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Fixed point theorems for α-ψ-quasi contractive mappings in metric spaces

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Abstract

A notion of α-ψ-quasi contractive mappings is introduced. Some new fixed point theorems for α-ψ-quasi contractive mappings are established. An application to integral equations is given.

MSC: 47H10; 54H25

Keywords: fixed point; α-ψ-quasi contractive mapping; metric space; ordered metric space

1 Introduction and preliminaries

Banach's contraction principle [1] is one of the pivotal results in nonlinear analysis. Banach's contraction principle and its generalizations have many applications in solving nonlinear functional equations. In a metric space setting, it can be stated as follows.

Theorem 1.1 *Let (X,d) be a complete metric space. Suppose that a mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \le kd(x,y) \quad \textit{for all } x,y \in X,$$

where $0 \le k < 1$.

Then T has a unique fixed point in X.

Ćirić [2] introduced quasi contraction, which is one of the most general contraction conditions.

Theorem 1.2 *Let (X,d) be a complete metric space. Suppose that a mapping $T : X \to X$ satisfies*

$$d(Tx,Ty) \le k \max\bigl\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\}$$

for all $x,y \in X$, where $0 \le k < 1$.

Then T has a unique fixed point in X.

In [3], Berinde generalized Ćirić's result.

Theorem 1.3 *Let (X, d) be a complete metric space. Suppose that a mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \le \phi\left(\max\left\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\right\}\right)$$

for all $x, y \in X$, where $\phi : [0,\infty) \to [0,\infty)$ is nondecreasing and continuous such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$.

If T has bounded orbits, then T has a unique fixed point in X.

Recently, Samet *et al.* [4] introduced the notion of α-ψ contractive mapping and gave some fixed point theorems for such mappings. Then, Asl *et al.* [5] gave generalizations of some of the results in [4], and Mohammadi *et al.* [6] generalized the results in [5].

The purpose of the paper is to introduce a concept of α-ψ-quasi contractive mappings and to give some new fixed point theorems for such mappings.

2 Fixed point theorems

Let Ψ be the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for all $t > 0$.

Lemma 2.1 *If $\psi \in \Psi$, then the following are satisfied.*

(a) *$\psi(t) < t$ for all $t > 0$;*
(b) *$\psi(0) = 0$;*
(c) *ψ is right continuous at $t = 0$.*

Remark 2.1

(a) If $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, then $\psi \in \Psi$.
(b) If $\psi : [0,\infty) \to [0,\infty)$ is upper semicontinuous such that $\psi(t) < t$ for all $t > 0$, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$.

Let (X, d) be a metric space, and let $\alpha : X \times X \to [0,\infty)$ be a function.

A mapping $T : X \to X$ is called *α-ψ-quasi contractive* if there exists $\psi \in \Psi$ such that, for all $x, y \in X$,

$$\alpha(x,y) d(Tx, Ty) \le \psi\left(M(x,y)\right),$$

where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$.

For $A \subset X$, we denote by $\delta(A)$ the diameter of A.

Let Λ be the family of all functions $\alpha : X \times X \to [0,\infty)$.

Theorem 2.1 *Let (X, d) be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$ be such that ψ is upper semicontinuous. Suppose that $T : X \to X$ is α-ψ-quasi contractive. Assume that there exists $x_0 \in X$ such that*

$$O(x_0, T; \infty) = \left\{T^n x_0 : n = 0, 1, 2, \ldots\right\} \text{ is bounded and } \alpha\left(T^i x_0, T^j x_0\right) \ge 1 \tag{2.1}$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

Suppose that either T is continuous or

$$\lim_{n\to\infty} \inf \alpha\left(T^n x_0, x\right) \geq 1 \tag{2.2}$$

for any cluster point x of $\{T^n x_0\}$.

Then T has a fixed point in X.

Proof Let $x_0 \in X$ be such that $O(x_0, T; \infty)$ is bounded and $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then x_n is a fixed point.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

We now show that $\{x_n\}$ is a Cauchy sequence.

Let $B_n = \{x_i : i \geq n\}$, for $n = 0, 1, 2, \dots$, and let $\delta(B_0) = B$.

We claim that for $n = 0, 1, 2, \dots$,

$$\delta(B_n) \leq \psi^n(B). \tag{2.3}$$

If $n = 0$, then obviously, (2.3) holds.

Suppose that (2.3) holds when $n = k$, *i.e.*, $\delta(B_k) \leq \psi^k(B)$.

Let $x_i, x_j \in B_{k+1}$ for any $i, j \geq k + 1$. Then

$$\begin{aligned}
d(x_i, x_j) &= d(Tx_{i-1}, Tx_{j-1}) \\
&\leq \alpha(x_{i-1}, x_{j-1}) d(Tx_{i-1}, Tx_{j-1}) \\
&\leq \psi\left(\max\left\{d(x_{i-1}, x_{j-1}), d(x_{i-1}, Tx_{i-1}), d(x_{j-1}, Tx_{j-1}), d(x_{i-1}, Tx_{j-1}), d(x_{j-1}, Tx_{i-1})\right\}\right) \\
&= \psi\left(\max\left\{d(x_{i-1}, x_{j-1}), d(x_{i-1}, x_i), d(x_{j-1}, x_j), d(x_{i-1}, x_j), d(x_{j-1}, x_i)\right\}\right) \\
&\leq \psi\left(\delta(B_k)\right) \\
&\leq \psi\left(\psi^k(B)\right) \\
&= \psi^{k+1}(B).
\end{aligned}$$

Therefore, (2.3) is true for $n = 0, 1, 2, \dots$.

Hence, from (2.3), we have $\lim_{n\to\infty} \delta(B_n) = 0$. Thus, $\{x_n\}$ is a Cauchy sequence in X. It follows from the completeness of X that there exists

$$x_* = \lim_{n\to\infty} x_n \in X.$$

If T is continuous, then $\lim_{n\to\infty} x_n = Tx_*$, and so $x_* = Tx_*$.

Assume that (2.2) is satisfied.

Then, $\lim_{n\to\infty} \inf \alpha(x_n, x_*) \geq 1$, and so there exists $N \in \mathbb{N}$ such that $\alpha(x_n, x_*) \geq 1$ for all $n > N$. Thus, we have

$$\begin{aligned}
&d(x_{n+1}, Tx_*) \\
&\quad = d(Tx_n, Tx_*)
\end{aligned}$$

$$\begin{aligned}&\le \frac{1}{\alpha(x_n,x_*)}\psi\bigl(M(x_n,x_*)\bigr)\\&\le \psi\bigl(M(x_n,x_*)\bigr)\end{aligned}\tag{2.4}$$

for all $n > N$, where

$$M(x_n,x_*) = \max\bigl\{d(x_n,x_*), d(x_n,x_{n+1}), d(x_*,Tx_*), d(x_n,Tx_*), d(x_*,x_{n+1})\bigr\}.$$

Assume that $d(x_*,Tx_*) > 0$.

We obtain $\lim_{n\to\infty} M(x_n,x_*) = d(x_*,Tx_*)$.

Using (2.4), and using upper semicontinuity of ψ, we have

$$\begin{aligned}d(x_*,Tx_*) &= \lim_{n\to\infty}\sup d(x_{n+1},Tx_*)\\&\le \lim_{n\to\infty}\sup \psi\bigl(M(x_n,x_*)\bigr) \le \psi\bigl(d(x_*,Tx_*)\bigr).\end{aligned}$$

Since $d(x_*,Tx_*) > 0$,

$$d(x_*,Tx_*) \le \psi\bigl(d(x_*,Tx_*)\bigr) < d(x_*,Tx_*),$$

which is a contradiction. Hence, $d(x_*,Tx_*) = 0$, and hence, $x_* = Tx_*$. □

Example 2.1 Let $X = [0,\infty)$ and $d(x,y) = |x-y|$ for all $x,y \in X$, and let

$$\psi(t) = \begin{cases}\frac{1}{3}t, & \text{if } t \in [1,2],\\ \frac{t}{1+t}, & \text{if otherwise.}\end{cases}$$

Then $\psi \in \Psi$.

Note that ψ is not continuous at $t = 1$, and $\sum_{n=1}^{\infty}\psi^n(\frac{1}{2}) = \sum_{n=1}^{\infty}\frac{1}{2+n} = \infty$.

Define a mapping $T : X \to X$ by

$$Tx = \begin{cases}x+1, & \text{if } x \in [0,1),\\ 2-\frac{1}{3}x, & \text{if } x \in [1,2],\\ 2x-3, & \text{if } x \in (2,\infty).\end{cases}$$

We define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases}1, & \text{if } x,y \in [1,2],\\ 0, & \text{if } x \notin [1,2] \text{ or } y \notin [1,2].\end{cases}$$

Clearly, T is an α-ψ quasi contractive mapping. Condition (2.1) holds with $x_0 = \frac{3}{2}$, and $O(x_0,T;\infty)$ is bounded. Obviously, (2.2) is satisfied.

Applying Theorem 2.1, T has a fixed point. Note that $\frac{3}{2}$ and 3 are two fixed points of T.

The following example shows that if we do not have the condition of which $O(x_0,T;\infty)$ is bounded for some $x_0 \in X$, then Theorem 2.1 does not hold. Thus, we have to have the condition above.

Example 2.2 Let $X = \mathbb{N}$, and let

$$d(x,y) = \begin{cases} \sum_{i=1}^{k} \frac{1}{i^2} & (x = n, y = n + k, k \geq 1), \\ 0 & (x = y) \end{cases}$$

for all $x, y \in X$.

Then (X, d) is a complete metric space.

Let $T : X \to X$ be a mapping defined by $T(n) = n + 1$, and let $\alpha(x, y) = 1$ for all $x, y \in X$.

Let $t_k = \sum_{i=1}^{k} \frac{1}{i^2}$ for $k = 1, 2, 3, \ldots$.

Define a function $\psi : [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \begin{cases} \frac{4}{5}t & (0 \leq t \leq \frac{5}{4}), \\ \frac{t_k - t_{k-1}}{t_{k+1} - t_k}(t - t_{k+1}) + t_k & (t_k \leq t \leq t_{k+1}, k = 2, 3, 4, \ldots). \end{cases}$$

Then, it easy to see that $\psi \in \Psi$.

We show that T is α-ψ-quasi contractive.

For $x_n, x_{n+k} \in X$ $(k \geq 1)$, we have

$$\begin{aligned} &\alpha(n, n + k)d\bigl(T(n), T(n + k)\bigr) \\ &\quad = d\bigl(T(n), T(n + k)\bigr) \\ &\quad = d(n + 1, n + k + 1) \\ &\quad = t_k \\ &\quad = \psi(t_{k+1}) \\ &\quad = \psi\bigl(d(n, n + k + 1)\bigr) \\ &\quad = \psi\bigl(n, T(n + k)\bigr) \\ &\quad \leq \psi\bigl(M(n, n + k)\bigr). \end{aligned}$$

Hence, T is α-ψ-quasi contractive. But the orbits are not bounded, and T has no fixed points.

Corollary 2.2 *Let (X, d) be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$ be such that ψ is upper semicontinuous. Suppose that $T : X \to X$ is α-ψ-quasi contractive. Assume that there exists $x_0 \in X$ such that*

$$d(x_0, Tx_0) < \varliminf_{t\to\infty} \bigl(t - \psi(t)\bigr) \quad \textit{and} \quad \alpha\bigl(T^i x_0, T^j x_0\bigr) \geq 1$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

Suppose that either T is continuous or $\lim_{n\to\infty} \inf \alpha(T^n x_0, x) \geq 1$ for any cluster point x of $\{T^n x_0\}$.

Then T has a fixed point in X.

Proof Define a sequence $\{x_n\} \subset X$ by $x_n = T^n x_0$ for $n = 1, 2, \ldots$.

By assumption, $d(x_0, x_1) < \underline{\lim}_{t\to\infty}(t - \phi(t))$. Hence, there exists $M > 0$ such that for all $t > M$,

$$d(x_0, x_1) < t - \phi(t). \tag{2.5}$$

If $1 \leq i < j \leq n$, then we have

$$\begin{aligned}
&\alpha(x_{i-1}, x_{j-1})d(x_i, x_j) \\
&\quad \leq \psi\big(\max\big\{d(x_{i-1}, x_{j-1}), d(x_{i-1}, x_i), d(x_{j-1}, x_j), d(x_{i-1}, x_j), d(x_{j-1}, x_i)\big\}\big) \\
&\quad \leq \psi(v_1),
\end{aligned}$$

where $v_1 = \max\{d(x_{i-1}, x_{j-1}), d(x_{i-1}, x_i), d(x_{j-1}, x_j), d(x_{i-1}, x_j), d(x_{j-1}, x_i)\} \leq \delta(O(x_{i-1}, T; n - i + 1))$.

Thus, we have

$$\begin{aligned}
&d(x_i, x_j) \\
&\quad \leq \alpha(x_{i-1}, x_{j-1})d(x_i, x_j) \\
&\quad \leq \psi\big(\delta\big(O(x_{i-1}, T; n - i + 1)\big)\big) \\
&\quad \leq \psi\big(\delta\big(O(x_0, T; n)\big)\big).
\end{aligned} \tag{2.6}$$

So we obtain

$$d(x_i, x_j) \leq \psi\big(\delta\big(O(x_0, T; n)\big)\big) < \delta\big(O(x_0, T; n)\big).$$

Hence, we have

$$\delta\big(O(x_0, T; n)\big) = \max\big\{d(x_0, x_k) : 1 \leq k \leq n\big\}.$$

From (2.6), we obtain

$$\begin{aligned}
\delta\big(O(x_0, T; n)\big) &= d(x_0, x_k) \\
&\leq d(x_0, x_1) + d(x_1, x_k) \\
&\leq d(x_0, x_1) + \psi\big(\delta\big(O(x_0, T; n)\big)\big).
\end{aligned}$$

Thus, we have

$$\delta\big(O(x_0, T; n)\big) - \psi\big(\delta\big(O(x_0, T; n)\big)\big) \leq d(x_0, x_1). \tag{2.7}$$

From (2.5) and (2.7), we have $\delta(O(x_0, T; n)) \leq M$ for all $n \in \mathbb{N}$. Hence, $O(x_0, T; n)$ is bounded. By Theorem 2.1, T has a fixed point in X. □

Corollary 2.3 *Let (X, d) be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$ be such that ψ is upper semicontinuous. Suppose that $T : X \to X$ is α-ψ-quasi contractive. Assume that*

$\lim_{t\to\infty}(t - \psi(t)) = \infty$, *and there exists* $x_0 \in X$ *such that*

$$\alpha\left(T^i x_0, T^j x_0\right) \geq 1$$

for all $i, j \in \mathbb{N} \cup \{0\}$ *with* $i < j$.

Suppose that either T *is continuous or* $\lim_{n\to\infty} \inf \alpha(T^n x_0, x) \geq 1$ *for any cluster point* x *of* $\{T^n x_0\}$.

Then T *has a fixed point in* X.

Corollary 2.4 *Let* $(X, \preceq, d)$ *be a complete ordered metric space, and let* $\psi \in \Psi$ *be such that* ψ *is upper semicontinuous.*

Suppose that a mapping $T : X \to X$ *satisfies*

$$d(Tx, Ty) \leq \psi\left(M(x, y)\right)$$

for all comparable elements $x, y \in X$. *Assume that there exists* $x_0 \in X$ *such that* $O(x_0, T; n)$ *is bounded, and* $T^i x_0$ *and* $T^j x_0$ *are comparable for all* $i, j \in \mathbb{N} \cup \{0\}$ *with* $i < j$.

Suppose that either T *is continuous, or* $T^n x_0$ *and* x *are comparable for all* $n \in \mathbb{N} \cup \{0\}$ *and for any cluster point* x *of* $\{T^n x_0\}$.

Then T *has a fixed point in* X.

Proof Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & (x \text{ and } y \text{ are comparable}), \\ 0 & (\text{otherwise}). \end{cases}$$

Using Theorem 2.1, T has a fixed point in X. □

Remark 2.2 Let (X, d) be a metric space, and let $\alpha \in \Lambda$.

Consider the following conditions:

(1) for each $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$;
(2) for each $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;
(3) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
(4) if $\{x_n\}$ is a sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_n = x \in X$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$;
(5) there exists $x_0 \in X$ such that $\alpha(T^i x_0, T^j x_0) \geq 1$ for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$;
(6) $\lim\inf \alpha(T^n x_0, x) \geq 1$ for all cluster point x of $\{T^n x_0\}$.

Then conditions (1), (2) and (3) imply (5), and condition (4) implies (6).

Remark 2.3 If we replace condition (2.1) of Theorem 2.1 with the conditions (1), (2) and (3) above and replace condition (2.2) of Theorem 2.1 with the condition (4) above, then T has a fixed point.

Corollary 2.5 *Let* $(X, \preceq, d)$ *be a complete ordered metric space, and let* $\psi \in \Psi$ *be such that* ψ *is upper semicontinuous. Suppose that a nondecreasing mapping* $T : X \to X$ *satisfies*

$$d(Tx, Ty) \leq \psi\left(M(x, y)\right)$$

for all $x, y \in X$ *with* $x \preceq y$.

Assume that there exists $x_0 \in X$ such that $O(x_0, T; n)$ is bounded, and $x_0 \preceq Tx_0$. Suppose that either T is continuous or if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof Define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & (x \preceq y), \\ 0 & (x \not\preceq y). \end{cases}$$

Using Remark 2.3, T has a fixed point in X. □

In Theorem 2.1, if $\alpha(x,y) = 1$ for all $x, y \in X$, we have the following corollary.

Corollary 2.6 *Let (X, d) be a complete metric space, and let $\psi \in \Psi$ be such that ψ is upper semicontinuous. Suppose that a mapping $T : X \to X$ satisfies*

$$d(Tx, Ty) \le \psi\bigl(M(x,y)\bigr)$$

for all $x, y \in X$. If there exists $x_0 \in X$ such that $O(x_0, T; n)$ is bounded, then T has a fixed point in X.

Remark 2.4 Corollary 2.6 is a generalization of Theorem 2 in [3].

Theorem 2.7 *Let (X, d) be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies*

$$\alpha(x,y)d(Tx, Ty) \le \psi\bigl(d(x,y)\bigr)$$

for all $x, y \in X$.

Assume that there exists $x_0 \in X$ such that $O(x_0, T; \infty)$ is bounded and

$$\alpha\bigl(T^i x_0, T^j x_0\bigr) \ge 1$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

Suppose that either T is continuous, or

$$\liminf \alpha\bigl(T^n x_0, x\bigr) > 0 \tag{2.8}$$

for any cluster point x of $\{T^n x_0\}$.

Then T has a fixed point in X.

Proof Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. As in the proof of Theorem 2.1, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $x_* \in X$ such that

$$\lim_{n\to\infty} x_n = x_*.$$

If T is continuous, then x_* is a fixed point of T.

Assume that (2.8) is satisfied.

Then, $L := \liminf \alpha(x_n, x_*) > 0$.

Let $\epsilon > 0$ be given.

Then there exists $N \in \mathbb{N}$ such that $d(x_n, x_*) < L\epsilon$ and $\alpha(x_n, x_*) > 0$ for all $n > N$.

Since ψ is nondecreasing, $\psi(d(x_n, x_*)) \leq \psi(L\epsilon)$ for all $n > N$.

Thus, we have

$$\begin{aligned} d(x_{n+1}, Tx_*) &= d(Tx_n, Tx_*) \\ &\leq \frac{1}{\alpha(x_n, x_*)} \psi\bigl(d(x_n, x_*)\bigr) \leq \frac{1}{\alpha(x_n, x_*)} \psi(L\epsilon) \end{aligned}$$

for all $n > N$.

Hence, we obtain

$$\begin{aligned} d(x_*, Tx_*) &= \limsup d(x_{n+1}, Tx_*) \\ &\leq \frac{1}{\liminf \alpha(x_n, x_*)} \psi(L\epsilon) = \frac{1}{L} \psi(L\epsilon) < \epsilon, \end{aligned}$$

and so $x_* = Tx_*$. □

Remark 2.5 Theorem 2.7 is a generalization of Theorem 2.1 and Theorem 2.2 in [4].

Corollary 2.8 *Let (X, d) be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies*

$$\alpha(x, y) d(Tx, Ty) \leq \psi\bigl(d(x, y)\bigr)$$

for all $x, y \in X$.

Assume that there exists $x_0 \in X$ such that

$$d(x_0, Tx_0) < \varliminf_{t\to\infty} \bigl(t - \psi(t)\bigr) \quad \textit{and} \quad \alpha\bigl(T^i x_0, T^j x_0\bigr) \geq 1$$

for all $i, j \in \mathbb{N} \cup \{0\}$ with $i < j$.

Suppose that either T is continuous, or $\lim_{n\to\infty} \inf \alpha(T^n x_0, x) > 0$ for any cluster point x of $\{T^n x_0\}$.

Then T has a fixed point in X.

Proof Define a sequence $\{x_n\} \subset X$ by $x_n = T^n x_0$ for $n = 1, 2, \ldots$.

As in the proof of Corollary 2.2, $O(x_0, T; n)$ is bounded. By Theorem 2.7, T has a fixed point in X. □

Corollary 2.9 *Let (X, d) be a complete metric space, $\alpha \in \Lambda$, and let $\psi \in \Psi$. Suppose that a mapping $T : X \to X$ satisfies*

$$\alpha(x, y) d(Tx, Ty) \leq \psi\bigl(d(x, y)\bigr)$$

for all $x, y \in X$.

Assume that $\lim_{t\to\infty}(t - \psi(t)) = \infty$, *and there exists* $x_0 \in X$ *such that*

$$\alpha\big(T^i x_0, T^j x_0\big) \geq 1$$

for all $i, j \in \mathbb{N} \cup \{0\}$ *with* $i < j$.

Suppose that either T *is continuous or* $\lim_{n\to\infty} \inf \alpha(T^n x_0, x) > 0$ *for any cluster point* x *of* $\{T^n x_0\}$.

Then T *has a fixed point in* X.

Corollary 2.10 *Let* $(X, \preceq, d)$ *be a complete ordered metric space, and let* $\psi \in \Psi$. *Suppose that a mapping* $T : X \to X$ *satisfies*

$$d(Tx, Ty) \leq \psi\big(d(x, y)\big)$$

for all comparable elements $x, y \in X$.

Assume that there exists $x_0 \in X$ *such that* $O(x_0, T; \infty)$ *is bounded, and* $T^i x_0$ *and* $T^j x_0$ *are comparable for all* $i, j \in \mathbb{N} \cup \{0\}$ *with* $i < j$.

Suppose that either T *is continuous or* $T^n x_0$ *and* x *are comparable for all* $n \in \mathbb{N} \cup \{0\}$ *and for any cluster point* X *of* $\{T^n x_0\}$.

Then T *has a fixed point in* X.

Corollary 2.11 *Let* $(X, \preceq, d)$ *be a complete ordered metric space, and let* $\psi \in \Psi$. *Suppose that a mapping* $T : X \to X$ *is nondecreasing such that* $d(Tx, Ty) \leq \psi(d(x, y))$ *for all* $x, y \in X$ *with* $x \preceq y$.

Assume that there exists $x_0 \in X$ *such that* $O(x_0, T; \infty)$ *is bounded, and* $x_0 \preceq Tx_0$. *Suppose that either* T *is continuous, or if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \preceq x_{n+1}$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = x$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

Then T *has a fixed point in* X.

Remark 2.6 Corollary 2.10 and Corollary 2.11 are generalizations of the results of [7].

3 An application to integral equations

We consider the following integral equation:

$$x(t) = \int_a^t K\big(t, s, x(s)\big)\, ds + g(t), \quad t \in I, \tag{3.1}$$

where $K : I \times I \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : I \to \mathbb{R}^n$ are continuous.

Recall that the Bielecki-type norm on X,

$$\|x\|_B = \max_{t\in I} \big|x(t)\big| e^{-\tau(t-a)} \quad \text{for all } x \in X,$$

where $\tau > 0$, is arbitrarily chosen.

Let $d_B(x, y) = \|x - y\|_B = \max_{t\in[a,b]} |x(t) - y(t)| e^{-\tau(t-a)}$ for all $x, y \in X$.

Then it is well known that (X, d_B) is a complete metric space.

Theorem 3.1 *Let* $\xi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *be a function. Suppose that the following conditions* (1)-(5) *are satisfied*:

(1) *for all* $u, v, w \in \mathbb{R}^n$, $\xi(u,v) \geq 0$ *and* $\xi(v,w) \geq 0$ *implies* $\xi(u,w) \geq 0$;

(2) *for all* $s, t \in I$ *and for all* $u, v \in \mathbb{R}^n$ *with* $\xi(u,v) \geq 0$

$$\bigl|K(t,s,u) - K(t,s,v)\bigr| \leq \psi\bigl(|u-v|\bigr),$$

where $\psi \in \Psi$ *such that* $\lim_{t\to\infty}(t - \psi(t)) = \infty$ *and* $\psi(\lambda t) \leq \lambda\psi(t)$ *for all* $t \geq 0$ *and for all* $\lambda \geq 1$;

(3) *there exists* $x_0 \in X$ *such that for all* $t \in I$,

$$\xi\left(x_0(t), \int_a^t K\bigl(t,s,x_0(s)\bigr)\,ds + g(t)\,ds\right) \geq 0;$$

(4) *for all* $x, y \in X$ *and for all* $t \in I$,

$$\xi\bigl(x(t), y(t)\bigr) \geq 0 \quad \textit{implies} \quad \xi\left(\int_a^t K\bigl(t,s,x(s)\bigr)\,ds + g(t), \int_a^t K\bigl(t,s,y(s)\bigr)\,ds + g(t)\right) \geq 0;$$

(5) *if* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} x_n = x \in X$ *and* $\xi(x_n, x_{n+1}) \geq 0$ *for all* $n \in \mathbb{N}$, *then* $\xi(x_n, x) \geq 0$ *for all* $n \in \mathbb{N}$.

Then the integral equation (3.1) *has at least one solution* $x^* \in X$.

Proof

$$Tx(t) = \int_a^t K\bigl(t,s,x(s)\bigr)\,ds + g(t) \quad \textit{for all } t \in I.$$

Let $x, y \in X$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$.
From (2), we have

$$\begin{aligned}
&\bigl|Tx(t) - Ty(t)\bigr| \\
&\quad = \left|\int_a^t \bigl[K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr]\,ds\right| \\
&\quad \leq \int_a^t \bigl|K\bigl(t,s,x(s)\bigr) - K\bigl(t,s,y(s)\bigr)\bigr|\,ds \\
&\quad \leq \int_a^t \psi\bigl(\bigl|x(s) - y(s)\bigr|\bigr)\,ds \\
&\quad = \int_a^t \psi\bigl(\bigl|x(s) - y(s)\bigr|e^{-\tau(s-a)}e^{\tau(s-a)}\bigr)\,ds \\
&\quad \leq \int_a^t e^{\tau(s-a)}\psi\bigl(\bigl|x(s) - y(s)\bigr|e^{-\tau(s-a)}\bigr)\,ds \\
&\quad \leq \psi\bigl(\|x-y\|_B\bigr)\int_a^t e^{\tau(s-a)}\,ds \\
&\quad \leq \frac{1}{\tau}\psi\bigl(\|x-y\|_B\bigr)e^{\tau(s-a)} \quad \text{for all } s \in I.
\end{aligned}$$

Hence, for $\tau \geq 1$, we obtain $d_B(Tx, Ty) \leq \psi(d_B(x,y))$ for all $x, y \in X$ with $\xi(x,y) \geq 0$.

We define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } \xi(x(t),y(t)) \geq 0, t \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for all $x, y \in X$, we have

$$\alpha(x,y)d_B(Tx,Ty) \leq \psi\bigl(d_B(x,y)\bigr).$$

It is easy to see that conditions (1)-(4) of Remark 2.2 are satisfied. By Corollary 2.9, T has a fixed point in X. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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