Common fuzzy fixed points for fuzzy mappings

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Abstract

Let (X, d) be a metric space and S, T be mappings from X to a set of all fuzzy subsets of X. We obtained sufficient conditions for the existence of a common α-fuzzy fixed point of S and T.

Keywords: analysis; fuzzy set; fuzzy mapping; α-fuzzy fixed point

1 Introduction

Fixed point theorems play a fundamental role in demonstrating the existence of solutions to a wide variety of problems arising in mathematics, physics, engineering, medicine and social sciences. The study of fixed point theorems in fuzzy mathematics was instigated by Weiss [1], Butnariu [2], Singh and Talwar [3], Estruch and Vidal [4], Wang *et al.* [5], Mihet [6], Qiu *et al.* [7] and Beg and Abbas [8]. Heilpern [9] introduced the concept of fuzzy contraction mappings and established the fuzzy Banach contraction principle on a complete metric linear spaces with the d_∞-metric for fuzzy sets. Azam and Beg [10] proved common fixed point theorems for a pair of fuzzy mappings satisfying Edelstein, Alber and Guerr-Delabriere type contractive conditions in a metric linear space. Azam *et al.* [11] presented some fixed point theorems for fuzzy mappings under Edelstein locally contractive conditions on a compact metric space with the d_∞-metric for fuzzy sets. Frigon and Regan [12] generalized the Heilpern theorem under a contractive condition for 1-level sets (*i.e.*, $[Tx]_1$) of a fuzzy contraction T on a complete metric space, where 1-level sets are not assumed to be convex and compact. Amemiya and Takahashi [13] studied some mathematical properties of contractive type set-valued and fuzzy mappings to obtain fixed points of fuzzy mappings by using the concept of w-distance (see [13]) in complete metric spaces. Recently, Zhang *et al.* [14] proved some common fixed point theorems for contraction mappings in a new fuzzy metric space.

The aim of this paper is to obtain a common α-fuzzy fixed point of a pair of fuzzy mappings S and T on a complete metric space under a generalized contractive condition for α-level sets (*i.e.*, $[Sx]_\alpha$, $[Tx]_\alpha$) of S and T in connection with Hausdorff metric for fuzzy sets. Our result (Theorem 5) generalizes the results proved by Azam and Arshad [15, Theorem 4], Bose and Sahani [16] and Vijayaraju and Marudai [17, Theorem 3.1] among others.

2 Preliminaries

Let (X, d) be a metric space and $CB(X)$ be the family of nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where

$$d(x, A) = \inf_{y \in A} d(x, y).$$

A fuzzy set in X is a function with domain X and values in $[0, 1]$. If A is a fuzzy set and $x \in X$, then the function-value $A(x)$ is called the grade of membership of x in A. The α-*level set* of A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \left\{ x : A(x) \geq \alpha \right\} \quad \text{if } \alpha \in (0, 1],$$
$$[A]_0 = \overline{\left\{ x : A(x) > 0 \right\}}.$$

Here, $\overline{B}$ denotes the closure of the set B. Let $\mathcal{F}(X)$ be the collection of all fuzzy sets in a metric space X. For $A, B \in \mathcal{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. We denote the fuzzy set $\chi_{\{x\}}$ by $\{x\}$ unless and until it is stated, where χ_A is the characteristic function of the crisp set A. If there exists an $\alpha \in [0, 1]$ such that $[A]_\alpha, [B]_\alpha \in CB(X)$, then define

$$p_\alpha(A, B) = \inf_{x \in [A]_\alpha, y \in [B]_\alpha} d(x, y),$$
$$D_\alpha(A, B) = H\big([A]_\alpha, [B]_\alpha\big).$$

If $[A]_\alpha, [B]_\alpha \in CB(X)$ for each $\alpha \in [0, 1]$, then define

$$p(A, B) = \sup_\alpha p_\alpha(A, B),$$
$$d_\infty(A, B) = \sup_\alpha D_\alpha(A, B).$$

We write $p(x, B)$ instead of $p(\{x\}, B)$. A fuzzy set A in a metric linear space V is said to be an *approximate quantity* if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and

$$\sup_{x \in V} A(x) = 1.$$

We denote the collection of all approximate quantities in a metric linear space X by $W(X)$. Let X be an arbitrary set, Y be a metric space. A mapping T is called a *fuzzy mapping* if T is a mapping from X into $\mathcal{F}(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with a membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$.

Definition 1 Let S, T be fuzzy mappings from X into $\mathcal{F}(X)$. A point z in X is called an α-*fuzzy fixed point* of T if $z \in [Tz]_\alpha$. The point z is called a common α-fuzzy fixed point of S and T if $z \in [Sz]_\alpha \cap [Tz]_\alpha$. When $\alpha = 1$, it is called a common fixed point of fuzzy maps.

For the sake of convenience, we first state some known results for subsequent use in the next section.

Lemma 2 [18] *Let (X, d) be a metric space and $A, B \in CB(X)$, then for each $a \in A$,*

$$d(a, B) \leq H(A, B).$$

Lemma 3 [18] *Let (X, d) be a metric space and $A, B \in CB(X)$, then for each $a \in A$, $\beta > 0$, there exists an element $b \in B$ such that $d(a, b) \leq H(A, B) + \beta$.*

Lemma 4 [19] *Let V be a metric linear space, $T : V \to W(V)$ be a fuzzy mapping and $x_0 \in V$. Then there exists $x_1 \in V$ such that $\{x_1\} \subset T(x_0)$.*

3 Common fuzzy fixed points

In this section, we establish Theorem 5 on the existence of an α-fuzzy fixed point of a fuzzy mapping and also obtain a fixed point of fuzzy mappings (see Corollaries 8 and 9) and multivalued mappings (see Corollary 7).

Theorem 5 *Let (X, d) be a complete metric space and let S, T be fuzzy mappings from X to $\mathcal{F}(X)$ satisfying the following conditions:*

(a) *for each $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that $[Sx]_{\alpha(x)}$, $[Tx]_{\alpha(x)}$ are nonempty closed bounded subsets of X and*

(b)

$$H\big([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\big) \leq a_1 d\big(x, [Sx]_{\alpha(x)}\big) + a_2 d\big(y, [Ty]_{\alpha(y)}\big)$$

$$+ a_3 d\big(x, [Ty]_{\alpha(y)}\big) + a_4 d\big(y, [Sx]_{\alpha(x)}\big) + a_5 d(x, y) \tag{1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers and $\sum_{i=1}^{5} a_i < 1$ and either $a_1 = a_2$ or $a_3 = a_4$. Then there exists $z \in X$ such that $z \in [Sz]_{\alpha(z)} \cap [Tz]_{\alpha(z)}$.

Proof We consider the following three possible cases:

(i) $a_1 + a_3 + a_5 = 0$;

(ii) $a_2 + a_4 + a_5 = 0$;

(iii) $a_1 + a_3 + a_5 \neq 0$, $a_2 + a_4 + a_5 \neq 0$.

Case (i): For $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that $[Sx]_{\alpha(x)}$ is a nonempty closed bounded subset of X. Take $y \in [Sx]_{\alpha(x)}$ and similarly $z \in [Ty]_{\alpha(y)}$. Then by Lemma 2, we obtain

$$d\big(y, [Ty]_{\alpha(y)}\big) \leq H\big([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\big).$$

Now, inequality (1) implies that

$$d\big(y, [Ty]_{\alpha(y)}\big) \leq a_1 d\big(x, [Sx]_{\alpha(x)}\big) + a_2 d\big(y, [Ty]_{\alpha(y)}\big) + a_3 d\big(x, [Ty]_{\alpha(y)}\big)$$

$$+ a_4 d\big(y, [Sx]_{\alpha(x)}\big) + a_5 d(x, y).$$

Using $a_1 + a_3 + a_5 = 0$ together with the fact that $d(y, [Sx]_{\alpha(x)}) = 0$, we obtain

$$d\big(y, [Ty]_{\alpha(y)}\big) \leq a_2 d\big(y, [Ty]_{\alpha(y)}\big).$$

It follows that $y \in [Ty]_{\alpha(y)}$, which further implies that

$$d\big(y, [Sy]_{\alpha(y)}\big) \leq H\big([Ty]_{\alpha(y)}, [Sy]_{\alpha(y)}\big).$$

Again, inequality (1) gives $d(y, [Sy]_{\alpha(y)}) = 0$. It follows that

$$y \in [Sy]_{\alpha(y)} \cap [Ty]_{\alpha(y)}.$$

Case (ii): It can be easily seen that

$$z \in [Sz]_{\alpha(z)} \cap [Tz]_{\alpha(z)}.$$

Case (iii): Let

$$\lambda = \left(\frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} \right), \qquad \mu = \left(\frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} \right).$$

If $a_3 = a_4$, then $\lambda, \mu < 1$ and so $0 < \lambda\mu < 1$. Moreover, if $a_1 = a_2$,

$$\begin{aligned}
0 < \lambda\mu &= \left(\frac{a_1 + a_3 + a_5}{1 - a_2 - a_3} \right)\left(\frac{a_2 + a_4 + a_5}{1 - a_1 - a_4} \right) \\
&= \left(\frac{a_1 + a_3 + a_5}{1 - a_1 - a_3} \right)\left(\frac{a_1 + a_4 + a_5}{1 - a_1 - a_4} \right) \\
&= \left(\frac{a_1 + a_3 + a_5}{1 - a_1 - a_4} \right)\left(\frac{a_1 + a_4 + a_5}{1 - a_1 - a_3} \right) < 1.
\end{aligned}$$

Choose $x_0 \in X$, then by hypotheses there exists $\alpha(x_0) \in (0,1]$ such that $[Sx_0]_{\alpha(x_0)}$ is a nonempty closed bounded subset of X. For convenience, we denote $\alpha(x_0)$ by α_1. Let $x_1 \in [Sx_0]_{\alpha_1}$; for this x_1, there exists $\alpha_2 \in (0,1]$ such that $[Tx_1]_{\alpha_2}$ is a nonempty closed bounded subset of X. Since $a_1 + a_3 + a_5 > 0$, by Lemma 3, there exists $x_2 \in [Tx_1]_{\alpha_2}$ such that

$$d(x_1, x_2) \leq H\big([Sx_0]_{\alpha_1}, [Tx_1]_{\alpha_2}\big) + a_1 + a_3 + a_5.$$

It implies that

$$\begin{aligned}
d(x_1, x_2) &\leq H\big([Sx_0]_{\alpha_1}, [Tx_1]_{\alpha_2}\big) + a_1 + a_3 + a_5 \\
&\leq a_1 d\big(x_0, [Sx_0]_{\alpha_1}\big) + a_2 d\big(x_1, [Tx_1]_{\alpha_2}\big) + a_3 d\big(x_0, [Tx_1]_{\alpha_2}\big) \\
&\quad + a_4 d\big(x_1, [Sx_0]_{\alpha_1}\big) + a_5 d(x_0, x_1) + a_1 + a_3 + a_5 \\
&\leq (a_1 + a_5)d(x_0, x_1) + a_2 d(x_1, x_2) + a_3 d(x_0, x_2) + a_1 + a_3 + a_5 \\
&\leq (a_1 + a_3 + a_5)d(x_0, x_1) + (a_2 + a_3)d(x_1, x_2) + a_1 + a_3 + a_5 \\
&\leq (a_1 + a_3 + a_5)d(x_0, x_1) + a_1 + a_3 + a_5 \\
&\leq \lambda d(x_0, x_1) + \lambda.
\end{aligned}$$

By the same argument, we can find $\alpha_3 \in (0,1]$ and $x_3 \in [Sx_2]_{\alpha_3}$ such that

$$
\begin{aligned}
d(x_2, x_3) &\leq H\big([Sx_2]_{\alpha_3}, [Tx_1]_{\alpha_2}\big) + \lambda(a_2 + a_4 + a_5) \\
&\leq a_1 d\big(x_2, [Sx_2]_{\alpha_3}\big) + a_2 d\big(x_1, [Tx_1]_{\alpha_2}\big) + a_3 d\big(x_2, [Tx_1]_{\alpha_2}\big) \\
&\quad + a_4 d\big(x_1, [Sx_2]_{\alpha_3}\big) + a_5 d(x_2, x_1) + \lambda(a_2 + a_4 + a_5) \\
&\leq a_1 d(x_2, x_3) + a_2 d(x_1, x_2) + a_3 d(x_2, x_2) \\
&\quad + a_4 d(x_1, x_3) + a_5 d(x_2, x_1) + \lambda(a_2 + a_4 + a_5) \\
&\leq (a_2 + a_4 + a_5) d(x_1, x_2) + (a_1 + a_4) d(x_2, x_3) + \lambda(a_2 + a_4 + a_5) \\
&\leq \mu d(x_1, x_2) + \lambda\mu.
\end{aligned}
$$

By induction, we produce a sequence $\{x_n\}$ of points of X,

$$
\begin{aligned}
&x_{2k+1} \in [Sx_{2k}]_{\alpha_{2k+1}}, \\
&x_{2k+2} \in [Tx_{2k+1}]_{\alpha_{2k+2}}, \quad k = 0, 1, 2, \ldots,
\end{aligned}
$$

such that

$$
\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &\leq H\big([Sx_{2k}]_{\alpha_{2k+1}}, [Tx_{2k+1}]_{\alpha_{2k+2}}\big) + (\lambda\mu)^k (a_1 + a_3 + a_5), \\
d(x_{2k+2}, x_{2k+3}) &\leq H\big([Sx_{2k+2}]_{\alpha_{2k+3}}, [Tx_{2k+1}]_{\alpha_{2k+2}}\big) + (\lambda\mu)^k \lambda(a_2 + a_4 + a_5).
\end{aligned}
$$

It implies that

$$
\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &\leq a_1 d\big(x_{2k}, [Sx_{2k}]_{\alpha_{2k+1}}\big) + a_2 d\big(x_{2k+1}, [Tx_{2k+1}]_{\alpha_{2k+2}}\big) \\
&\quad + a_3 d\big(x_{2k}, [Tx_{2k+1}]_{\alpha_{2k+2}}\big) + a_4 d\big(x_{2k+1}, [Sx_{2k}]_{\alpha_{2k+1}}\big) \\
&\quad + a_5 d(x_{2k}, x_{2k+1}) + (\lambda\mu)^k (a_1 + a_3 + a_5) \\
&\leq (a_1 + a_3 + a_5) d(x_{2k}, x_{2k+1}) + (a_2 + a_3) d(x_{2k+1}, x_{2k+2}) \\
&\quad + (\lambda\mu)^k (a_1 + a_3 + a_5) \\
&\leq \lambda d(x_{2k}, x_{2k+1}) + \lambda(\lambda\mu)^k.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
d(x_{2k+2}, x_{2k+3}) &\leq a_1 d\big(x_{2k+2}, [Sx_{2k+2}]_{\alpha_{2k+3}}\big) + a_2 d\big(x_{2k+1}, [Tx_{2k+1}]_{\alpha_{2k+2}}\big) \\
&\quad + a_3 d\big(x_{2k+2}, [Tx_{2k+1}]_{\alpha_{2k+2}}\big) + a_4 d\big(x_{2k+1}, [Sx_{2k+2}]_{\alpha_{2k+3}}\big) \\
&\quad + a_5 d(x_{2k+2}, x_{2k+1}) + (\lambda\mu)^k \lambda(a_2 + a_4 + a_5) \\
&\leq (a_2 + a_4 + a_5) d(x_{2k+1}, x_{2k+2}) + (a_1 + a_4) d(x_{2k+2}, x_{2k+3}) \\
&\quad + (\lambda\mu)^k \lambda(a_2 + a_4 + a_5) \\
&\leq \mu d(x_{2k+1}, x_{2k+2}) + (\lambda\mu)^{k+1}.
\end{aligned}
$$

It follows that for each $k = 0, 1, 2, \ldots,$

$$
\begin{aligned}
d(x_{2k+1}, x_{2k+2}) &\leq \lambda d(x_{2k}, x_{2k+1}) + \lambda(\lambda\mu)^k \\
&\leq \lambda \left[\mu d(x_{2k-1}, x_{2k}) + (\lambda\mu)^k \right] + \mu^k \lambda^{k+1} \\
&\leq \lambda\mu d(x_{2k-1}, x_{2k}) + 2\lambda(\lambda\mu)^k \\
&\leq \lambda\mu\lambda d(x_{2k-2}, x_{2k-1}) + 3\lambda(\lambda\mu)^k \\
&\leq \cdots \leq \lambda(\lambda\mu)^k d(x_0, x_1) + (2k+1)\lambda(\lambda\mu)^k
\end{aligned}
$$

and

$$
\begin{aligned}
d(x_{2k+2}, x_{2k+3}) &\leq \mu d(x_{2k+1}, x_{2k+2}) + (\lambda\mu)^{k+1} \\
&\leq \cdots \leq (\lambda\mu)^{k+1} d(x_0, x_1) + (2k+2)(\lambda\mu)^{k+1}.
\end{aligned}
$$

Then for $p < q$, we have

$$
\begin{aligned}
d(x_{2p+1}, x_{2q+1}) &\leq d(x_{2p+1}, x_{2p+2}) + d(x_{2p+2}, x_{2p+3}) + d(x_{2p+3}, x_{2p+4}) \\
&\quad + \cdots + d(x_{2q}, x_{2q+1}) \\
&\leq \left[\lambda \sum_{i=p}^{q-1}(\lambda\mu)^i + \sum_{i=p+1}^{q}(\lambda\mu)^i \right] d(x_0, x_1) \\
&\quad + \lambda \sum_{i=p}^{q-1}(2i+1)(\lambda\mu)^i + \sum_{i=p+1}^{q} 2i(\lambda\mu)^i.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
d(x_{2p}, x_{2q+1}) &\leq \left[\sum_{i=p}^{q}(\lambda\mu)^i + \lambda \sum_{i=p}^{q-1}(\lambda\mu)^i \right] d(x_0, x_1) \\
&\quad + \sum_{i=p}^{q} 2i(\lambda\mu)^i + \lambda \sum_{i=p}^{q-1}(2i+1)(\lambda\mu)^i,
\end{aligned}
$$

$$
\begin{aligned}
d(x_{2p}, x_{2q}) &\leq \left[\sum_{i=p}^{q-1}(\lambda\mu)^i + \lambda \sum_{i=p}^{q-1}(\lambda\mu)^i \right] d(x_0, x_1) \\
&\quad + \sum_{i=p}^{q-1} 2i(\lambda\mu)^i + \lambda \sum_{i=p}^{q-1}(2i+1)(\lambda\mu)^i
\end{aligned}
$$

and

$$
\begin{aligned}
d(x_{2p+1}, x_{2q}) &\leq \left[\lambda \sum_{i=p}^{q-1}(\lambda\mu)^i + \sum_{i=p+1}^{q-1}(\lambda\mu)^i \right] d(x_0, x_1) \\
&\quad + \sum_{i=p}^{q-1}(2i+1)(\lambda\mu)^i + \sum_{i=p+1}^{q-1} 2i(\lambda\mu)^i.
\end{aligned}
$$

Since $(\lambda\mu) < 1$, it follows from Cauchy's root test $\sum 2i(\lambda\mu)^i$, $\sum(2i+1)(\lambda\mu)^i$ are convergent series. Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that $x_n \to u$. Now,

$$
\begin{aligned}
d\big(u, [Su]_{\alpha(u)}\big) &\leq d(u, x_{2n}) + d\big(x_{2n}, [Su]_{\alpha(u)}\big) \\
&\leq d(u, x_{2n}) + H\big([Tx_{2n-1}]^{\alpha_{2n}}, [Su]_{\alpha(u)}\big) \\
&\leq d(u, x_{2n}) + a_1 d\big(u, [Su]_{\alpha(u)}\big) + a_2 d\big(x_{2n-1}, [Tx_{2n-1}]^{\alpha_{2n}}\big) \\
&\quad + a_3 d\big(u, [Tx_{2n-1}]^{\alpha_{2n}}\big) + a_4 d\big(x_{2n-1}, [Su]_{\alpha(u)}\big) + a_5 d(u, x_{2n-1}) \\
&\leq d(u, x_{2n}) + a_1 d\big(u, [Su]_{\alpha(u)}\big) + a_2 d(x_{2n-1}, x_{2n}) \\
&\quad + a_3 d(u, x_{2n}) + a_4\big[d(x_{2n-1}, u) + d\big(u, [Su]_{\alpha(u)}\big)\big] + a_5 d(u, x_{2n-1}) \\
&\leq (1 + a_3)d(u, x_{2n}) + (a_5 + a_4)d(u, x_{2n-1}) + a_2 d(x_{2n-1}, x_{2n}) \\
&\quad + (a_1 + a_4)d\big(u, [Su]_{\alpha(u)}\big).
\end{aligned}
$$

It further implies that

$$
\begin{aligned}
d\big(u, [Su]_{\alpha(u)}\big) &\leq \left[\frac{1 + a_3}{1 - a_1 - a_4}\right]d(u, x_{2n}) + \left[\frac{a_4 + a_5}{1 - a_1 - a_4}\right]d(u, x_{2n-1}) \\
&\quad + \left[\frac{a_2}{1 - a_1 - a_4}\right]d(x_{2n-1}, x_{2n}).
\end{aligned}
$$

Letting $n \to \infty$, we have $d(u, [Su]_{\alpha(u)}) = 0$. It implies that $u \in [Su]_{\alpha(u)}$. Similarly, by using

$$
d\big(u, [Tu]_{\alpha(u)}\big) \leq d(u, x_{2n+1}) + d\big(x_{2n+1}, [Tu]_{\alpha(u)}\big),
$$

we can show that $u \in [Tu]_{\alpha(u)}$, which implies that $u \in [Su]_{\alpha(u)} \cap [Tu]_{\alpha(u)}$.

The proof of the following corollary illustrates a link between multivalued mappings and fuzzy mappings. It is well known [20] that

$$
(X, d) \mapsto \big(CB(X), H\big) \mapsto \big(E(X), d_\infty\big)
$$

are isometric embeddings under $x \to \{x\}$ and $A \to \chi_A$, respectively, where

$$
E(X) = \big\{A \in F(X) : [A]_\alpha \in CB(X), \forall \alpha \in [0, 1]\big\}
$$

and $\{x\}$ is a crisp set. $\qquad\square$

In the following, we furnish an illustrative example to highlight the utility of Theorem 5.

Example 6 Let $X = \{1, 2, 3\}, \{1\}, \{2\}, \{3\}$ be crisp sets. Define $d : X \times X \to \mathbb{R}$ as follows:

$$
d(x, y) = \begin{cases}
0 & \text{if } x = y, \\
\frac{5}{17} & \text{if } x \neq y \text{ and } x, y \in X - \{2\}, \\
1 & \text{if } x \neq y \text{ and } x, y \in X - \{3\}, \\
\frac{4}{17} & \text{if } x \neq y \text{ and } x, y \in X - \{1\}.
\end{cases}
$$

Define fuzzy mappings $S, T : X \to \mathcal{F}(X)$ as follows:

$$T(1)(t) = T(3)(t) = \begin{cases} \frac{3}{4} & \text{if } t = 1, \\ \frac{1}{2} & \text{if } t = 2, \\ 0 & \text{if } t = 3, \end{cases}$$

$$T(2)(t) = \begin{cases} 0 & \text{if } t = 1, \\ \frac{1}{3} & \text{if } t = 2, \\ \frac{3}{4} & \text{if } t = 3, \end{cases}$$

and

$$S(1)(t) = S(2)(t) = S(3)(t) = \begin{cases} \frac{3}{4} & \text{if } t = 1, \\ \frac{1}{19} & \text{if } t = 2, \\ 0 & \text{if } t = 3. \end{cases}$$

Then, for $\alpha(x) = \frac{3}{4}$,

$$[Tx]_{\alpha(x)} = \left\{ t : T(x)(t) = \alpha(x) \right\} = \begin{cases} \{1\} & \text{if } x \neq 2, \\ \{3\} & \text{if } x = 2 \end{cases}$$

and

$$H\big(\{1\}, \{3\}\big) = \max \left\{ \sup_{u \in \{1\}} d\big(u, \{3\}\big), \sup_{v \in \{3\}} d\big(\{1\}, v\big) \right\} = \frac{5}{17}.$$

Now,

$$d_\infty\big(T(3), T(2)\big) \geq D_\alpha\big(T(3), T(2)\big) = H\big(\{1\}, \{3\}\big) = \frac{5}{17}, \qquad d(3, 2) = \frac{4}{17}.$$

Since $d_\infty(T(3), T(2)) > \alpha d(3, 2)$ for each $\alpha < 1$ and X is not linear, therefore [12, Theorems 2.1, 2.2] and main results in [9–11, 16] are not applicable to find $1 \in [T1]_{\frac{3}{4}}$. Now, $[Sx]_{\alpha(x)} = \{t : S(x)(t) = \frac{3}{4}\} = \{1\}$ for all $x \in X$ and

$$H\big([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\big) = \begin{cases} H(\{1\}, \{1\}) = 0 & \text{if } y \neq 2, \\ H(\{1\}, \{3\}) = \frac{5}{17} & \text{if } y = 2. \end{cases}$$

It follows that for

$$a_1 = a_2 = a_3 = a_5 = 0, \qquad a_4 = \frac{5}{17},$$

$a_1 d(x, [Sx]_{\alpha(x)}) + a_2 d(y, [Ty]_{\alpha(y)}) + a_3 d(x, [Ty]_{\alpha(y)}) + a_4 d(y, [Sx]_{\alpha(x)}) + a_5 d(x, y) = \frac{5}{17}$ if $y = 2$. Hence, for $a_1 = a_2 = a_3 = a_5 = 0$, $a_4 = \frac{5}{17}$, the conditions of Theorem 5 are satisfied to obtain $1 \in [S1]_{\frac{3}{4}} \cap [T1]_{\frac{3}{4}}$. Since $a_3 \neq a_4$, the results proved by Vijayaraju and Marudai [17, Theorem 3.1] and Azam and Arshad [15, Theorem 4] are also not applicable.

Corollary 7 *Let (X, d) be a complete metric space and $F, G : X \to CB(X)$ be multivalued mappings. Suppose that for all $x, y \in X$,*

$$H(Fx, Gy) \leq a_1 d(x, Fx) + a_2 d(y, Gy) + a_3 d(x, Gy) + a_4 d(y, Fx) + a_5 d(x, y),$$

where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers and $\sum_{i=1}^{5} a_i < 1$ and either $a_1 = a_2$ or $a_3 = a_4$. Then there exists $u \in X$ such that $u \in Fu \cap Gu$.

Proof Consider a mapping $\alpha : X \to (0, 1]$ and a pair of fuzzy mappings $S, T : X \to \mathcal{F}(X)$ defined by

$$S(x)(t) = \begin{cases} \alpha(x), & t \in Fx, \\ 0, & t \notin Fx, \end{cases}$$

$$T(x)(t) = \begin{cases} \alpha(x), & t \in Gx, \\ 0, & t \notin Gx. \end{cases}$$

Then

$$[Sx]_{\alpha(x)} = \left\{ t : S(x)(t) \geq \alpha(x) \right\} = Fx, \qquad [Tx]_{\alpha(x)} = Gx.$$

Thus, Theorem 5 can be applied to obtain $u \in X$ such that

$$u \in [Su]_{\alpha(u)} \cap [Tu]_{\alpha(u)} = Fu \cap Gu. \qquad \square$$

Corollary 8 [16] *Let (X, d) be a complete metric linear space and $S, T : X \to W(X)$ be fuzzy mappings, and for all $x, y \in X$,*

$$d_\infty \big(S(x), T(y) \big) \leq a_1 p\big(x, S(x)\big) + a_2 p\big(y, T(y)\big) + a_3 p\big(x, T(y)\big)$$
$$+ a_4 p\big(y, S(x)\big) + a_5 d(x, y),$$

where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers and $\sum_{i=1}^{5} a_i < 1$ and either $a_1 = a_2$ or $a_3 = a_4$. Then there exists $u \in X$ such that $\{u\} \subset T(u)$, $\{u\} \subset S(u)$.

Proof Let $x \in X$, then by Lemma 4 there exists $y \in X$ such that $y \in [Sx]_1$. Similarly, we can find $z \in X$ such that $z \in [Tx]_1$. It follows that for each $x \in X$, $[Sx]_{\alpha(x)}$, $[Tx]_{\alpha(x)}$ are nonempty closed bounded subsets of X. As $\alpha(x) = \alpha(y) = 1$, by the definition of a d_∞-metric for fuzzy sets, we have

$$H\big([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\big) \leq d_\infty \big(S(x), T(y)\big)$$

for all $x, y \in X$. It implies that

$$H\big([Sx]_{\alpha(x)}, [Ty]_{\alpha(y)}\big) \leq d_\infty \big(S(x), T(y)\big)$$
$$\leq a_1 p\big(x, S(x)\big) + a_2 p\big(y, T(y)\big)$$
$$+ a_3 p\big(x, T(y)\big) + a_4 p\big(y, S(x)\big) + a_5 d(x, y)$$

for all $x, y \in X$. Since $[Sx]_1 \subseteq [Sx]_\alpha$ for each $\alpha \in [0, 1]$, therefore $d(x, [Sx]_\alpha) \leq d(x, [Sx]_1)$ for each $\alpha \in [0, 1]$. It implies that $p(x, S(x)) \leq d(x, [Sx]_1)$, similarly, $p(x, T(x)) \leq d(x, [Tx]_1)$. This further implies that for all $x, y \in X$,

$$H\big([Sx]_1, [Ty]_1\big) \leq a_1 d\big(x, [Sx]_1\big) + a_2 d\big(y, [Ty]_1\big) + a_3 d\big(x, [Ty]_1\big)$$
$$+ a_4 d\big(y, [Sx]_1\big) + a_5 d(x, y).$$

Now, by Theorem 5, we obtain $u \in X$ such that $u \in [Su]_1 \cap [Tu]_1$, *i.e.*, $\{u\} \subset T(u)$, $\{u\} \subset S(u)$. □

In the following, we suppose that $\widehat{T}$ (for details, see [2, 21]) is the set-valued mapping induced by fuzzy mappings $T : X \to \mathcal{F}(X)$, *i.e.*,

$$\widehat{T}x = \Big\{ y : T(x)(y) = \max_{t \in X} T(x)(t) \Big\}.$$

Corollary 9 *Let (X, d) be a complete metric space and $S, T : X \to \mathcal{F}(X)$ be fuzzy mappings such that for all $x \in X$, $\widehat{S}(x)$, $\widehat{T}(x)$ nonempty closed bounded subsets of X and*

$$H(\widehat{S}x, \widehat{T}x) \leq a_1 d(x, \widehat{S}x) + a_2 d(y, \widehat{T}y) + a_3 d(x, \widehat{T}y)$$
$$+ a_4 d(y, \widehat{S}x) + a_5 d(x, y)$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4, a_5 are nonnegative real numbers and $\sum_{i=1}^{5} a_i < 1$ and either $a_1 = a_2$ or $a_3 = a_4$. Then there exists a point $x^* \in X$ such that $T(x^*)(x^*) \geq T(x^*)(x)$ and $S(x^*)(x^*) \geq S(x^*)(x)$ for all $x \in X$.

Proof By Corollary 7, there exists $x^* \in X$ such that $x^* \in \widehat{S}x^* \cap \widehat{T}x^*$. Then by [21, Lemma 1], we obtain

$$T\big(x^*\big)\big(x^*\big) \geq T\big(x^*\big)(x), \qquad S\big(x^*\big)\big(x^*\big) \geq S\big(x^*\big)(x)$$

for all $x \in X$. □

Remark 10 The result proved by Vijayaraju and Marudai [17, Theorem 3.1] and Azam and Arshad [15, Theorem 4] is the case when $a_1 \neq a_2$ and $a_3 = a_4$ in Theorem 5. Corollary 9 also generalizes the results proved by Heilpern [9] and Frigon and Regan [12, Theorems 2.1, 2.2] for

$$S = T,$$

$$a_i = 0, \quad i = 1, 2, 3, 4 \quad \text{and} \quad a_5 \neq 0.$$

4 Conclusion

In this paper, we obtained fixed point results for fuzzy set-valued mappings under a generalized contractive condition related to the d_∞-metric which is useful for computing Hausdorff dimensions. These dimensions help us to understand e^∞-space which is used in high energy physics. Our results are also useful in geometric problems arising in high energy physics. This is because events in this case are mostly fuzzy sets.

Authors' contributions

IB gave the idea and AA wrote the initial draft. Both authors read and agreed upon the draft and finalized the manuscript. Correspondence was mainly done by IB. All authors read and approved the final manuscript.

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