be self-mappings such that $T(X) \subseteq F(X)$, and $F(X)$ is a closed subspace of X, and that the following condition holds*:

$$\psi\big(d(Tx,Ty)\big) \leq \alpha\big(d(Fx,Fy)\big) - \beta\big(d(Fx,Fy)\big) \tag{2}$$

for all $x,y \in X$, where $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma$, and these mappings satisfy the condition

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \textit{for all } t > 0. \tag{3}$$

Then T and F have a unique coincidence point in X. Moreover, if T and F are weakly compatible, then T and F have a unique common fixed point.

Remark 1.1 Let (X,d) be *RMS*. Then d is continuous (see, *e.g.*, Proposition 2 in [5]).

2 Main results

We start this section with the following theorem which is a slightly improved version of Theorem 1.1, obtained by replacing the continuity condition of ϕ with a lower semi-continuity.

Theorem 2.1 *Let (X,d) be a Hausdorff and complete RMS, and let $T : X \to X$ be a self-map satisfying*

$$\psi\big(d(Tx,Ty)\big) \leq \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \tag{4}$$

for all $x,y \in X$, where $\psi \in \Psi$ and $\phi \in \Gamma$. Then T has a unique fixed point in X.

Proof Let $x_0 \in X$ and $x_{n+1} = Tx_n$ for $n = 0,1,2,\ldots$. Following the lines of the proof of Theorem 1.1 in [1], we conclude that there exists $r \geq 0$ such that

$$d(x_n,x_{n+1}) \to r \quad \text{as } n \to \infty. \tag{5}$$

We can easily derive that

$$\psi\big(d(x_n,x_{n+1})\big) \le \psi\big(d(x_{n-1},x_n)\big) - \phi\big(d(x_{n-1},x_n)\big) \tag{6}$$

by replacing $x = x_{n-1}$ and $y = x_n$ in inequality (4).

Taking lim sup in inequality (6) as $n \to \infty$, we find that

$$\limsup_{n\to\infty} \psi\big(d(x_n,x_{n+1})\big) \le \limsup_{n\to\infty} \psi\big(d(x_{n-1},x_n)\big) - \liminf_{n\to\infty} \phi\big(d(x_{n-1},x_n)\big), \tag{7}$$

and using the continuity of ψ and lower semi-continuity of ϕ, thus, we get

$$\psi(r) \le \psi(r) - \phi(r), \tag{8}$$

which implies that $\phi(r) = 0$ and then $r = 0$. Consequently, we have $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$.

Next, we shall prove that

$$d(x_n,x_{n+2}) \to 0 \quad \text{as } n \to \infty. \tag{9}$$

By using inequality (4), we derive that

$$\begin{aligned}\psi\big(d(x_n,x_{n+2})\big) &= \psi\big(d(Tx_{n-1},Tx_{n+1})\big)\\ &\le \psi\big(d(x_{n-1},x_{n+1})\big) - \phi\big(d(x_{n-1},x_{n+1})\big)\\ &\le \psi\big(d(x_{n-1},x_{n+1})\big).\end{aligned} \tag{10}$$

From the monotone property of the function ψ, it follows that $\{d(x_n,x_{n+2})\}$ is monotone decreasing. Thus, there exists $s \ge 0$ such that

$$d(x_n,x_{n+2}) \to s \quad \text{as } n \to \infty. \tag{11}$$

Taking lim sup of inequality (10) as $n \to \infty$, we derive that

$$\limsup_{n\to\infty} \psi\big(d(x_n,x_{n+2})\big) \le \limsup_{n\to\infty} \psi\big(d(x_{n-1},x_{n+1})\big) - \liminf_{n\to\infty} \phi\big(d(x_{n-1},x_{n+1})\big). \tag{12}$$

Then, by using the continuity of ψ and lower semi-continuity of ϕ, we find

$$\psi(s) \le \psi(s) - \phi(s), \tag{13}$$

which implies that $\phi(s) = 0$. So, we conclude that $s = 0$ and hence $d(x_n,x_{n+2}) \to 0$ as $n \to \infty$.

As in Theorem 1.1 in [1], we notice that T has no periodic point.

We assert that $\{x_n\}$ is a Cauchy sequence. Suppose, on the contrary, that there exists $\varepsilon > 0$ for which we can obtain subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) \ge \varepsilon. \tag{14}$$

Again, repeating the steps of Theorem 1.1 in [1], we obtain that

$$\psi\left(d(x_{n(k)},x_{m(k)})\right) \le \psi\left(d(x_{n(k)-1},x_{m(k)-1})\right) - \phi\left(d(x_{n(k)-1},x_{m(k)-1})\right). \tag{15}$$

Now, letting lim sup in inequality (15) as $n \to \infty$, we observe that

$$\begin{aligned}
&\limsup_{n\to\infty} \psi\left(d(x_{n(k)},x_{m(k)})\right) \\
&\quad \le \limsup_{n\to\infty} \psi\left(d(x_{n(k)-1},x_{m(k)-1})\right) - \liminf_{n\to\infty} \phi\left(d(x_{n(k)-1},x_{m(k)-1})\right).
\end{aligned} \tag{16}$$

Using the continuity of ψ and lower semi-continuity of ϕ, we get

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon), \tag{17}$$

which implies that $\phi(\varepsilon) = 0$ and then $\varepsilon = 0$, a contradiction with $\varepsilon > 0$. Hence, $\{x_n\}$ is a Cauchy sequence. The rest of the proof is the mimic of the proof of Theorem 1.1 in [1] and hence we omit the details. □

Inspired by Theorem 1.2, one can state the following theorem.

Theorem 2.2 *Let (X,d) be a Hausdorff and complete RMS, and let $T : X \to X$ be self-mappings such that*

$$\psi\left(d(Tx,Ty)\right) \le \alpha\left(d(x,y)\right) - \beta\left(d(x,y)\right) \tag{18}$$

for all $x, y \in X$, where $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma$ and these mappings satisfy the condition

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \textit{for all } t > 0. \tag{19}$$

Then T has a unique fixed point in X.

Since the proof is the mimic of the proof of Theorem 1.2, we omit it.

We first prove that the above theorem is equivalent to Theorem 2.1.

Theorem 2.3 *Theorem* 2.2 *is a consequence of Theorem* 2.1.

Proof Taking $\alpha = \psi$ in Theorem 2.2, we obtain immediately Theorem 2.1. Now, we shall prove that Theorem 2.2 can be deduced from Theorem 2.1. Indeed, let $T : X \to X$ be a mapping satisfying (18) with $\psi \in \Psi$, $\alpha \in \Phi$, $\beta \in \Gamma$, and let these mappings satisfy condition (19). From (18), for all $x, y \in X$, we have

$$\begin{aligned}
\psi\left(d(Tx,Ty)\right) &\le \alpha\left(d(x,y)\right) - \beta\left(d(x,y)\right) \\
&= \psi\left(d(x,y)\right) - \left[\beta\left(d(x,y)\right) - \alpha\left(d(x,y)\right) + \psi\left(d(x,y)\right)\right].
\end{aligned} \tag{20}$$

Define $\theta : [0,\infty) \to [0,\infty)$ by $\theta(t) = \beta(t) - \alpha(t) + \psi(t)$, $t \ge 0$. Then we have

$$\psi\left(d(Tx,Ty)\right) \le \psi\left(d(x,y)\right) - \theta\left(d(x,y)\right) \tag{21}$$

for all $x, y \in X$. Due to the definition of θ, we observe that $\theta \in \Gamma$. Now, Theorem 2.2 follows immediately from Theorem 2.1. □

By regarding the techniques in [3], we conclude the following result.

Theorem 2.4 *Theorem* 1.2 *is a consequence of Theorem* 2.2.

Proof By Lemma 1.1, there exists $E \subseteq X$ such that $F(E) = F(X)$ and $F : E \to X$ is one-to-one. Now, define a map $h : F(E) \to F(E)$ by $h(Fx) = Tx$. Since F is one-to-one on E, h is well defined. Note that $\psi(d(h(Fx), h(Fy))) \leq \alpha(d(Fx, Fy)) - \beta(d(Fx, Fy))$ for all $Fx, Fy \in F(E)$. Since $F(E) = F(X)$ is complete, by using Theorem 2.2, there exists $x_0 \in X$ such that $h(Fx_0) = Fx_0$. Hence, T and F have a point of coincidence, which is also unique. It is clear that T and F have a unique common fixed point whenever T and F are weakly compatible. □

Theorem 2.5 *Theorem* 1.2 *is a consequence of Theorem* 2.1.

Proof It is evident from Theorem 2.3 and Theorem 2.4. □

3 Conclusion

In this paper, we first slightly improve the main result of Lakzian and Samet, Theorem 1.1. Then, we conclude that the main result (Theorem 1.2) of Isik-Turkoglu [2] is a consequence of our improved result, Theorem 2.1.

Competing interests
The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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