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A new common coupled fixed point theorem in generalized metric space and applications to integral equations

Feng Gu* and Yun Yin

*Correspondence:
gufeng99@sohu.com
Department of Mathematics,
Institute of Applied Mathematics,
Hangzhou Normal University,
Hangzhou, Zhejiang 310036, China

Abstract

In the present paper, we prove a common coupled fixed point theorem in the setting of a generalized metric space in the sense of Mustafa and Sims. Our results improve and extend the corresponding results of Shatanawi. We also present an application to integral equations.

Keywords: *G*-metric space; common coupled coincidence fixed point; common fixed point; integral equation

1 Introduction and preliminaries

The study of fixed points of mappings satisfying certain contractive conditions has been in the center of rigorous research activity. For a survey of common fixed point theory in metric and cone metric spaces, we refer the reader to [1–9]. In 2006, Bhaskar and Lakshmikantham [10] initiated the study of a coupled fixed point in ordered metric spaces and applied their results to prove the existence and uniqueness of solutions for a periodic boundary value problem. For more works in coupled and coincidence point theorems, we refer the reader to [11–13].

Some authors generalized the concept of metric spaces in different ways. Mustafa and Sims [14] introduced the notion of G-metric space, in which the real number is assigned to every triplet of an arbitrary set as a generalization of the notion of metric spaces. Based on the notion of G-metric spaces, many authors (for example, [15–33]) obtained some fixed point and common fixed point theorems for mappings satisfying various contractive conditions. Fixed point problems have also been considered in partially ordered G-metric spaces [34–39].

The purpose of this paper is to obtain some common coupled coincidence point theorems in G-metric spaces satisfying some contractive conditions.

The following definitions and results will be needed in the sequel.

Definition 1.1 [14] Let X be a nonempty set, and let $G : X \times X \times X \to R^+$ be a function satisfying the following axioms:

(G1) $G(x,y,z) = 0$ if $x = y = z$;

(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ (symmetry in all three variables);

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality),

then the function G is called a generalized metric, or more specifically, a G-metric on X, and the pair (X,G) is called a G-metric space.

Definition 1.2 [14] Let (X,G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x,x_n,x_m) = 0$, and one says that the sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X,G), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \epsilon$ for all $n,m \ge N$.

Proposition 1.3 [14] *Let (X,G) be a G-metric space, then the following are equivalent*:

(1) $\{x_n\}$ *is G-convergent to x.*

(2) $G(x_n,x_n,x) \to 0$ *as* $n \to \infty$.

(3) $G(x_n,x,x) \to 0$ *as* $n \to \infty$.

(4) $G(x_n,x_m,x) \to 0$ *as* $n,m \to \infty$.

Definition 1.4 [14] Let (X,G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if for each $\epsilon > 0$, there exists a positive integer $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \epsilon$ for all $n,m,l \ge N$; *i.e.*, if $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

Definition 1.5 [14] A G-metric space (X,G) is said to be G-complete if every G-Cauchy sequence in (X,G) is G-convergent in X.

Proposition 1.6 [14] *Let (X,G) be a G-metric space, then the following are equivalent*:

(1) *The sequence $\{x_n\}$ is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \epsilon$ for all $n,m \ge k$.*

Proposition 1.7 [14] *Let (X,G) be a G-metric space. Then the function $G(x,y,z)$ is jointly continuous in all three of its variables.*

Definition 1.8 [14] Let (X,G) and (X',G') be G-metric space, and let $f:(X,G) \to (X',G')$ be a function. Then f is said to be G-continuous at a point $a \in X$ if and only if for every $\epsilon > 0$, there is $\delta > 0$ such that $x,y \in X$ and $G(a,x,y) < \delta$ implies that $G'(f(a),f(x),f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Proposition 1.9 [14] *Let (X,G) and (X',G') be G-metric spaces, then a function $f: X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x, $(f(x_n))$ is G-convergent to $f(x)$.*

Proposition 1.10 [14] *Let (X,G) be a G-metric space. Then for any x, y, z, a in X, it follows that*

(i) *if $G(x,y,z) = 0$, then $x = y = z$;*

(ii) $G(x,y,z) \le G(x,x,y) + G(x,x,z)$;

(iii) $G(x,y,y) \le 2G(y,x,x)$;

(iv) $G(x,y,z) \le G(x,a,z) + G(a,y,z)$;

(v) $G(x,y,z) \leq \frac{2}{3}(G(x,y,a) + G(x,a,z) + G(a,y,z))$;
(vi) $G(x,y,z) \leq G(x,a,a) + G(y,a,a) + G(z,a,a)$.

Definition 1.11 [10] An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 1.12 [11] An element $(x,y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

Definition 1.13 [11] Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if $gF(x,y) = F(gx,gy)$.

2 Main results

We start our work by proving the following crucial lemma.

Lemma 2.1 *Let (X,G) be a G-metric space. Let $F_1, F_2, F_3 : X \times X \to X$ and $g : X \to X$ be four mappings such that*

$$\begin{aligned} G\bigl(F_1(x,y),F_2(u,v),F_3(w,z)\bigr) \leq{}& a_1G(gx,gu,gw) + a_2G(gy,gv,gz) + a_3G(gx,gu,gu) \\ &+ a_4G(gy,gv,gv) + a_5G(gu,gw,gw) + a_6G(gv,gz,gz) \\ &+ a_7G(gw,gx,gx) + a_8G(gz,gy,gy) \end{aligned} \tag{2.1}$$

for all $x,y,u,v,w,z \in X$, where $a_i \geq 0$, $i = 1,2,\ldots,8$ and $a_1 + a_2 + a_3 + a_4 + a_7 + a_8 < 1$. Suppose that (x,y) is a common coupled coincidence point of the mappings pair (F_1,g), (F_2,g) and (F_3,g). Then

$$F_1(x,y) = F_2(x,y) = F_3(x,y) = gx = gy = F_1(y,x) = F_2(y,x) = F_3(y,x).$$

Proof Since (x,y) is a common coupled coincidence point of the mappings pair (F_1,g), (F_2,g) and (F_3,g), we have $gx = F_1(x,y) = F_2(x,y) = F_3(x,y)$ and $gy = F_1(y,x) = F_2(y,x) = F_3(y,x)$. Assume that $gx \neq gy$. Then by (2.1), we get

$$\begin{aligned} G(gx,gy,gy) &= G\bigl(F_1(x,y),F_2(y,x),F_3(y,x)\bigr) \\ &\leq a_1G(gx,gy,gy) + a_2G(gy,gx,gx) + a_3G(gx,gy,gy) + a_4G(gy,gx,gx) \\ &\quad + a_5G(gy,gy,gy) + a_6G(gx,gx,gx) + a_7G(gy,gx,gx) + a_8G(gx,gy,gy) \\ &= (a_1 + a_3 + a_8)G(gx,gy,gy) + (a_2 + a_4 + a_7)G(gy,gx,gx). \end{aligned}$$

Also by (2.1), we have

$$\begin{aligned} G(gy,gx,gx) &= G\bigl(F_1(y,x),F_2(x,y),F_3(x,y)\bigr) \\ &\leq a_1G(gy,gx,gx) + a_2G(gx,gy,gy) + a_3G(gy,gx,gx) + a_4G(gx,gy,gy) \\ &\quad + a_5G(gx,gx,gx) + a_6G(gy,gy,gy) + a_7G(gx,gy,gy) + a_8G(gy,gx,gx) \\ &= (a_1 + a_3 + a_8)G(gy,gx,gx) + (a_2 + a_4 + a_7)G(gx,gy,gy). \end{aligned}$$

Therefore,

$$G(gx,gy,gy)+G(gy,gx,gx) \\ \quad \leq (a_1+a_2+a_3+a_4+a_7+a_8)\big[G(gx,gy,gy)+G(gy,gx,gx)\big].$$

Since $0 \leq a_1+a_2+a_3+a_4+a_7+a_8<1$, we get

$$G(gx,gy,gy)+G(gy,gx,gx)<G(gx,gy,gy)+G(gy,gx,gx),$$

which is a contradiction. So, $gx=gy$, and hence,

$$F_1(x,y)=F_2(x,y)=F_3(x,y)=gx=gy=F_1(y,x)=F_2(y,x)=F_3(y,x). \qquad \square$$

Theorem 2.1 *Let (X,G) be a G-metric space. Let $F_1,F_2,F_3:X\times X\to X$ and $g:X\to X$ be four mappings such that*

$$\begin{aligned} G\big(F_1(x,y),F_2(u,v),F_3(w,z)\big) \leq{}& a_1G(gx,gu,gw)+a_2G(gy,gv,gz)+a_3G(gx,gu,gu) \\ &+a_4G(gy,gv,gv)+a_5G(gu,gw,gw)+a_6G(gv,gz,gz) \\ &+a_7G(gw,gx,gx)+a_8G(gz,gy,gy) \end{aligned} \tag{2.2}$$

for all $x,y,u,v,w,z\in X$, where $a_i\geq 0$, $i=1,2,\ldots,8$ and $a_1+a_2+a_3+a_4+2a_5+2a_6+a_7+a_8<1$. Suppose that F_1, F_2, F_3 and g satisfy the following conditions:

(i) $F_1(X\times X)\subseteq gX$, $F_2(X\times X)\subseteq gX$, $F_3(X\times X)\subseteq gX$;
(ii) *gX is G-complete*;
(iii) *g is G-continuous and commutes with F_1, F_2, F_3.*

Then there exist unique $x\in X$ such that

$$gx=F_1(x,x)=F_2(x,x)=F_3(x,x)=x.$$

Proof Let $x_0,y_0\in X$. Since $F_1(X\times X)\subseteq gX$, $F_2(X\times X)\subseteq gX$, $F_3(X\times X)\subseteq gX$, we can choose $x_1,x_2,x_3,y_1,y_2,y_3\in X$ such that $gx_1=F_1(x_0,y_0)$, $gy_1=F_1(y_0,x_0)$, $gx_2=F_2(x_1,y_1)$, $gy_2=F_2(y_1,x_1)$, $gx_3=F_3(x_2,y_2)$ and $gy_3=F_3(y_2,x_2)$. Combining this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned} &gx_{3n}=F_3(x_{3n-1},y_{3n-1}), \qquad gy_{3n}=F_3(y_{3n-1},x_{3n-1}), \quad n=1,2,3,\ldots, \\ &gx_{3n+1}=F_1(x_{3n},y_{3n}), \qquad gy_{3n+1}=F_1(y_{3n},x_{3n}), \quad n=0,1,2,3,\ldots, \\ &gx_{3n+2}=F_2(x_{3n+1},y_{3n+1}), \qquad gy_{3n+2}=F_2(y_{3n+1},x_{3n+1}), \quad n=0,1,2,3,\ldots. \end{aligned}$$

If $gx_{3n}=gx_{3n+1}$, then $gx=F_1(x,y)$, where $x=x_{3n}$, $y=y_{3n}$. If $gx_{3n+1}=gx_{3n+2}$, then $gx=F_2(x,y)$, where $x=x_{3n+1}$, $y=y_{3n+1}$. If $gx_{3n+2}=gx_{3n+3}$, then $gx=F_3(x,y)$, where $x=x_{3n+2}$, $y=y_{3n+2}$. On the other hand, if $gy_{3n}=gy_{3n+1}$, then $gy=F_1(y,x)$, where $y=y_{3n}$, $x=x_{3n}$. If $gy_{3n+1}=gy_{3n+2}$, then $gy=F_2(y,x)$, where $y=y_{3n+1}$, $x=x_{3n+1}$. If $gy_{3n+2}=gy_{3n+3}$, then $gy=F_3(y,x)$, where $y=y_{3n+2}$, $x=x_{3n+2}$. Without loss of generality, we can assume that $gx_n\neq gx_{n+1}$ and $gy_n\neq gy_{n+1}$, for all $n=0,1,2,\ldots$.

By (2.2) and (G3), we have

$$\begin{aligned} G(gx_{3n},gx_{3n+1},gx_{3n+2}) &= G\bigl(F_3(x_{3n-1},y_{3n-1}),F_1(x_{3n},y_{3n}),F_2(x_{3n+1},y_{3n+1})\bigr) \\ &= G\bigl(F_1(x_{3n},y_{3n}),F_2(x_{3n+1},y_{3n+1}),F_3(x_{3n-1},y_{3n-1})\bigr) \\ &\le a_1G(gx_{3n},gx_{3n+1},gx_{3n-1}) + a_2G(gy_{3n},gy_{3n+1},gy_{3n-1}) \\ &\quad + a_3G(gx_{3n},gx_{3n+1},gx_{3n+1}) + a_4G(gy_{3n},gy_{3n+1},gy_{3n+1}) \\ &\quad + a_5G(gx_{3n+1},gx_{3n-1},gx_{3n-1}) + a_6G(gy_{3n+1},gy_{3n-1},gy_{3n-1}) \\ &\quad + a_7G(gx_{3n-1},gx_{3n},gx_{3n}) + a_8G(gy_{3n-1},gy_{3n},gy_{3n}) \\ &\le (a_1+a_3+a_5+a_7)G(gx_{3n-1},gx_{3n},gx_{3n+1}) \\ &\quad + (a_2+a_4+a_6+a_8)G(gy_{3n-1},gy_{3n},gy_{3n+1}). \end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned} G(gy_{3n},gy_{3n+1},gy_{3n+2}) &\le (a_1+a_3+a_5+a_7)G(gy_{3n-1},gy_{3n},gy_{3n+1}) \\ &\quad + (a_2+a_4+a_6+a_8)G(gx_{3n-1},gx_{3n},gx_{3n+1}). \end{aligned} \tag{2.4}$$

By combining (2.3) and (2.4), we get

$$\begin{aligned} &G(gx_{3n},gx_{3n+1},gx_{3n+2}) + G(gy_{3n},gy_{3n+1},gy_{3n+2}) \\ &\quad \le \left(\sum_{i=1}^{8} a_i\right)\bigl[G(gx_{3n-1},gx_{3n},gx_{3n+1}) + G(gy_{3n-1},gy_{3n},gy_{3n+1})\bigr]. \end{aligned} \tag{2.5}$$

In the same way, we can show that

$$\begin{aligned} &G(gx_{3n-1},gx_{3n},gx_{3n+1}) + G(gy_{3n-1},gy_{3n},gy_{3n+1}) \\ &\quad \le \left(\sum_{i=1}^{8} a_i\right)\bigl[G(gx_{3n-2},gx_{3n-1},gx_{3n}) + G(gy_{3n-2},gy_{3n-1},gy_{3n})\bigr] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &G(gx_{3n-2},gx_{3n-1},gx_{3n}) + G(gy_{3n-2},gy_{3n-1},gy_{3n}) \\ &\quad \le \left(\sum_{i=1}^{8} a_i\right)\bigl[G(gx_{3n-3},gx_{3n-2},gx_{3n-1}) + G(gy_{3n-3},gy_{3n-2},gy_{3n-1})\bigr]. \end{aligned} \tag{2.7}$$

It follows from (2.5), (2.6) and (2.7) that for all $n \in \mathbb{N}$, we have

$$\begin{aligned} &G(gx_n,gx_{n+1},gx_{n+2}) + G(gy_n,gy_{n+1},gy_{n+2}) \\ &\quad \le \left(\sum_{i=1}^{8} a_i\right)\bigl[G(gx_{n-1},gx_n,gx_{n+1}) + G(gy_{n-1},gy_n,gy_{n+1})\bigr] \\ &\quad = k\bigl[G(gx_{n-1},gx_n,gx_{n+1}) + G(gy_{n-1},gy_n,gy_{n+1})\bigr] \end{aligned}$$

$$
\begin{aligned}
&\le k^2\big[G(gx_{n-2},gx_{n-1},gx_n)+G(gy_{n-2},gy_{n-1},gy_n)\big]\\
&\quad\vdots\\
&\le k^n\big[G(gx_0,gx_1,gx_2)+G(gy_0,gy_1,gy_2)\big].
\end{aligned}
\tag{2.8}
$$

Where $k=\sum_{i=1}^{8}a_i\in[0,1)$. From (G3), we have $G(gx_n,gx_{n+1},gx_{n+1})\le G(gx_n,gx_{n+1},gx_{n+2})$ and $G(gy_n,gy_{n+1},gy_{n+1})\le G(gy_n,gy_{n+1},gy_{n+2})$. Hence, by the (G3) and (2.8), we get

$$
\begin{aligned}
&G(gx_n,gx_{n+1},gx_{n+1})+G(gy_n,gy_{n+1},gy_{n+1})\\
&\quad\le G(gx_n,gx_{n+1},gx_{n+2})+G(gy_n,gy_{n+1},gy_{n+2})\\
&\quad\le k^n\big[G(gx_0,gx_1,gx_2)+G(gy_0,gy_1,gy_2)\big].
\end{aligned}
\tag{2.9}
$$

Therefore, for all $n,m\in\mathbb{N}$, $n<m$, by (G5) and (2.9), we have

$$
\begin{aligned}
&G(gx_n,gx_m,gx_m)+G(gy_n,gy_m,gy_m)\\
&\quad\le\big[G(gx_n,gx_{n+1},gx_{n+1})+G(gy_n,gy_{n+1},gy_{n+1})\big]\\
&\qquad+\big[G(gx_{n+1},gx_{n+2},gx_{n+2})+G(gy_{n+1},gy_{n+2},gy_{n+2})\big]\\
&\qquad+\cdots+\big[G(gx_{m-1},gx_m,gx_m)+G(gy_{m-1},gy_m,gy_m)\big]\\
&\quad\le\big(k^n+k^{n+1}+\cdots+k^{m-1}\big)\big[G(gx_0,gx_1,gx_2)+G(gy_0,gy_1,gy_2)\big]\\
&\quad\le\frac{k^n}{1-k}\big[G(gx_0,gx_1,gx_2)+G(gy_0,gy_1,gy_2)\big]\to 0\quad\text{as } n,m\to\infty.
\end{aligned}
\tag{2.10}
$$

Which implies that

$$
G(gx_n,gx_m,gx_m)\to 0\quad\text{and}\quad G(gy_n,gy_m,gy_m)\to 0\quad\text{as } n,m\to\infty.
$$

Thus, $\{gx_n\}$ and $\{gy_n\}$ are all G-Cauchy in gX. Since gX is G-complete, we get that $\{gx_n\}$ and $\{gy_n\}$ are G-convergent to some $x\in gX$ and $y\in gX$, respectively. Since g is G-continuous, we have $\{ggx_n\}$ is G-convergent to gx and $\{ggy_n\}$ is G-convergent to gy. That is,

$$
ggx_n\to gx\quad\text{and}\quad ggy_n\to gy\quad\text{as } n\to\infty.
\tag{2.11}
$$

Also, since g commutes with F_1, F_2 and F_3, respectively, we have

$$
\begin{aligned}
&ggx_{3n}=gF_3(x_{3n-1},y_{3n-1})=F_3(gx_{3n-1},gy_{3n-1}),\\
&ggy_{3n}=gF_3(y_{3n-1},x_{3n-1})=F_3(gy_{3n-1},gx_{3n-1}),\\
&ggx_{3n+1}=gF_1(x_{3n},y_{3n})=F_1(gx_{3n},gy_{3n}),\\
&ggy_{3n+1}=gF_1(y_{3n},x_{3n})=F_1(gy_{3n},gx_{3n}),\\
&ggx_{3n+2}=gF_2(x_{3n+1},y_{3n+1})=F_2(gx_{3n+1},gy_{3n+1}),\\
&ggy_{3n+2}=gF_2(y_{3n+1},x_{3n+1})=F_2(gy_{3n+1},gx_{3n+1}).
\end{aligned}
$$

Thus, from condition (2.2), we have

$$\begin{aligned}
&G\big(ggx_{3n}, ggx_{3n+1}, F_2(x,y)\big) \\
&\quad = G\big(F_1(gx_{3n}, gy_{3n}), F_2(x,y), F_3(gx_{3n-1}, gy_{3n-1})\big) \\
&\quad \le a_1 G(ggx_{3n}, gx, ggx_{3n-1}) + a_2 G(ggy_{3n}, gy, ggy_{3n-1}) + a_3 G(ggx_{3n}, gx, gx) \\
&\qquad + a_4 G(ggy_{3n}, gy, gy) + a_5 G(gx, ggx_{3n-1}, ggx_{3n-1}) + a_6 G(gy, ggy_{3n-1}, ggy_{3n-1}) \\
&\qquad + a_7 G(ggx_{3n-1}, ggx_{3n}, ggx_{3n}) + a_8 G(ggy_{3n-1}, ggy_{3n}, ggy_{3n}).
\end{aligned}$$

Letting $n \to \infty$, using (2.11) and the fact that G is continuous on its variables, we get that

$$G\big(gx, gx, F_2(x,y)\big) = 0.$$

Hence, $gx = F_2(x,y)$. Similarly, we may show that $gy = F_2(y,x)$. Also for the same reason, we may show that $gx = F_1(x,y)$, $gy = F_1(y,x)$, $gx = F_3(x,y)$ and $gy = F_3(y,x)$. Therefore, (x,y) is a common coupled coincidence point of the pair (F_1,g), (F_2,g) and (F_3,g). By Lemma 2.1, we obtain

$$gx = F_1(x,y) = F_2(x,y) = F_3(x,y) = F_1(y,x) = F_2(y,x) = F_3(y,x) = gy. \tag{2.12}$$

Since the sequences $\{gx_{3n-1}\}$, $\{gx_{3n}\}$ and $\{gx_{3n+1}\}$ are all a subsequence of $\{gx_n\}$, then they are all G-convergent to x. Similarly, we may show that $\{gy_{3n-1}\}$, $\{gy_{3n}\}$ and $\{gy_{3n+1}\}$ are all G-convergent to y. From (2.2), we have

$$\begin{aligned}
G(gx_{3n}, gx, gx) &= G\big(F_1(x,y), F_2(x,y), F_3(x_{3n-1}, y_{3n-1})\big) \\
&\le a_1 G(gx, gx, gx_{3n-1}) + a_2 G(gy, gy, gy_{3n-1}) + a_3 G(gx, gx, gx) \\
&\quad + a_4 G(gy, gy, gy) + a_5 G(gx, gx_{3n-1}, gx_{3n-1}) + a_6 G(gy, gy_{3n-1}, gy_{3n-1}) \\
&\quad + a_7 G(gx_{3n-1}, gx, gx) + a_8 G(gy_{3n-1}, gy, gy).
\end{aligned}$$

Letting $n \to \infty$, and using the fact that G is continuous on its variables, we get that

$$G(x, gx, gx) \le (a_1 + a_7) G(gx, gx, x) + (a_2 + a_8) G(gy, gy, y) + a_5 G(gx, x, x) + a_6 G(gy, y, y).$$

Similarly, we may show that

$$G(y, gy, gy) \le (a_1 + a_7) G(gy, gy, y) + (a_2 + a_8) G(gx, gx, x) + a_5 G(gy, y, y) + a_6 G(gx, x, x).$$

Thus, using the Proposition 1.10(iii), we have

$$\begin{aligned}
G(x, gx, gx) + G(y, gy, gy) &\le (a_1 + a_2 + a_7 + a_8)\big[G(gx, gx, x) + G(gy, gy, y)\big] \\
&\quad + (a_5 + a_6)\big[G(gx, x, x) + G(gy, y, y)\big] \\
&\le (a_1 + a_2 + 2a_5 + 2a_6 + a_7 + a_8)\big[G(gx, gx, x) + G(gy, gy, y)\big].
\end{aligned}$$

Since $0 \leq a_1 + a_2 + a_3 + a_4 + 2a_5 + 2a_6 + a_7 + a_8 < 1$, so the last inequality happens only if $G(x, gx, gx) = 0$ and $G(y, gy, gy) = 0$. Hence, $x = gx$ and $y = gy$. From (2.12), we have $x = gx = gy = y$, thus, we get

$$gx = F_1(x,x) = F_2(x,x) = F_3(x,x) = x.$$

To prove the uniqueness, let $z \in X$ with $z \neq x$ such that

$$z = gz = F_1(z,z) = F_2(z,z) = F_3(z,z).$$

Again using condition (2.2) and Proposition 1.10(iii), we have

$$\begin{aligned} G(z,z,x) &= G\big(F_1(z,z), F_2(z,z), F_3(x,x)\big) \\ &\leq a_1 G(gz,gz,gx) + a_2 G(gz,gz,gx) + a_3 G(gz,gz,gz) + a_4 G(gz,gz,gz) \\ &\quad + a_5 G(gz,gx,gx) + a_6 G(gz,gx,gx) + a_7 G(gx,gz,gz) + a_8 G(gx,gz,gz) \\ &\leq (a_1 + a_2 + 2a_5 + 2a_6 + a_7 + a_8) G(z,z,x). \end{aligned}$$

Since $0 \leq a_1 + a_2 + a_3 + a_4 + 2a_5 + 2a_6 + a_7 + a_8 < 1$, we get $G(z,z,x) < G(z,z,x)$, which is a contradiction. Thus, F_1, F_2, F_3 and g have a unique common fixed point. □

Remark 2.1 Theorem 2.1 extends and improves Theorem 3.2 of Shatanawi [26].

The following corollary can be obtained from Theorem 2.1 immediately.

Corollary 2.1 *Let (X, G) be a G-metric space. Let $F_1, F_2, F_3 : X \times X \to X$ and $g : X \to X$ be mappings such that*

$$G\big(F_1(x,y), F_2(u,v), F_3(w,z)\big) \leq a_1 G(gx,gu,gw) + a_2 G(gy,gv,gz) \tag{2.13}$$

for all $x, y, u, v, w, z \in X$, where $a_i \geq 0$, $i = 1, 2$ and $a_1 + a_2 < 1$. Suppose that F_1, F_2, F_3 and g satisfy the following conditions:

(1) $F_1(X \times X) \subseteq gX$, $F_2(X \times X) \subseteq gX$, $F_2(X \times X) \subseteq gX$;
(2) *gX is G-complete*;
(3) *g is G-continuous and commutes with F_1, F_2, F_3.*

Then there exist unique $x \in X$ such that

$$gx = F_1(x,x) = F_2(x,x) = F_3(x,x) = x.$$

Remark 2.2 If $F_1(x,y) = F_2(x,y) = F_3(x,y)$ and $a_1 = a_2 = k$, then Corollary 2.1 is reduced to Theorem 3.2 of Shatanawi [26].

Now, we give an example to support Corollary 2.1.

Example 2.1 Let $X = [0,1]$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|$$

for all $x, y, z \in X$. Then (X, G) is a complete G-metric space. Define a map

$$F_1, F_2, F_3 : X \times X \to X$$

by

$$F_1(x, y) = F_2(x, y) = F_3(x, y) = \frac{x + y}{8}$$

for all $x, y \in X$. Also, define $g : X \to X$ by $gx = \frac{x}{2}$ for $x \in X$. Then $F(X \times X) \subseteq gX$. Through calculation, we have

$$\begin{aligned}
&G\bigl(F_1(x,y), F_2(u,v), F_3(w,z)\bigr) \\
&\quad \le G\left(\frac{x+y}{8}, \frac{u+v}{8}, \frac{w+z}{8}\right) \\
&\quad = \frac{1}{8}\bigl(|x-u+y-v| + |u-w+v-z| + |w-x+z-y|\bigr) \\
&\quad \le \frac{1}{8}\bigl(|x-u| + |y-v| + |u-w| + |v-z| + |w-x| + |z-y|\bigr) \\
&\quad = \frac{1}{4}\bigl(G(gx, gu, gw) + G(gy, gv, gz)\bigr).
\end{aligned}$$

Then the mappings F_1, F_2, F_3 and g are satisfying condition (2.13) of Corollary 2.1 with $a_1 = a_2 = \frac{1}{4}$. So that all the conditions of Corollary 2.1 are satisfied. By Corollary 2.4, F_1, F_2, F_3 and g have a unique common fixed point. Moreover, 0 is the unique common fixed point for all of the mappings F_1, F_2, F_3 and g.

If $a_1 = a_2 = 0$, then Theorem 2.1 is reduced to the following.

Corollary 2.2 *Let (X, G) be a G-metric space. Let $F_1, F_2, F_3 : X \times X \to X$ and $g : X \to X$ be four mappings such that*

$$\begin{aligned}
G\bigl(F_1(x,y), F_2(u,v), F_3(w,z)\bigr) &\le c_1 G(gx, gu, gu) + c_2 G(gy, gv, gv) \\
&\quad + c_3 G(gu, gw, gw) + c_4 G(gv, gz, gz) \\
&\quad + c_5 G(gw, gx, gx) + c_6 G(gz, gy, gy) \qquad (2.14)
\end{aligned}$$

for all $x, y, u, v, w, z \in X$, where $c_i \ge 0$, $i = 1, 2, \ldots, 6$ and $c_1 + c_2 + 2c_3 + 2c_4 + c_5 + c_6 < 1$. Suppose that F_1, F_2, F_3 and g satisfy the following conditions:

(i) $F_1(X \times X) \subseteq gX$, $F_2(X \times X) \subseteq gX$, $F_3(X \times X) \subseteq gX$;
(ii) *gX is G-complete*;
(iii) *g is G-continuous and commutes with F_1, F_2, F_3.*

Then there exist unique $x \in X$ such that

$$gx = F_1(x, x) = F_2(x, x) = F_3(x, x) = x.$$

If we take $F_1(x, y) = F_2(x, y) = F_3(x, y)$ in Corollary 2.2, then the following corollary is obtained.

Corollary 2.3 *Let (X,G) be a G-metric space. Let $F:X\times X\to X$ and $g:X\to X$ be four mappings such that*

$$G\big(F(x,y),F(u,v),F(w,z)\big)\le c_1G(gx,gu,gu)+c_2G(gy,gv,gv)+c_3G(gu,gw,gw)+c_4G(gv,gz,gz)+c_5G(gw,gx,gx)+c_6G(gz,gy,gy) \tag{2.15}$$

for all $x,y,u,v,w,z\in X$, where $c_i\ge 0$, $i=1,2,\dots,6$ and $c_1+c_2+2c_3+2c_4+c_5+c_6<1$. Suppose that F and g satisfy the following conditions:

(i) *$F(X\times X)\subseteq gX$*;

(ii) *gX is G-complete*;

(iii) *g is G-continuous and commutes with F.*

Then there exist unique $x\in X$ such that

$$gx=F(x,x)=x.$$

Now, we give an example to support Corollary 2.3.

Example 2.2 Let $X=[0,1]$. Define $G:X\times X\times X\to\mathbb{R}^+$ by

$$G(x,y,z)=|x-y|+|y-z|+|z-x|$$

for all $x,y,z\in X$. Then (X,G) is a complete G-metric space. Define a map $F:X\times X\to X$ by

$$F(x,y)=\frac{xy}{8}$$

for all $x,y\in X$. Also, define $g:X\to X$ by $gx=x$ for $x\in X$. Then $F(X\times X)\subseteq gX$. Through calculation, we have

$$\begin{aligned}
&G\big(F(x,y),F(u,v),F(w,z)\big)\\
&\quad=\frac{1}{8}\big(|xy-uv|+|uv-wz|+|wz-xy|\big)\\
&\quad\le\frac{1}{8}\big(|y||x-u|+|u||y-v|+|v||u-w|+|w||v-z|+|z||w-x|+|x||z-y|\big)\\
&\quad\le\frac{1}{8}\big(|x-u|+|y-v|+|u-w|+|v-z|+|w-x|+|z-y|\big)\\
&\quad=\frac{1}{16}\big(G(gx,gu,gu)+G(gy,gv,gv)+G(gu,gw,gw)+G(gv,gz,gz)\\
&\qquad+G(gw,gx,gx)+c_6G(gz,gy,gy)\big).
\end{aligned}$$

Then the mappings F_1, F_2, F_3 and g are satisfying condition (2.15) of Corollary 2.3 with $c_1=c_2=c_3=c_4=c_5=c_6=\frac{1}{16}$. So that all the conditions of Corollary 2.3 are satisfied. By Corollary 2.3, F and g have a unique common fixed point. Moreover, 0 is the unique common fixed point for all of the mappings F and g.

If we take $F_1(x,y) = F_2(x,y) = F_3(x,y)$ in Theorem 2.1, then the following corollary is obtained.

Corollary 2.4 *Let (X,G) be a G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that*

$$
\begin{aligned}
&G\bigl(F(x,y),F(u,v),F(w,z)\bigr) \\
&\quad \le a_1 G(gx,gu,gw) + a_2 G(gy,gv,gz) + a_3 G(gx,gu,gu) \\
&\qquad + a_4 G(gy,gv,gv) + a_5 G(gu,gw,gw) + a_6 G(gv,gz,gz) \\
&\qquad + a_7 G(gw,gx,gx) + a_8 G(gz,gy,gy)
\end{aligned}
\tag{2.16}
$$

for all $x,y,u,v,w,z \in X$, where $a_i \ge 0$, $i = 1,2,\dots,8$ and $a_1 + a_2 + a_3 + a_4 + 2a_5 + 2a_6 + a_7 + a_8 < 1$. Suppose that F and g satisfy the following conditions:

1. *$F(X \times X) \subseteq gX$;*
2. *gX is G-complete;*
3. *g is G-continuous and commutes with F.*

Then there exist unique $x \in X$ such that $gx = F(x,x) = x$.

Now, we introduce an example to support Corollary 2.4.

Example 2.3 Let $X = [-1,1]$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x,y,z) = |x-y| + |y-z| + |z-x|$$

for all $x,y,z \in X$. Then (X,G) is a complete G-metric space. Define a map

$$F : X \times X \to X$$

by

$$F(x,y) = \frac{1}{16}x^2 + \frac{1}{16}y^2 - 1$$

for all $x,y \in X$. Also, define $g : X \to X$ by $gx = x$ for $x \in X$.

Clearly, we can get $F(X \times X) = [-1, -\frac{7}{8}] \subseteq gX$, and g is G-continuous and commutes with F.

By the definition of the mappings of F and g, for all $x,y,z,u,v,w \in [-1,1]$, we have

$$
\begin{aligned}
&G\bigl(F(x,y),F(u,v),F(w,z)\bigr) \\
&\quad \le G\biggl(\frac{1}{16}x^2 + \frac{1}{16}y^2 - 1, \frac{1}{16}u^2 + \frac{1}{16}v^2 - 1, \frac{1}{16}w^2 + \frac{1}{16}z^2 - 1\biggr) \\
&\quad = \frac{1}{16}\bigl(\bigl|x^2 - u^2 + y^2 - v^2\bigr| + \bigl|u^2 - w^2 + v^2 - z^2\bigr| + \bigl|w^2 - x^2 + z^2 - y^2\bigr|\bigr) \\
&\quad \le \frac{1}{16}\bigl(\bigl|x^2 - u^2\bigr| + \bigl|y^2 - v^2\bigr| + \bigl|u^2 - w^2\bigr| + \bigl|v^2 - z^2\bigr| + \bigl|w^2 - x^2\bigr| + \bigl|z^2 - y^2\bigr|\bigr) \\
&\quad \le \frac{1}{16}\bigl(2|x-u| + 2|y-v| + 2|u-w| + 2|v-z| + 2|w-x| + 2|z-y|\bigr)
\end{aligned}
$$

$$= \frac{1}{16}G(gx, gu, gu) + \frac{1}{16}G(gy, gv, gv) + \frac{1}{16}G(gu, gw, gw) + \frac{1}{16}G(gv, gz, gz) + \frac{1}{16}G(gw, gx, gx) + \frac{1}{16}G(gz, gy, gy).$$

Then the mappings F and g are satisfying condition (2.16) of Corollary 2.4 with $a_1 = a_2 = 0$, $a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = \frac{1}{16}$. So that all the conditions of Corollary 2.4 are satisfied. By Corollary 2.4, F and g have a unique common fixed point. Here $x = 4 - 2\sqrt{6}$ is the unique common fixed point of mappings F and g; that is, $F(x, x) = gx = x$.

3 Application to integral equations

Throughout this section, we assume that $X = C[0, 1]$ is the set of all continuous functions defined on $[0, 1]$. Define $G : X \times X \times X \to \mathbb{R}^+$ by

$$G(x, y, z) = \sup_{t\in[0,1]} \bigl|x(t) - y(t)\bigr| + \sup_{t\in[0,1]} \bigl|y(t) - z(t)\bigr| + \sup_{t\in[0,1]} \bigl|z(t) - x(t)\bigr|$$

for all $x, y, z \in X$. Then (X, G) is a G-complete metric space.

Consider the following integral equations:

$$F_i(x, y)(t) = \int_0^1 k(t, s)\bigl(f_i\bigl(s, x(s)\bigr) + g_i\bigl(s, y(s)\bigr)\bigr)\, ds, \quad t \in [0, 1]\ (i = 1, 2, 3). \tag{3.1}$$

Next, we will analyze (3.1) under the following conditions:

(i) $k : [0, 1] \times [0, 1] \to \mathbb{R}^+$ is continuous.

(ii) $f_i, g_i : [0, 1] \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3)$ are continuous functions.

(iii) There exist constants $\lambda_i, \mu_i > 0$ $(i = 1, 2, 3)$ such that

$$\begin{cases} |f_1(t, x) - f_2(t, y)| \le \lambda_1|x - y|, \\ |f_2(t, x) - f_3(t, y)| \le \lambda_2|x - y|, \\ |f_3(t, x) - f_1(t, y)| \le \lambda_3|x - y| \end{cases} \quad \text{and} \quad \begin{cases} |g_1(t, x) - g_2(t, y)| \le \mu_1|x - y|, \\ |g_2(t, x) - g_3(t, y)| \le \mu_2|x - y|, \\ |g_3(t, x) - g_1(t, y)| \le \mu_3|x - y| \end{cases}$$

for all $t \in [0, 1]$ and $x, y \in \mathbb{R}$.

(iv) $\|k\|_\infty(\max\{\lambda_1, \mu_1\} + 2\max\{\lambda_2, \mu_2\} + \max\{\lambda_3, \mu_3\}) < 1$, where

$$\|k\|_\infty = \sup\bigl\{k(t, s) : t, s \in [0, 1]\bigr\}.$$

The aim of this section is to give an existence theorem for a solution of the above integral equations by using the obtained result given by Theorem 2.1.

Theorem 3.1 *Under conditions* (i)-(iv), *integral equation* (3.1) *has a unique common solution in* $C[0, 1]$.

Proof First, we consider $F_i : X \times X \to X$ $(i = 1, 2, 3)$. By virtue of our assumptions, F_i is well defined (this means that for $x, y \in X$ then $F_i(x, y) \in X$ $(i = 1, 2, 3)$). Then we can get

$$G\bigl(F_1(x, y), F_2(u, v), F_3(w, z)\bigr) = \sup_{t\in[0,1]} \bigl|F_1(x, y) - F_2(u, v)\bigr| + \sup_{t\in[0,1]} \bigl|F_2(u, v) - F_3(w, z)\bigr| + \sup_{t\in[0,1]} \bigl|F_3(w, z) - F_1(x, y)\bigr|$$

$$\begin{aligned}
&= \sup_{t\in[0,1]} \left| \int_0^1 k(t,s)\big(f_1(s,x(s)) + g_1(s,y(s))\big)\,ds - \int_0^1 k(t,s)\big(f_2(s,u(s)) + g_2(s,v(s))\big)\,ds \right| \\
&\quad + \sup_{t\in[0,1]} \left| \int_0^1 k(t,s)\big(f_2(s,u(s)) + g_2(s,v(s))\big)\,ds - \int_0^1 k(t,s)\big(f_3(s,w(s)) + g_3(s,z(s))\big)\,ds \right| \\
&\quad + \sup_{t\in[0,1]} \left| \int_0^1 k(t,s)\big(f_3(s,w(s)) + g_3(s,z(s))\big)\,ds - \int_0^1 k(t,s)\big(f_1(s,x(s)) + g_1(s,y(s))\big)\,ds \right| \\
&= \sup_{t\in[0,1]} \left| \int_0^1 k(t,s)\Big(\big(f_1(s,x(s)) - f_2(s,u(s))\big) + \big(g_1(s,y(s)) - g_2(s,v(s))\big)\Big)\,ds \right| \\
&\quad + \sup_{t\in[0,1]} \left| \int_0^1 k(t,s)\Big(\big(f_2(s,u(s)) - f_3(s,w(s))\big) + \big(g_2(s,v(s)) - g_3(s,z(s))\big)\Big)\,ds \right| \\
&\quad + \sup_{t\in[0,1]} \left| \int_0^1 k(t,s)\Big(\big(f_3(s,w(s)) - f_1(s,x(s))\big) + \big(g_3(s,z(s)) - g_1(s,y(s))\big)\Big)\,ds \right| \\
&\le \sup_{t\in[0,1]} \int_0^1 k(t,s)\Big(\big|f_1(s,x(s)) - f_2(s,u(s))\big| + \big|g_1(s,y(s)) - g_2(s,v(s))\big|\Big)\,ds \\
&\quad + \sup_{t\in[0,1]} \int_0^1 k(t,s)\Big(\big|f_2(s,u(s)) - f_3(s,w(s))\big| + \big|g_2(s,v(s)) - g_3(s,z(s))\big|\Big)\,ds \\
&\quad + \sup_{t\in[0,1]} \int_0^1 k(t,s)\Big(\big|f_3(s,w(s)) - f_1(s,x(s))\big| + \big|g_3(s,z(s)) - g_1(s,y(s))\big|\Big)\,ds. \qquad (3.2)
\end{aligned}$$

By conditions (iii),

$$\begin{cases}
|f_1(s,x(s)) - f_2(s,u(s))| \le \lambda_1 |x(s) - u(s)|, \\
|f_2(s,u(s)) - f_3(s,w(s))| \le \lambda_2 |u(s) - w(s)|, \\
|f_3(s,w(s)) - f_1(s,x(s))| \le \lambda_3 |w(s) - x(s)|
\end{cases}$$

and

$$\begin{cases}
|g_1(s,y(s)) - g_2(s,v(s))| \le \mu_1 |y(s) - v(s)|, \\
|g_2(s,v(s)) - g_3(s,z(s))| \le \mu_2 |v(s) - z(s)|, \\
|g_3(s,z(s)) - g_1(s,y(s))| \le \mu_3 |z(s) - y(s)|.
\end{cases}$$

Taking these inequalities into (3.2), we obtain

$$\begin{aligned}
&G\big(F_1(x,y), F_2(u,v), F_3(w,z)\big) \\
&\quad\le \sup_{t\in[0,1]} \int_0^1 k(t,s)\big(\lambda_1 |x(s) - u(s)| + \mu_1 |y(s) - v(s)|\big)\,ds \\
&\qquad + \sup_{t\in[0,1]} \int_0^1 k(t,s)\big(\lambda_2 |u(s) - w(s)| + \mu_2 |v(s) - z(s)|\big) \\
&\qquad + \sup_{t\in[0,1]} \int_0^1 k(t,s)\big(\lambda_3 |w(s) - x(s)| + \mu_3 |z(s) - y(s)|\big) \\
&\quad\le \max\{\lambda_1, \mu_1\} \sup_{t\in[0,1]} \int_0^1 k(t,s)\big(|x(s) - u(s)| + |y(s) - v(s)|\big)\,ds
\end{aligned}$$

$$+ \max\{\lambda_2, \mu_2\} \sup_{t\in[0,1]} \int_0^1 k(t,s)\big(|u(s) - w(s)| + |v(s) - z(s)|\big)\,ds$$

$$+ \max\{\lambda_3, \mu_3\} \sup_{t\in[0,1]} \int_0^1 k(t,s)\big(|w(s) - x(s)| + |z(s) - y(s)|\big)\,ds. \tag{3.3}$$

Using the Cauchy-Schwartz inequality in (3.3), we get

$$\begin{aligned}
&\int_0^1 k(t,s)\big(|x(s) - u(s)| + |y(s) - v(s)|\big)\,ds \\
&\quad\le \left(\int_0^1 k^2(t,s)\,ds\right)^{\frac{1}{2}} \left(\int_0^1 \big(|x(s) - u(s)| + |y(s) - v(s)|\big)^2\,ds\right)^{\frac{1}{2}} \\
&\quad\le \|k\|_\infty \Big(\sup_{t\in[0,1]} |x(t) - u(t)| + \sup_{t\in[0,1]} |y(t) - v(t)|\Big).
\end{aligned} \tag{3.4}$$

Similarly, we can obtain the following estimate

$$\begin{aligned}
&\int_0^1 k(t,s)\big(|u(s) - w(s)| + |v(s) - z(s)|\big)\,ds \\
&\quad\le \|k\|_\infty \Big(\sup_{t\in[0,1]} |u(t) - w(t)| + \sup_{t\in[0,1]} |v(t) - z(t)|\Big),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\int_0^1 k(t,s)\big(|w(s) - x(s)| + |z(s) - y(s)|\big)\,ds \\
&\quad\le \|k\|_\infty \Big(\sup_{t\in[0,1]} |w(t) - x(t)| + \sup_{t\in[0,1]} |z(t) - y(t)|\Big).
\end{aligned} \tag{3.6}$$

Substituting (3.4), (3.5) and (3.6) into (3.3), we obtain that

$$\begin{aligned}
&G\big(F_1(x,y), F_2(u,v), F_3(w,z)\big) \\
&\quad\le \max\{\lambda_1, \mu_1\}\|k\|_\infty \Big(\sup_{t\in[0,1]} |x(t) - u(t)| + \sup_{t\in[0,1]} |y(t) - v(t)|\Big) \\
&\qquad + \max\{\lambda_2, \mu_2\}\|k\|_\infty \Big(\sup_{t\in[0,1]} |u(t) - w(t)| + \sup_{t\in[0,1]} |v(t) - z(t)|\Big) \\
&\qquad + \max\{\lambda_3, \mu_3\}\|k\|_\infty \Big(\sup_{t\in[0,1]} |w(t) - x(t)| + \sup_{t\in[0,1]} |z(t) - y(t)|\Big) \\
&\quad= \frac{1}{2}\max\{\lambda_1, \mu_1\}\|k\|_\infty \cdot 2 \sup_{t\in[0,1]} |x(t) - u(t)| \\
&\qquad + \frac{1}{2}\max\{\lambda_1, \mu_1\}\|k\|_\infty \cdot 2 \sup_{t\in[0,1]} |y(t) - v(t)| \\
&\qquad + \frac{1}{2}\max\{\lambda_2, \mu_2\}\|k\|_\infty \cdot 2 \sup_{t\in[0,1]} |u(t) - w(t)| \\
&\qquad + \frac{1}{2}\max\{\lambda_2, \mu_2\}\|k\|_\infty \cdot 2 \sup_{t\in[0,1]} |v(t) - z(t)| \\
&\qquad + \frac{1}{2}\max\{\lambda_3, \mu_3\}\|k\|_\infty \cdot 2 \sup_{t\in[0,1]} |w(t) - x(t)| \\
&\qquad + \frac{1}{2}\max\{\lambda_3, \mu_3\}\|k\|_\infty \cdot 2 \sup_{t\in[0,1]} |z(t) - y(t)|
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\max\{\lambda_1,\mu_1\}\|k\|_\infty G(x,u,u) + \frac{1}{2}\max\{\lambda_1,\mu_1\}\|k\|_\infty G(y,v,v) \\
&\quad + \frac{1}{2}\max\{\lambda_2,\mu_2\}\|k\|_\infty G(u,w,w) + \frac{1}{2}\max\{\lambda_2,\mu_2\}\|k\|_\infty G(v,z,z) \\
&\quad + \frac{1}{2}\max\{\lambda_3,\mu_3\}\|k\|_\infty G(w,x,x) + \frac{1}{2}\max\{\lambda_3,\mu_3\}\|k\|_\infty G(z,y,y). \qquad (3.7)
\end{aligned}$$

Taking $gx = x$ for all $x \in X$, and

$$\begin{aligned}
&a_1 = a_2 = 0, \qquad a_3 = a_4 = \frac{1}{2}\max\{\lambda_1,\mu_1\}\|k\|_\infty, \\
&a_5 = a_6 = \frac{1}{2}\max\{\lambda_2,\mu_2\}\|k\|_\infty, \qquad a_7 = a_8 = \frac{1}{2}\max\{\lambda_3,\mu_3\}\|k\|_\infty,
\end{aligned}$$

then inequality (3.7) becomes

$$\begin{aligned}
G\bigl(F_1(x,y),F_2(u,v),F_3(w,z)\bigr) &\le a_1G(gx,gu,gw) + a_2G(gy,gv,gz) + a_3G(gx,gu,gu) \\
&\quad + a_4G(gy,gv,gv) + a_5G(gu,gw,gw) + a_6G(gv,gz,gz) \\
&\quad + a_7G(gw,gx,gx) + a_8G(gz,gy,gy). \qquad (3.8)
\end{aligned}$$

By condition (iv), we know that

$$\begin{aligned}
&a_1 + a_2 + a_3 + a_4 + 2(a_5 + a_6) + a_7 + a_8 \\
&\quad = \|k\|_\infty\bigl(\max\{\lambda_1,\mu_1\} + 2\max\{\lambda_2,\mu_2\} + \max\{\lambda_3,\mu_3\}\bigr) < 1.
\end{aligned}$$

This proves that the operator F_i $(i = 1,2,3)$ and $g = I$ satisfy contractive condition (2.2) appearing in Theorem 2.1 with $g = I$. Therefore, F_1, F_2, F_3 have a unique common coupled fixed point, that is, $F_1(x,x) = F_2(x,x) = F_3(x,x) = x$, and so, (x,x) is the unique solution of equation (3.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

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