Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Dislocated metric space to metric spaces with some fixed point theorems

Erdal Karapınar[1]* and Peyman Salimi[2]

*Correspondence:
erdalkarapinar@yahoo.com;
ekarapinar@atilim.edu.tr
[1]Department of Mathematics,
Atilim University, Incek, Ankara
06836, Turkey
Full list of author information is
available at the end of the article

Abstract

In this paper, we notice the notions metric-like space and dislocated metric space are exactly the same. After this historical remark, we discuss the existence and uniqueness of a fixed point of a cyclic mapping in the context of metric-like spaces. We consider some examples to illustrate the validity of the derived results of this paper.

MSC: 47H10; 54H25

Keywords: dislocated metric spaces; metric-like spaces; fixed point

1 Introduction and preliminaries

Fixed point theory is one of the most dynamic research subjects in nonlinear sciences. Regarding the feasibility of application of it to the various disciplines, a number of authors have contributed to this theory with a number of publications. The most impressing result in this direction was given by Banach, called the Banach contraction mapping principle: Every contraction in a complete metric space has a unique fixed point. In fact, Banach demonstrated how to find the desired fixed point by offering a smart and plain technique. This elementary technique leads to increasing of the possibility of solving various problems in different research fields. This celebrated result has been generalized in many abstract spaces for distinct operators. In particular, Hitzler [1] obtained one of interesting characterizations of the Banach contraction mapping principle by introducing dislocated metric spaces, which is rediscovered by Amini-Harandi [2].

Definition 1.1 A dislocated (metric-like) on a nonempty set X is a function $\sigma : X \times X \to [0,+\infty)$ such that for all $x, y, z \in X$:

(σ1) if $\sigma(x,y) = 0$ then $x = y$,
(σ2) $\sigma(x,y) = \sigma(y,x)$,
(σ3) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$,

and the pair (X,σ) is called a dislocated (metric-like) space.

The motivation of defining this new notion is to get better results in logic programming semantics (see, *e.g.*, [1, 3]). Following these initial reports, many authors paid attention to the subject and have published several papers (see, *e.g.*, [4–12]). Another interesting generalization of the Banach contraction mapping principle was given by Kirk *et al.* [13] via a cyclic mapping (see, *e.g.*, [14–16]). In this remarkable paper, the mappings, for which the existence and uniqueness of a fixed point were discussed, do not need to be continuous.

A mapping $T : A \cup B \to A \cup B$ is called *cyclic* if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Theorem 1.2 (See [13]) *Let A and B be two nonempty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is cyclic and satisfies the following*:

(C) *There exists a constant $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) \quad \textit{for all } x \in A, y \in B.$$

Then T has a unique fixed point that belongs to $A \cap B$.

Cyclic mappings and related fixed point theorems have been considered by many authors (see, *e.g.*, [13–28]). In this paper, we discuss the existence and uniqueness of fixed point theory of a cyclic mapping with certain properties in the context of metric-like spaces.

We recall some basic definitions and crucial results on the topic. In this paper, we follow the notations of Amini-Harandi [2].

Definition 1.3 (See [2]) Let (X, σ) be a metric-like space and U be a subset of X. We say U is a σ-open subset of X if for all $x \in X$ there exists $r > 0$ such that $B_\sigma(x, r) \subseteq U$. Also, $V \subseteq X$ is a σ-closed subset of X if $(X \backslash V)$ is a σ-open subset of X.

Lemma 1.4 *Let (X, σ) be a metric-like space and V be a σ-closed subset of X. Let $\{x_n\}$ be a sequence in V. If $x_n \to x$ as $n \to \infty$, then $x \in V$.*

Proof Let $x \notin V$. By Definition 1.3, $(X \backslash V)$ is a σ-open set. Then there exists $r > 0$ such that $B_\sigma(x, r) \subseteq X \backslash V$. On the other hand, we have $\lim_{n\to\infty} |\sigma(x_n, x) - \sigma(x, x)| = 0$ since $x_n \to x$ as $n \to \infty$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\sigma(x_n, x) - \sigma(x, x)\right| < r$$

for all $n \geq n_0$. So, we conclude that $\{x_n\} \subseteq B_\sigma(x, r) \subseteq X \backslash V$ for all $n \geq n_0$. This is a contradiction since $\{x_n\} \subseteq V$ for all $n \in \mathbb{N}$. $\square$

Lemma 1.5 *Let (X, σ) be a metric-like space and $\{x_n\}$ be a sequence in X such that $x_n \to x$ as $n \to \infty$ and $\sigma(x, x) = 0$. Then $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$ for all $y \in X$.*

Proof From $(\sigma 3)$ we have

$$\sigma(x, y) - \sigma(x_n, x) \leq \sigma(x_n, y) \leq \sigma(x_n, x) + \sigma(x, y).$$

Letting $n \to \infty$ in the above inequalities, we get $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$. $\square$

Lemma 1.6 *Let (X, σ) be a metric-like space. Then*

(A) *if $\sigma(x, y) = 0$, then $\sigma(x, x) = \sigma(y, y) = 0$;*

(B) *if $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$, then we have*

$$\lim_{n\to\infty} \sigma(x_n, x_n) = \lim_{n\to\infty} \sigma(x_{n+1}, x_{n+1}) = 0;$$

(C) *if* $x \neq y$, *then* $\sigma(x,y) > 0$;
(D) $\sigma(x,x) \leq \frac{2}{n}\sum_{i=1}^{i=n}\sigma(x,x_i)$ *holds for all* $x_i, x \in X$, *where* $1 \leq i \leq n$.

Proof We skip the proof (A) since it is evident.

(B) Due to the triangle inequality, we have $\sigma(x_n,x_n) \leq \sigma(x_n,x_{n+1}) + \sigma(x_{n+1},x_n) = 2\sigma(x_{n+1}, x_n)$. So, we find

$$0 \leq \lim_{n\to\infty}\sigma(x_n,x_n) \leq 2\lim_{n\to\infty}\sigma(x_n,x_{n+1}) = 0.$$

Analogously, we derive

$$0 \leq \lim_{n\to\infty}\sigma(x_{n+1},x_{n+1}) \leq 2\lim_{n\to\infty}\sigma(x_n,x_{n+1}) = 0.$$

(C) If $x \neq y$ and $\sigma(x,y) = 0$, then by $(\sigma1)$ we have $x = y$, which is a contradiction.

(D) Again from $(\sigma3)$ we get

$$\sigma(x,x) \leq 2\sigma(x,x_i),$$

where $1 \leq i \leq n$. Then we observe that

$$\sum_{i=1}^{i=n}\sigma(x,x) \leq 2\sum_{i=1}^{i=n}\sigma(x,x_i).$$

Hence, we derive that

$$\sigma(x,x) \leq \frac{2}{n}\sum_{i=1}^{i=n}\sigma(x,x_i).$$

□

At first, we define the class of Φ and Ψ by the following ways:

$$\Psi = \{\psi : [0,\infty) \to [0,\infty) \text{ such that } \psi \text{ is non-decreasing and continuous}\}$$

and

$$\Phi = \{\phi : [0,\infty) \to [0,\infty) \text{ such that } \phi \text{ is lower semicontinuous}\}.$$

Definition 1.7 Let (X,σ) be a metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be σ-closed nonempty subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. We say that T is called a cyclic generalized ϕ-ψ-contractive mapping if

(1) $Y = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T;

(2)

$$\psi(t) - \psi(s) + \phi(s) > 0 \quad \text{for all } t > 0 \text{ and } s = t \text{ or } s = 0$$

and

$$\psi\bigl(\sigma(Tx,Ty)\bigr) \leq \psi\bigl(M_\sigma(x,y)\bigr) - \phi\bigl(M_\sigma(x,y)\bigr) \tag{1}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$, $\phi \in \Phi$, $\psi \in \Psi$ and

$$M_\sigma(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty) + \sigma(y,Tx)}{4}\right\}.$$

Let X be a nonempty set and $T : X \to X$ be a given map. The set of all fixed points of T will be denoted by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in X; x = Tx\}$.

Theorem 1.8 *Let (X,σ) be a complete metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be nonempty σ-closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ is a cyclic generalized ϕ-ψ-contractive mapping. Then T has a fixed point in $\bigcap_{i=1}^n A_i$. Moreover, if $\sigma(x,y) \geq \sigma(x,x)$ for all $x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point in $\bigcap_{i=1}^n A_i$.*

Proof Let x_0 be an arbitrary point of Y. So, there exists some i_0 such that $x_0 \in A_{i_0}$. Since $T(A_{i_0}) \subseteq A_{i_0+1}$, we conclude that $Tx_0 \in A_{i_0+1}$. Thus, there exists x_1 in A_{i_0+1} such that $Tx_0 = x_1$. Recursively, $Tx_n = x_{n+1}$, where $x_n \in A_{i_n}$. Hence, for $n \geq 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$. In case $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \ldots$, then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \neq x_{n+1}$ for all n. Hence, by Lemma 1.6(C) we have $\sigma(x_{n-1}, x_n) > 0$ for all n. We shall show that the sequence $\{\sigma_n\}$ is non-increasing where $\sigma_n = \sigma(x_n, x_{n+1})$. Assume that there exists some $n_0 \in \mathbb{N}$ such that

$$\sigma(x_{n_0-1}, x_{n_0}) \leq \sigma(x_{n_0}, x_{n_0+1}).$$

Hence

$$\psi\bigl(\sigma(x_{n_0-1}, x_{n_0})\bigr) \leq \psi\bigl(\sigma(x_{n_0}, x_{n_0+1})\bigr). \tag{2}$$

By taking $x = x_{n-1}$ and $y = x_n$ in condition (1) together with (2), we get

$$\begin{aligned}
&\psi\bigl(\sigma(x_n, x_{n+1})\bigr) \\
&\quad = \psi\bigl(\sigma(Tx_{n-1}, Tx_n)\bigr) \\
&\quad \leq \psi\left(\max\left\{\sigma(x_{n-1}, x_n), \sigma(x_{n-1}, Tx_{n-1}), \sigma(x_n, Tx_n), \frac{\sigma(x_{n-1}, Tx_n) + \sigma(x_n, Tx_{n-1})}{4}\right\}\right) \\
&\qquad - \phi\left(\max\left\{\sigma(x_{n-1}, x_n), \sigma(x_{n-1}, Tx_{n-1}), \sigma(x_n, Tx_n), \frac{\sigma(x_{n-1}, Tx_n) + \sigma(x_n, Tx_{n-1})}{4}\right\}\right) \\
&\quad \leq \psi\left(\max\left\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)}{4}\right\}\right) \\
&\qquad - \phi\left(\max\left\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)}{4}\right\}\right).
\end{aligned} \tag{3}$$

On the other hand, from Lemma 1.6(D) we have

$$\sigma(x_n, x_n) \leq \sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}),$$

and by $(\sigma 3)$ we have

$$\sigma(x_{n-1}, x_{n+1}) \leq \sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}).$$

That is,

$$\begin{aligned}\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\} &\le \max\left\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1}),\frac{\sigma(x_{n-1},x_{n+1})+\sigma(x_n,x_n)}{4}\right\}\\ &\le \max\left\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1}),\frac{\sigma(x_{n-1},x_n)+\sigma(x_n,x_{n+1})}{2}\right\}\\ &= \max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}.\end{aligned}$$

Then

$$\max\left\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1}),\frac{\sigma(x_{n-1},x_{n+1})+\sigma(x_n,x_n)}{4}\right\} = \max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}.$$

Therefore from (3) we get

$$\psi(\sigma(x_n,x_{n+1})) \le \psi(\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}) - \phi(\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\}).$$

Now, if $\max\{\sigma(x_{n-1},x_n),\sigma(x_n,x_{n+1})\} = \sigma(x_n,x_{n+1})$, then

$$\psi(\sigma(x_n,x_{n+1})) \le \alpha(\sigma(x_n,x_{n+1})) - \beta(\sigma(x_n,x_{n+1})),$$

a contradiction. Hence, we have

$$\psi(\sigma(x_n,x_{n+1})) \le \psi(\sigma(x_{n-1},x_n)) - \phi(\sigma(x_{n-1},x_n)) \tag{4}$$

for all $n \in \mathbb{N}$. By taking $x = x_{n_0-1}$ and $y = x_{n_0}$ in (4) and keeping (2) in mind, we deduce that

$$\psi(\sigma(x_{n_0-1},x_{n_0})) \le \psi(\sigma(x_{n_0-1},x_{n_0})) - \phi(\sigma(x_{n_0-1},x_{n_0})),$$

a contradiction. Hence, we conclude that $\sigma_n < \sigma_{n-1}$ holds for all $n \in \mathbb{N}$. Thus, there exists $r \ge 0$ such that $\lim_{n\to\infty}\sigma_n = r$. We shall show that $r = 0$ by the method of *reductio ad absurdum*. For this purpose, we assume that $r > 0$. By (4), together with the properties of ϕ, ψ, we have

$$\psi(r) = \limsup_{n\to\infty}\psi(\sigma_n) \le \limsup_{n\to\infty}[\psi(\sigma_{n-1}) - \phi(\sigma_{n-1})] \le \psi(r) - \phi(r),$$

which yields that $\phi(r) \le 0$. This is a contradiction. Hence, we obtain that

$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\sigma(x_n,x_{n+1}) = 0. \tag{5}$$

We shall show that $\{x_n\}$ is a σ-Cauchy sequence. To reach this goal, we shall follow the standard techniques that can be found in, *e.g.*, [22]. For the sake of completeness, we shall adopt the techniques used in [22]. First, we prove the following claim:

(K) For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \ge n$ with $r - q \equiv 1(m)$, then $\sigma(x_r,x_q) < \varepsilon$.

Suppose, on the contrary, that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying

$$\sigma(x_{q_n}, x_{r_n}) \geq \varepsilon. \tag{6}$$

Now, we take $n > 2m$. Then, corresponding to $q_n \geq n$, we can choose r_n in such a way that it is the smallest integer with $r_n > q_n$ satisfying $r_n - q_n \equiv 1(m)$ and $\sigma(x_{q_n}, x_{r_n}) \geq \varepsilon$. Therefore, $\sigma(x_{q_n}, x_{r_{n-m}}) \leq \varepsilon$. By using the triangular inequality, we obtain

$$\varepsilon \leq \sigma(x_{q_n}, x_{r_n}) \leq \sigma(x_{q_n}, x_{r_{n-m}}) + \sum_{i=1}^{m} \sigma(x_{r_{n-i}}, x_{r_{n-i+1}}) < \varepsilon + \sum_{i=1}^{m} p(x_{r_{n-i}}, x_{r_{n-i+1}}).$$

Passing to the limit as $n \to \infty$ in the last inequality and taking (5) into account, we obtain that

$$\lim_{n\to\infty} \sigma(x_{q_n}, x_{r_n}) = \varepsilon. \tag{7}$$

Again, by ($\sigma 3$), we derive that

$$\begin{aligned}
\varepsilon &\leq \sigma(x_{q_n}, x_{r_n}) \\
&\leq \sigma(x_{q_n}, x_{q_n+1}) + \sigma(x_{q_n+1}, x_{r_n+1}) + \sigma(x_{r_n+1}, x_{r_n}) \\
&\leq \sigma(x_{q_n}, x_{q_n+1}) + \sigma(x_{q_n+1}, x_{q_n}) + \sigma(x_{q_n}, x_{r_n}) + \sigma(x_{r_n}, x_{r_n+1}) + \sigma(x_{r_n+1}, x_{r_n}) \\
&= 2\sigma(x_{q_n}, x_{q_n+1}) + \sigma(x_{q_n}, x_{r_n}) + 2\sigma(x_{r_n}, x_{r_n+1}).
\end{aligned}$$

Taking (5) and (7) into account, we get

$$\lim_{n\to\infty} \sigma(x_{q_n+1}, x_{r_n+1}) = \varepsilon. \tag{8}$$

By ($\sigma 3$), we have the following inequalities:

$$\sigma(x_{q_n}, x_{r_n+1}) \leq \sigma(x_{q_n}, x_{r_n}) + \sigma(x_{r_n}, x_{r_n+1}) \tag{9}$$

and

$$\sigma(x_{q_n}, x_{r_n}) \leq \sigma(x_{q_n}, x_{r_n+1}) + \sigma(x_{r_n}, x_{r_n+1}). \tag{10}$$

Letting $n \to \infty$ in (9) and (10), we derive that

$$\lim_{n\to\infty} \sigma(x_{q_n}, x_{r_n+1}) = \varepsilon. \tag{11}$$

Again by ($\sigma 3$) we have

$$\sigma(x_{r_n}, x_{q_n+1}) \leq \sigma(x_{r_n}, x_{r_n+1}) + \sigma(x_{r_n+1}, x_{q_n+1}) \tag{12}$$

and

$$\sigma(x_{r_n+1}, x_{q_n+1}) \le \sigma(x_{r_n+1}, x_{r_n}) + \sigma(x_{r_n}, x_{q_n+1}). \tag{13}$$

Letting $n \to \infty$ in (12) and (13), we derive that

$$\lim_{n\to\infty} \sigma(x_{r_n}, x_{q_n+1}) = \varepsilon. \tag{14}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \le i \le m$, by using (5), (7), (8), (11), (14) together with the fact that T is a generalized cyclic ϕ-ψ-contractive mapping, we find that

$$\begin{aligned}
&\psi\bigl(\sigma(x_{q_n+1}, x_{r_n+1})\bigr)\\
&\quad = \psi\bigl(\sigma(Tx_{q_n}, Tx_{r_n})\bigr)\\
&\quad \le \psi\left(\max\left\{\sigma(x_{q_n}, x_{r_n}), \sigma(x_{q_n}, Tx_{q_n}), \sigma(x_{r_n}, Tx_{r_n}), \frac{\sigma(x_{q_n}, Tx_{r_n}) + \sigma(x_{r_n}, Tx_{q_n})}{4}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\sigma(x_{q_n}, x_{r_n}), \sigma(x_{q_n}, Tx_{q_n}), \sigma(x_{r_n}, Tx_{r_n}), \frac{\sigma(x_{q_n}, Tx_{r_n}) + \sigma(x_{r_n}, Tx_{q_n})}{4}\right\}\right)\\
&\quad = \psi\left(\max\left\{\sigma(x_{q_n}, x_{r_n}), \sigma(x_{q_n}, x_{q_n+1}), \sigma(x_{r_n}, x_{r_n+1}), \frac{\sigma(x_{q_n}, x_{r_n+1}) + \sigma(x_{r_n}, x_{q_n+1})}{4}\right\}\right)\\
&\qquad - \phi\left(\max\left\{\sigma(x_{q_n}, x_{r_n}), \sigma(x_{q_n}, x_{q_n+1}), \sigma(x_{r_n}, x_{r_n+1}), \frac{\sigma(x_{q_n}, x_{r_n+1}) + \sigma(x_{r_n}, x_{q_n+1})}{4}\right\}\right).
\end{aligned}$$

Regarding the properties of ϕ, ψ in the last inequality, we obtain that

$$\psi(\varepsilon) \le \psi(\varepsilon) - \phi(\varepsilon),$$

a contradiction. Hence, the condition (K) is satisfied. Fix $\varepsilon > 0$. By the claim, we find $n_0 \in \mathbb{N}$ such that if $r, q \ge n_0$ with $r - q \equiv 1(m)$,

$$\sigma(x_r, x_q) \le \frac{\varepsilon}{2}. \tag{15}$$

Since $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$\sigma(x_n, x_{n+1}) \le \frac{\varepsilon}{2m} \tag{16}$$

for any $n \ge n_1$. Suppose that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s - r + \varphi \equiv 1(m)$ for $\varphi = m - k + 1$. So, we have for $j \in \{1, \ldots, m\}$, $s + j - r \equiv 1(m)$

$$\sigma(x_r, x_s) \le \sigma(x_r, x_{s+j}) + \sigma(x_{s+j}, x_{s+j-1}) + \cdots + \sigma(x_{s+1}, x_s).$$

By (15) and (16) and from the last inequality, we get

$$\sigma(x_r, x_s) \le \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \le \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon.$$

This proves that $\{x_n\}$ is a σ-Cauchy sequence. Since ε is arbitrary, $\{x_n\}$ is a Cauchy sequence. Since Y is σ-closed in (X,σ), then (Y,σ) is also complete, there exists $x \in Y = \bigcup_{i=1}^m A_i$ such that $\lim_{n\to\infty} x_n = x$ in (Y,σ); equivalently

$$\sigma(x,x) = \lim_{n\to\infty} \sigma(x,x_n) = \lim_{n,m\to\infty} \sigma(x_n,x_m) = 0. \tag{17}$$

In what follows, we prove that x is a fixed point of T. In fact, since $\lim_{n\to\infty} x_n = x$ and, as $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1,2,\dots,m\}$. Suppose that $x \in A_i$, $Tx \in A_{i+1}$, and we take a subsequence x_{n_k} of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the above-mentioned comment). By using the contractive condition, we can obtain

$$\begin{aligned}
&\psi\bigl(\sigma(Tx, Tx_{n_k})\bigr) \\
&\quad \le \psi\left(\max\left\{\sigma(x,x_{n_k}), \sigma(x,Tx), \sigma(x_{n_k},Tx_{n_k}), \frac{\sigma(x,Tx_{n_k}) + \sigma(x_{n_k},Tx)}{4}\right\}\right) \\
&\qquad - \phi\left(\max\left\{\sigma(x,x_{n_k}), \sigma(x,Tx), \sigma(x_{n_k},Tx_{n_k}), \frac{\sigma(x,Tx_{n_k}) + \sigma(x_{n_k},Tx)}{4}\right\}\right).
\end{aligned}$$

Passing to the limit as $n \to \infty$ and using $x_{n_k} \to x$, lower semi-continuity of φ, we have

$$\psi\bigl(\sigma(x,Tx)\bigr) \le \psi\bigl(\sigma(x,Tx)\bigr) - \phi\bigl(\sigma(x,Tx)\bigr).$$

So, $\sigma(x,Tx) = 0$ and, therefore, x is a fixed point of T. Finally, to prove the uniqueness of the fixed point, suppose that $y,z \in X$ are two distinct fixed points of T. The cyclic character of T and the fact that $y,z \in X$ are fixed points of T imply that $x,y \in \bigcap_{i=1}^m A_i$. Suppose that $x \neq y$ and for all $u,w \in \mathrm{Fix}(T)$, $\sigma(u,w) \ge \sigma(u,u)$. Using the contractive condition, we obtain

$$\begin{aligned}
\psi\bigl(\sigma(Tx,Ty)\bigr) &\le \psi\left(\max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty) + \sigma(y,Tx)}{4}\right\}\right) \\
&\quad - \phi\left(\max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty) + \sigma(y,Tx)}{4}\right\}\right).
\end{aligned}$$

Then we have

$$\psi\bigl(\sigma(x,y)\bigr) \le \psi\bigl(\sigma(x,y)\bigr) - \phi\bigl(\sigma(x,y)\bigr),$$

which is a contradiction. Thus, we derive that $\sigma(y,z) = 0 \iff y = z$, which finishes the proof. □

If in Theorem 1.8 we take $A_i = X$ for all $0 \le i \le m$, then we deduce the following theorem.

Theorem 1.9 *Let (X,σ) be a complete metric-like space and T be a self-map on X. Assume that there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi\bigl(\sigma(Tx,Ty)\bigr) \le \psi\bigl(M_\sigma(x,y)\bigr) - \phi\bigl(M_\sigma(x,y)\bigr)$$

for all $x, y \in X$, where

$$M_\sigma(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\right\}.$$

Then T has a fixed point. Moreover, if $\sigma(x,y) \geq \sigma(x,x)$ for all $x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point.

If in Theorem 1.8 we take $\psi(t) = t$ and $\phi(t) = (1-r)t$, where $r \in [0,1)$, then we deduce the following corollary.

Corollary 1.10 *Let (X,σ) be a complete metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be nonempty σ-closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ is an operator such that*

(i) *$Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T;*
(ii) *there exists $r \in [0,1)$ such that*

$$\sigma(Tx,Ty) \leq r \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\right\}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$. Then T has a fixed point $z \in \bigcap_{i=1}^m A_i$. Moreover, if $\sigma(x,y) \geq \sigma(x,x)$ for all $x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point.

Example 1.11 Let $X = \mathbb{R}$ with the metric-like $\sigma(x,y) = \max\{|x|,|y|\}$ for all $x, y \in X$. Suppose $A_1 = [-1,0]$ and $A_2 = [0,1]$ and $Y = \bigcup_{i=1}^2 A_i$. Define $T : Y \to Y$ by

$$Tx = \begin{cases} \frac{-x}{3} & \text{if } x \in [-1,0], \\ \frac{-x^3}{2} & \text{if } x \in [0,1]. \end{cases}$$

It is clear that $\bigcup_{i=1}^2 A_i$ is a cyclic representation of Y with respect to T. Let $x \in A_1 = [-1,0]$ and $y \in A_2 = [0,1]$. Then

$$\begin{aligned} \sigma(Tx,Ty) &= \max\left\{\left|\frac{-x}{3}\right|, \left|\frac{-y^3}{2}\right|\right\} = \max\left\{\frac{-x}{3}, \frac{y^3}{2}\right\} \leq \max\left\{\frac{-x}{2}, \frac{y}{2}\right\} \\ &= \frac{1}{2}\max\{-x, y\} = \frac{1}{2}\sigma(x,y), \end{aligned}$$

and so

$$\sigma(Tx,Ty) \leq \frac{1}{2}\max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\right\}.$$

Hence, the conditions of Corollary 1.10 (Theorem 1.8) hold and T has a fixed point in $A_1 \cap A_2$. Here, $x = 0$ is a fixed point of T.

If in the above corollary we take $A_i = X$ for all $0 \leq i \leq m$, then we deduce the following corollary.

Corollary 1.12 *Let* (X,σ) *be a complete metric-like space and* T *be a self-map on* X. *Assume that there exists* $r \in [0,1)$ *such that*

$$\sigma(Tx,Ty) \leq r \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\right\}$$

holds for all $x,y \in X$. *Then* T *has a fixed point. Moreover, if* $\sigma(x,y) \geq \sigma(x,x)$ *for all* $x,y \in$ $\mathrm{Fix}(T)$, *then* T *has a unique fixed point.*

Example 1.13 Let $X = \mathbb{R}$ with the metric-like $\sigma(x,y) = \max\{x,y\}$ for all $x,y \in X$. Let $T:$ $X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{5}x^2 & \text{if } 0 \leq x < 1/3, \\ (1-x)/2 & \text{if } 1/3 \leq x \leq 1, \\ \frac{1}{6}x & \text{if } x > 1. \end{cases}$$

Proof To show the existence and uniqueness point of T, we need to consider the following cases.

- Let $0 \leq x,y < 1/3$. Then

$$\sigma(Tx,Ty) = \frac{1}{5}\max\{x^2,y^2\} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}\sigma(x,y).$$

- Let $1/3 \leq x,y \leq 1$. Then

$$\sigma(Tx,Ty) = \frac{1}{2}\max\{1-x,1-y\} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}\sigma(x,y).$$

- Let $x,y > 1$. Then

$$\sigma(Tx,Ty) = \frac{1}{6}\max\{x,y\} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}\sigma(x,y).$$

- Let $0 \leq x < 1/3$ and $1/3 \leq y \leq 1$. Then

$$\sigma(Tx,Ty) = \max\left\{\frac{1}{5}x^2, (1-y)/2\right\} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}\sigma(x,y).$$

- Let $0 \leq x < 1/3$ and $y > 1$. Then

$$\sigma(Tx,Ty) = \max\left\{\frac{1}{5}x^2, \frac{1}{6}y\right\} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}\sigma(x,y).$$

- Let $1/3 \leq x \leq 1$ and $y > 1$. Then

$$\sigma(Tx,Ty) = \max\left\{(1-x)/2, \frac{1}{6}y\right\} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}\sigma(x,y),$$

and so

$$\sigma(Tx,Ty) \leq \frac{1}{2}\max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\right\}.$$

Hence, we conclude that all the conditions of Corollary 1.12 (Theorem 1.9) hold and hence T has a fixed point 0 in $[0,\infty)$. □

By Corollary 1.10 we deduce the following result.

Corollary 1.14 *Let (X,σ) be a complete metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be nonempty σ-closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ is an operator such that*

(i) *$Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T;*
(ii) *there exists $r \in [0,1)$ such that*

$$\int_0^{\sigma(Tx,Ty)} \rho(t)\,dt \le r \int_0^{\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}} \rho(t)\,dt$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,m$, where $A_{m+1} = A_1$, and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)\,dt > 0$ for $\varepsilon > 0$. Then T has a fixed point $z \in \bigcap_{i=1}^m A_i$. Moreover, if $\sigma(x,y) \ge \sigma(x,x)$ for all $x,y \in \mathrm{Fix}(T)$, then T has a unique fixed point.

If in the above corollary we take $A_i = X$ for all $0 \le i \le m$, then we deduce the following corollary.

Corollary 1.15 *Let (X,σ) be a complete metric-like space and let $T : X \to X$ be a mapping such that for any $x,y \in X$,*

$$\int_0^{\sigma(Tx,Ty)} \rho(t)\,dt \le r \int_0^{\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\}} \rho(t)\,dt,$$

where $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)\,dt$ for $\varepsilon > 0$ and the constant $\beta \in [0,\frac{1}{4})$. Then T has a unique fixed point.

Definition 1.16 Let $T : X \to X$ and $\psi : X \to [0,\infty)$ and $\gamma \in [0,1]$. A mapping T is said to be a γ-ψ-subadmissible mapping if

$$\psi(x) \le \gamma \quad \text{implies} \quad \psi(Tx) \le \gamma, \quad x \in X.$$

Example 1.17 Let $T : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}_+$ be defined by $Tx = x^3$ and $\psi(x) = \frac{1}{2}e^x$. Then T is a γ-ψ-subadmissible mapping where $\gamma = \frac{1}{6}$. Indeed, if $\psi(x) = \frac{1}{6}e^x \le \frac{1}{6}$, then $x \le 0$, and hence $Tx \le 0$. That is, $\psi(Tx) = \frac{1}{6}e^{Tx} \le \frac{1}{6}$.

Example 1.18 Let $T : [-\pi,\pi] \to [-\pi,\pi]$ and $\psi : [-\pi,\pi] \to \mathbb{R}_+$ be defined by $Tx = \frac{\pi}{2}\sin(x)$ and $\psi(x) = |x - \frac{1}{2}\pi| + \frac{1}{2}$. Then T is a γ-ψ-subadmissible mapping where $\gamma = \frac{1}{2}$. Indeed, if $\psi(x) = |x - \frac{1}{2}\pi| + \frac{1}{2} \le \frac{1}{2}$, then $x = \frac{1}{2}\pi$, and hence $Tx = \frac{1}{2}\pi$. That is, $\psi(Tx) = \frac{1}{2}$.

Let Λ be the class of all the functions $\varphi : [0,+\infty)^3 \to [0,+\infty)$ that are a continuous with the following property:

$$\varphi(x,y,z) = 0 \quad \text{if and only if} \quad x = y = z = 0.$$

Definition 1.19 Let (X,σ) be a metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be σ-closed nonempty subsets of (X, d_p) and $Y = \bigcup_{i=1}^m A_i$. Assume that $T : Y \to Y$ is a γ-ψ-subadmissible mapping where $\gamma = \frac{1}{6}$. Then T is called a ψ-cyclic generalized weakly C-contraction if

(1) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T;

(2)

$$\begin{aligned}\sigma(Tx, Ty) &\le \psi(x)\sigma(x, Tx) + \psi(Tx)\sigma(y, Ty) + \psi\big(T^2x\big)\sigma(x, Ty) + \psi\big(T^3x\big)\sigma(y, Tx) \\ &\quad - \varphi\biggl(\sigma(x, Tx), \sigma(x, Ty), \frac{1}{2}\bigl[\sigma(x, Ty) + \sigma(y, Tx)\bigr]\biggr) \qquad (18)\end{aligned}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \ldots, m$, where $A_{m+1} = A_1$ and $\varphi \in \Lambda$.

Theorem 1.20 *Let (X,σ) be a complete metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \ldots, A_m$ be nonempty σ-closed subsets of (X, p) and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ is a ψ-cyclic generalized weakly C-contraction. If there exists $x_0 \in Y$ such that $\psi(x_0) \le \frac{1}{6}$, then T has a fixed point $z \in \bigcap_{i=1}^n A_i$. Moreover, if $\psi(z) \le \frac{1}{6}$, then z is unique.*

Proof Let $x_0 \in Y$ be such that $\psi(x_0) \le \frac{1}{6}$. Since T is a sub ψ-admissible mapping with respect to $\frac{1}{6}$, then $\psi(Tx_0) \le \frac{1}{6}$. $\psi(T^n x_0) \le \frac{1}{6}$ for all $n \in \mathbb{N} \cup 0$. Also, there exists some i_0 such that $x_0 \in A_{i_0}$. Now $T(A_{i_0}) \subseteq A_{i_0+1}$ implies that $Tx_0 \in A_{i_0+1}$. Thus there exists x_1 in A_{i_0+1} such that $Tx_0 = x_1$. Similarly, $Tx_n = x_{n+1}$, where $x_n \in A_{i_n}$. Hence, for $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. In case $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \ldots$, then it is clear that x_{n_0} is a fixed point of T. Now assume that $x_n \ne x_{n+1}$ for all n. Since $T : Y \to Y$ is a cyclic generalized weak C-contraction, we have that for all $n \in \mathbb{N}^*$,

$$\begin{aligned}&\sigma(x_n, x_{n+1}) \\ &\quad = \sigma(Tx_{n-1}, Tx_n) \\ &\quad \le \psi(x_{n-1})\sigma(x_{n-1}, Tx_{n-1}) + \psi(Tx_{n-1})\sigma(x_n, Tx_n) + \psi\big(T^2x_{n-1}\big)\sigma(x_{n-1}, Tx_n) \\ &\qquad + \psi\big(T^3x_{n-1}\big)\sigma(x_n, Tx_{n-1}) \\ &\qquad - \varphi\biggl(\sigma(x_{n-1}, Tx_{n-1}), \sigma(x_n, Tx_n), \frac{1}{2}\bigl[\sigma(x_{n-1}, Tx_n) + \sigma(x_n, Tx_{n-1})\bigr]\biggr) \\ &\quad = \psi(x_{n-1})\sigma(x_{n-1}, x_n) + \psi(x_n)\sigma(x_n, x_{n+1}) + \psi(x_{n+1})\sigma(x_{n-1}, x_{n+1}) + \psi(x_{n+2})\sigma(x_n, x_n) \\ &\qquad - \varphi\biggl(\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{1}{2}\bigl[\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\bigr]\biggr) \\ &\quad \le \frac{1}{6}\bigl[\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}) + \sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\bigr] \\ &\qquad - \varphi\biggl(\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1}), \frac{1}{2}\bigl[\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\bigr]\biggr),\end{aligned}$$

and so

$$\sigma(x_n, x_{n+1}) \le \frac{1}{6}\bigl[\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}) + \sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\bigr]. \qquad (19)$$

On the other hand, from ($\sigma 3$) we have

$$\sigma(x_{n-1}, x_{n+1}) \leq \sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}),$$

and by Lemma 1.6(D) we have

$$\sigma(x_n, x_n) \leq \sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1}).$$

Then by (19) we get

$$\sigma(x_n, x_{n+1}) \leq \frac{1}{2}\big[\sigma(x_{n-1}, x_n) + \sigma(x_n, x_{n+1})\big].$$

Therefore,

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n) \tag{20}$$

for any $n \geq 1$. Set $t_n = \varphi(x_n, x_{n-1})$. On the occasion of the facts above, $\{t_n\}$ is a nonincreasing sequence of nonnegative real numbers. Consequently, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = L. \tag{21}$$

We shall prove that $L = 0$. Since $\sigma(x_n, x_n) \leq 2\varphi(x_n, x_{n+1})$, then $\lim_{n\to\infty} \sigma(x_n, x_n) \leq 2L$. Similarly, $\lim_{n\to\infty} \sigma(x_{n-1}, x_{n+1}) \leq 2L$. Then

$$\lim_{n\to\infty} \big[\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})\big] \leq 4L.$$

On the other hand, by taking limit as $n \to \infty$ in (19), we have

$$L \leq \frac{1}{6}\Big[2L + \lim_{n\to\infty}\big[\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})\big]\Big],$$

which implies

$$4L \leq \lim_{n\to\infty} \big[\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})\big].$$

Hence,

$$\lim_{n\to\infty} \big[\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})\big] = 4L.$$

Now, from (18) we have

$$\begin{aligned} t_{n+1} &\leq \psi(x_{n-1})t_n + \psi(x_n)t_{n+1} + \psi(x_{n+1})\sigma(x_{n-1}, x_{n+1}) + \psi(x_{n+2})\sigma(x_n, x_n) \\ &\quad - \varphi\left(t_n, t_{n+1}, \frac{1}{2}\big[\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\big]\right) \\ &\leq \frac{1}{6}\big[t_n + t_{n+1} + \sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\big] \\ &\quad - \varphi\left(t_n, t_{n+1}, \frac{1}{2}\big[\sigma(x_{n-1}, x_{n+1}) + \sigma(x_n, x_n)\big]\right). \end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality, we deduce

$$L \leq L - \varphi(L, L, 2L),$$

and so $\varphi(L, L, 2L) = 0$. Since $\varphi(x, y, z) = 0 \iff x = y = z = 0$, we get $L = 0$. Due to $\lim_{n\to\infty} \sigma(x_n, x_n) \leq 2L$ and $\lim_{n\to\infty} \sigma(x_{n-1}, x_{n+1}) \leq 2L$, we have

$$\lim_{n\to\infty} \sigma(x_n, x_n) = \lim_{n\to\infty} \sigma(x_{n-1}, x_{n+1}) = \lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{22}$$

We shall show that $\{x_n\}$ is a σ-Cauchy sequence. At first, we prove the following fact:

(K) For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $r, q \geq n$ with $r - q \equiv 1(m)$, then $\sigma(x_r, x_q) < \varepsilon$.

Suppose to the contrary that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, we can find $r_n > q_n \geq n$ with $r_n - q_n \equiv 1(m)$ satisfying

$$\sigma(x_{q_n}, x_{r_n}) \geq \varepsilon. \tag{23}$$

Following the related lines of the proof of Theorem 1.8, we have

$$\lim_{n\to\infty} \sigma(x_{q_n}, x_{r_n}) = \varepsilon;$$

$$\lim_{n\to\infty} \sigma(x_{q_n+1}, x_{r_n+1}) = \varepsilon;$$

$$\lim_{n\to\infty} \sigma(x_{q_n}, x_{r_n+1}) = \varepsilon$$

and

$$\lim_{n\to\infty} \sigma(x_{r_n}, x_{q_n+1}) = \varepsilon. \tag{24}$$

Since x_{q_n} and x_{r_n} lie in different adjacently labeled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, using the fact that T is a ψ-cyclic generalized weakly C-contraction, we have

$$\begin{aligned}
\sigma(x_{q_n+1}, x_{r_n+1}) &= \sigma(Tx_{q_n}, Tx_{r_n}) \\
&\leq \psi(x_{q_n})\sigma(x_{q_n}, Tx_{q_n}) + \psi(Tx_{q_n})\sigma(x_{r_n}, Tx_{r_n}) \\
&\quad + \psi(T^2x_{q_n})\sigma(x_{q_n}, Tx_{r_n}) + \psi(T^3x_{q_n})\sigma(x_{r_n}, Tx_{q_n}) \\
&\quad - \varphi\left(\sigma(x_{q_n}, Tx_{q_n}), \sigma(x_{r_n}, Tx_{r_n}), \frac{1}{2}\left[\sigma(x_{q_n}, Tx_{r_n}) + \sigma(x_{r_n}, Tx_{q_n})\right]\right) \\
&\leq \frac{1}{6}\left[\sigma(x_{q_n}, x_{q_n+1}) + \sigma(x_{r_n}, x_{r_n+1}) + \sigma(x_{q_n}, x_{r_n+1}) + \sigma(x_{r_n}, x_{q_n+1})\right] \\
&\quad - \varphi\left(\sigma(x_{q_n}, x_{q_n+1}), \sigma(x_{r_n}, x_{r_n+1}), \frac{1}{2}\left[\sigma(x_{q_n}, x_{r_n+1}) + \sigma(x_{r_n}, x_{q_n+1})\right]\right).
\end{aligned}$$

Now, by taking limit as $n \to \infty$ in the above inequality, we derive that

$$\varepsilon \leq \frac{1}{6}[0 + 0 + \varepsilon + \varepsilon] - \varphi(0, 0, \varepsilon) \leq \frac{1}{3}\varepsilon,$$

which is a contradiction. Hence, condition (K) holds. We are ready to show that the sequence $\{x_n\}$ is Cauchy. Fix $\varepsilon > 0$. By the claim, we find $n_0 \in \mathbb{N}$ such that if $r, q \geq n_0$ with $r - q \equiv 1(m)$,

$$\sigma(x_r, x_q) \leq \frac{\varepsilon}{2}. \tag{25}$$

Since $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$\sigma(x_n, x_{n+1}) \leq \frac{\varepsilon}{2m} \tag{26}$$

for any $n \geq n_1$. Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \dots, m\}$ such that $s - r \equiv k(m)$. Therefore, $s - r + \varphi \equiv 1(m)$ for $\varphi = m - k + 1$. So, we have, for $j \in \{1, \dots, m\}$, $s + j - r \equiv 1(m)$,

$$\sigma(x_r, x_s) \leq \sigma(x_r, x_{s+j}) + \sigma(x_{s+j}, x_{s+j-1}) + \cdots + \sigma(x_{s+1}, x_s).$$

By (25) and (26) and from the last inequality, we get

$$\sigma(x_r, x_s) \leq \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \leq \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon.$$

This proves that $\{x_n\}$ is a σ-Cauchy sequence.

Since Y is σ-closed in (X, σ), then (Y, σ) is also complete, there exists $z \in Y = \bigcup_{i=1}^m A_i$ such that $\lim_{n\to\infty} x_n = z$ in (Y, p); equivalently

$$\sigma(z, z) = \lim_{n\to\infty} \sigma(z, x_n) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{27}$$

In what follows, we prove that x is a fixed point of T. In fact, since $\lim_{n\to\infty} x_n = z$ and, as $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T, the sequence (x_n) has infinite terms in each A_i for $i \in \{1, 2, \dots, m\}$. Suppose that $x \in A_i$, $Tx \in A_{i+1}$, and we take a subsequence x_{n_k} of (x_n) with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by the above-mentioned comment). By using the contractive condition, we can obtain

$$\begin{aligned}
\sigma(x_{n_{k+1}}, Tx) &= \sigma(Tx_{n_k}, Tx) \\
&\leq \psi(x_{n_k})\sigma(x_{n_k}, Tx_{n_k}) + \psi(Tx_{n_k})\sigma(x, Tx) \\
&\quad + \psi\left(T^2 x_{n_k}\right)\sigma(x_{n_k}, Tx) + \psi\left(T^3 x_{n_k}\right)\sigma(x, Tx_{n_k}) \\
&\quad - \varphi\left(\sigma(x_{n_k}, Tx_{n_k}), \sigma(x, Tx), \frac{1}{2}\left[\sigma(x_{n_k}, Tx) + \sigma(x, Tx_{n_k})\right]\right) \\
&\leq \frac{1}{6}\left[\sigma(x_{n_k}, x_{n_k+1}) + \sigma(x, Tx) + \sigma(x_{n_k}, Tx) + \sigma(x, x_{n_k+1})\right] \\
&\quad - \varphi\left(\sigma(x_{n_k}, x_{n_k+1}), \sigma(x, Tx), \frac{1}{2}\left[\sigma(x_{n_k}, Tx) + \sigma(x, x_{n_k+1})\right]\right).
\end{aligned}$$

Passing to the limit as $n \to \infty$ and using $x_{n_k} \to x$, lower semi-continuity of φ, we have

$$\sigma(x, Tx) \leq \frac{1}{3}\sigma(x, Tx) - \varphi\left(0, \sigma(x, Tx), \frac{1}{2}\sigma(x, Tx)\right) \leq \frac{1}{3}\sigma(x, Tx).$$

So, $\sigma(x,Tx) = 0$ and, therefore, x is a fixed point of T. Finally, to prove the uniqueness of the fixed point, suppose that $y,z \in X$ are fixed points of T. The cyclic character of T and the fact that $y,z \in X$ are fixed points of T imply that $y,z \in \bigcap_{i=1}^m A_i$. Also, suppose that $\psi(y) \le \frac{1}{6}$. By using the contractive condition, we derive that

$$\begin{aligned}\sigma(y,z) &= \sigma(Ty,Tx)\\ &\le \psi(y)\sigma(y,Ty) + \psi(Ty)\sigma(z,Tz) + \psi\bigl(T^2y\bigr)\sigma(y,Tz) + \psi\bigl(T^2y\bigr)\sigma(z,Ty)\\ &\quad - \varphi\biggl(\sigma(y,Ty), \sigma(z,Tz), \frac{1}{2}\bigl[\sigma(y,Tz) + \sigma(z,Ty)\bigr]\biggr).\end{aligned}$$

Then

$$\begin{aligned}\sigma(y,z) &\le \frac{1}{6}\bigl[2\sigma(y,z) + \sigma(y,y)\bigr] - \varphi\biggl(0,0,\frac{1}{2}\bigl[\sigma(y,z) + \sigma(z,y)\bigr]\biggr)\\ &\le \frac{1}{6}\bigl[2\sigma(y,z) + 2\sigma(y,z)\bigr] - \varphi\biggl(0,0,\frac{1}{2}\bigl[\sigma(y,z) + \sigma(z,y)\bigr]\biggr)\\ &= \frac{2}{3}\sigma(y,z) - \varphi\biggl(0,0,\frac{1}{2}\bigl[\sigma(y,z) + \sigma(z,y)\bigr]\biggr) \le \frac{2}{3}\sigma(y,z).\end{aligned}$$

This gives us $\sigma(y,z) = 0$, that is, $y = z$. This finishes the proof. □

Corollary 1.21 *Let (X,σ) be a complete metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \dots, A_m$ be nonempty σ-closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ is an operator such that*

(i) *$Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T;*
(ii) *there exists $\beta \in [0, \frac{1}{6})$ such that*

$$\sigma(Tx,Ty) \le \beta\bigl[\sigma(x,Tx) + \sigma(y,Ty) + \sigma(x,Ty) + \sigma(y,Tx)\bigr] \tag{28}$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\dots,m$, where $A_{m+1} = A_1$. Then T has a fixed point $z \in \bigcap_{i=1}^n A_i$.

Proof Let $\psi(t) = \frac{1}{6}$ and $\beta \in [0, \frac{1}{6})$. Here, it suffices to take the function $\varphi : [0,+\infty)^4 \to [0,+\infty)$ defined by $\varphi(a,b,c,e) = (\frac{1}{6} - \beta)(a+b+c+e)$. Obviously, φ satisfies that $\varphi(a,b,c,e) = 0$ if and only if $a = b = c = e = 0$, and $\varphi(x,y,z,t) = (\frac{1}{6} - \beta)(x+y+z+t) = \varphi(x+y+z+t, 0)$. Then we apply Theorem 1.20 to finish the proof. □

Example 1.22 Let $X = \mathbb{R}$ with the metric-like $\sigma(x,y) = \max\{|x|,|y|\}$ for all $x,y \in X$. Suppose $A_1 = [-1,0]$ and $A_2 = [0,1]$ and $Y = \bigcup_{i=1}^2 A_i$. Define $T : Y \to Y$ by

$$Tx = \begin{cases} -\frac{1}{24}x & \text{if } x \in [-1,0], \\ -\frac{1}{12}x & \text{if } x \in [0,1]. \end{cases}$$

It is clear that $\bigcup_{i=1}^2 A_i$ is a cyclic representation of Y with respect to T.

Let $x \in A_1 = [-1, 0]$ and $y \in A_2 = [0, 1]$. Then

$$\begin{aligned}\sigma(Tx, Ty) &= \max\left\{\left|-\frac{1}{24}x\right|, \left|-\frac{1}{12}y\right|\right\} = \max\left\{-\frac{1}{24}x, \frac{1}{12}y\right\} \leq \max\left\{\frac{-x}{12}, \frac{y}{12}\right\} \\ &= \frac{1}{12}\max\{-x, y\} = \frac{1}{12}\sigma(x, y),\end{aligned}$$

and so

$$\sigma(Tx, Ty) \leq \frac{1}{12}\big[\sigma(x, Tx) + \sigma(y, Ty) + \sigma(x, Ty) + \sigma(y, Tx)\big].$$

Hence, the conditions of Corollary 1.21 (Theorem 1.20) hold and T has a fixed point in $A_1 \cap A_2$. Here, $x = 0$ is a fixed point of T.

If in Theorem 1.20 we take $A_i = X$ for all $0 \leq i \leq m$, then we deduce the following theorem.

Theorem 1.23 *Let (X, σ) be a complete metric-like space and let $T : X \to X$ be a sub-ψ-admissible mapping such that*

$$\begin{aligned}\sigma(Tx, Ty) \leq{}& \psi(x)\sigma(x, Tx) + \psi(Tx)\sigma(y, Ty) + \psi\big(T^2x\big)\sigma(x, Ty) + \psi\big(T^3x\big)\sigma(y, Tx) \\ &- \varphi\left(\sigma(x, Tx), \sigma(x, Ty), \frac{1}{2}\big[\sigma(x, Ty) + \sigma(y, Tx)\big]\right)\end{aligned}$$

for any $x, y \in X$, where $\psi \in \Psi$ and $\varphi \in \Lambda$. Then T has a unique fixed point in X.

Corollary 1.24 *Let (X, σ) be a complete metric-like space and let $T : X \to X$ be a sub-ψ-admissible mapping such that*

$$\sigma(Tx, Ty) \leq \beta\big[\sigma(x, Tx) + \sigma(y, Ty) + \sigma(x, Ty) + \sigma(y, Tx)\big]$$

for any $x, y \in X$, where $\beta \in [0, \frac{1}{6})$. Then T has a unique fixed point in X.

Example 1.25 Let $X = \mathbb{R}_+$ with the metric-like $\sigma(x, y) = \max\{x, y\}$ for all $x, y \in X$. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{14}(x^3 + x) & \text{if } 0 \leq x < 1, \\ \frac{1}{10}x^2 & \text{if } x \geq 1. \end{cases}$$

Proof To show the existence and uniqueness point of T, we investigate the following cases:

- Let $0 \leq x, y < 1$. Then we get

$$\sigma(Tx, Ty) = \max\left\{\frac{1}{14}\big(x^3 + x\big), \frac{1}{14}\big(y^3 + y\big)\right\} \leq \frac{1}{7}\max\{x, y\} = \frac{1}{7}\sigma(x, y).$$

- Let $x, y \geq 1$. So we have

$$\sigma(Tx, Ty) = \frac{1}{10}\max\big\{x^2, y^2\big\} \leq \frac{1}{10}\max\{x, y\} \leq \frac{1}{7}\max\{x, y\} = \frac{1}{7}\sigma(x, y).$$

- Let $0 \le x < 1$ and $y \ge 1$. Then we obtain

$$\sigma(Tx, Ty) = \max\left\{\frac{1}{14}\left(x^2 + x\right), \frac{1}{10}y^2\right\} \le \max\left\{\frac{1}{7}x, \frac{1}{10}y\right\} \le \frac{1}{7}\max\{x, y\} = \frac{1}{7}\sigma(x, y),$$

and hence

$$\sigma(Tx, Ty) \le \frac{1}{7}\left[\sigma(x, Tx) + \sigma(y, Ty) + \sigma(x, Ty) + \sigma(y, Tx)\right].$$

Then all the conditions of Corollary 1.24 (Theorem 1.23) are satisfied. Thus, T has a unique fixed point X. Indeed, 0 is the unique fixed point of T. □

Corollary 1.26 *Let (X,σ) be a complete metric-like space, $m \in \mathbb{N}$, let $A_1, A_2, \dots, A_m$ be nonempty σ-closed subsets of X and $Y = \bigcup_{i=1}^m A_i$. Suppose that $T : Y \to Y$ is an operator such that*

(i) *$Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of X with respect to T;*
(ii) *there exists $\beta \in [0, \frac{1}{6})$ such that*

$$\int_0^{\sigma(Tx,Ty)} \rho(t)\,dt \le \beta \int_0^{\sigma(x,Tx)+\sigma(y,Ty)+\sigma(x,Ty)+\sigma(y,Tx)} \rho(t)\,dt$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$, where $A_{m+1} = A_1$, and $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)\,dt$ for $\varepsilon > 0$. Then T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

If in Corollary 1.26, we take $A_i = X$ for $i = 1, 2, \dots, m$, we obtain the following result.

Corollary 1.27 *Let (X,σ) be a complete metric-like space and let $T : X \to X$ be a mapping such that for any $x, y \in X$,*

$$\int_0^{\sigma(Tx,Ty)} \rho(t)\,dt \le \beta \int_0^{\sigma(x,Tx)+\sigma(y,Ty)+\sigma(x,Ty)+\sigma(y,Tx)} \rho(t)\,dt,$$

where $\rho : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping satisfying $\int_0^\varepsilon \rho(t)\,dt$ for $\varepsilon > 0$ and the constant $\beta \in [0, \frac{1}{6})$. Then T has a unique fixed point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Atilim University, Incek, Ankara 06836, Turkey. [2]Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran.

Acknowledgements
The authors thanks to anonymous referee for his remarkable comments, suggestion and ideas that helps to improve this paper.

Received: 6 February 2013 Accepted: 7 August 2013 Published: 21 August 2013

References

1. Hitzler, P: Generalized metrics and topology in logic programming semantics. PhD thesis, School of Mathematics, Applied Mathematics and Statistics, National University Ireland, University College Cork (2001)
2. Amini-Harandi, A: Metric-like spaces, partial metric spaces and fixed points. Fixed Point Theory Appl. **2012**, 204 (2012)
3. Hitzler, P, Seda, AK: Dislocated topologies. J. Electr. Eng. **51**(12), 3-7 (2000)
4. Aage, CT, Salunke, JN: The results on fixed points in dislocated and dislocated quasi-metric space. Appl. Math. Sci. **2**(59), 2941-2948 (2008)
5. Aage, CT, Salunke, JN: Some results of fixed point theorem in dislocated quasi-metric spaces. Bull. Marathwada Math. Soc. **9**, 1-5 (2008)
6. Daheriya, RD, Jain, R, Ughade, M: Some fixed point theorem for expansive type mapping in dislocated metric space. ISRN Math. Anal. **2012**, Article ID 376832 (2012)
7. Sarma, IR, Kumari, PS: On dislocated metric spaces. Int. J. Math. Arch. **3**(1), 72-77 (2012)
8. Shrivastava, R, Ansari, ZK, Sharma, M: Some results on fixed points in dislocated and dislocated quasi-metric spaces. J. Adv. Stud. Topol. **3**(1), 25-31 (2012)
9. Zeyada, FM, Hassan, GH, Ahmed, MA: A generalization of a fixed point theorem due to Hitzler and Seda in dislocated quasi-metric spaces. Arab. J. Sci. Eng., Sect. A **31**(1), 111-114 (2005)
10. Zoto, K, Hoxha, E, Isufati, A: Some new results in dislocated and dislocated quasi-metric spaces. Appl. Math. Sci. **6**(71), 3519-3526 (2012)
11. Zoto, K, Hoxha, E: Fixed point theorems for ψ-contractive type mappings in dislocated quasi-metric spaces. Int. Math. Forum **7**(51), 2503-2508 (2012)
12. Zoto, K, Hoxha, E: Fixed point theorems in dislocated and dislocated quasi-metric spaces. J. Adv. Stud. Topol. **3**(4), 119-124 (2012)
13. Kirk, WA, Srinavasan, PS, Veeramani, P: Fixed points for mapping satisfying cyclical contractive conditions. Fixed Point Theory **4**, 79-89 (2003)
14. Sintunavarat, W, Kumam, P: Common fixed point theorem for cyclic generalized multi-valued contraction mappings. Appl. Math. Lett. **25**(11), 1849-1855 (2012)
15. Nashine, HK, Sintunavarat, W, Kumam, P: Cyclic generalized contractions and fixed point results with applications to integral equation. Fixed Point Theory Appl. **2012**, 217 (2012)
16. Mongkolkeha, C, Kumam, P: Best proximity point theorems for generalized cyclic contractions in ordered metric spaces. J. Optim. Theory Appl. (2012). doi:10.1007/s10957-012-9991-y
17. Eldered, AA, Veeramani, P: Convergence and existence for best proximity points. J. Math. Anal. Appl. **323**, 1001-1006 (2006)
18. Eldered, AA, Veeramani, P: Proximal pointwise contraction. Topol. Appl. **156**, 2942-2948 (2009)
19. Chen, CM: Fixed point theorems for cyclic Meir-Keeler type mappings in complete metric spaces. Fixed Point Theory Appl. **2012**, 41 (2012)
20. Karapınar, E: Fixed point theory for cyclic weak ϕ-contraction. Appl. Math. Lett. **24**, 822-825 (2011)
21. Karapınar, E, Erhan, IM, Ulus, AY: Fixed point theorem for cyclic maps on partial metric spaces. Appl. Math. Inf. Sci. **6**(1), 239-244 (2012)
22. Karapınar, E, Sadarangani, K: Fixed point theory for cyclic $(\psi - \phi)$ contractions. Fixed Point Theory Appl. **2011**, 69 (2011)
23. Karpagam, S, Agrawal, S: Best proximity points theorems for cyclic Meir-Keeler contraction maps. Nonlinear Anal. **74**, 1040-1046 (2011)
24. Karpagam, S, Agrawal, S: Existence of best proximity points of p-cyclic contractions. Fixed Point Theory **13**(1), 99-105 (2012)
25. Păcurar, M, Rus, IA: Fixed point theory for cyclic φ-contractions. Nonlinear Anal. **72**(3-4), 1181-1187 (2010)
26. Petruşel, G: Cyclic representations and periodic points. Stud. Univ. Babeş-Bolyai, Math. **50**, 107-112 (2005)
27. Rezapour, S, Derafshpour, M, Shahzad, N: Best proximity point of cyclic φ-contractions in ordered metric spaces. Topol. Methods Nonlinear Anal. **37**, 193-202 (2011)
28. Rus, IA: Cyclic representations and fixed points. Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity **3**, 171-178 (2005)

doi:10.1186/1687-1812-2013-222
Cite this article as: Karapınar and Salimi: **Dislocated metric space to metric spaces with some fixed point theorems.** *Fixed Point Theory and Applications* 2013 **2013**:222.