Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem

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Abstract

In this paper, we introduce a new iterative algorithm by the relaxed extragradient-like method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of solutions of a more general system of variational inequalities for finite inverse strongly monotone mappings and the set of solutions of a fixed point problem of a strictly pseudocontractive mapping in a Hilbert space. Then we prove strong convergence of the scheme to a common element of the three above described sets.

Keywords: generalized mixed equilibrium problems; more general system of variational inequalities; fixed point; inverse strongly monotone mapping; strictly pseudocontractive mapping

1 Introduction

In this paper, we assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$ and C is a nonempty closed convex subset of H. P_C denotes the metric projection of H onto C and $\mathcal{F}(T)$ denotes the fixed points set of a mapping T. The sequence $\{x_n\}$ converges weakly to x which is denoted by $x_n \rightharpoonup x$.

Let $\varphi : C \to \mathbb{R}$ be a real-valued function and let $A : C \to H$ be a nonlinear mapping. Suppose that $F : C \times C \to \mathbb{R}$ is a bifunction.

The generalized mixed equilibrium problem is to find $x \in C$ (see, *e.g.*, [1–6]) such that

$$F(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{GMEP}(F,\varphi,A)$.

If $\varphi \equiv 0$, then problem (1.1) reduces to the generalized equilibrium problem, which is to find $x \in C$ (see, *e.g.*, [7–9]) such that

$$F(x,y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{GEP}(F,A)$.

If $A \equiv 0$, then problem (1.1) reduces to the mixed equilibrium problem, which is to find $x \in C$ (see, *e.g.*, [10–13]) such that

$$F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{MEP}(F,\varphi)$.

If $F \equiv 0$, then problem (1.1) reduces to the mixed variational inequality of Browder type, which is to find $x \in C$ (see, *e.g.*, [3, 14]) such that

$$\varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by $\mathrm{MVI}(C,\varphi,A)$.

If $\varphi \equiv 0$, $A \equiv 0$, then problem (1.1) reduces to the equilibrium problem, which is to find $x \in C$ (see, *e.g.*, [15–17]) such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.5}$$

The set of solutions of (1.5) is denoted by $\mathrm{EP}(F)$.

If $F \equiv 0$, $\varphi \equiv 0$, then problem (1.1) reduces to the variational inequality, which is to find $x \in C$ (see, *e.g.*, [18–29]) such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of (1.6) is denoted by $\mathrm{VI}(C,A)$.

If $F \equiv 0$, $A \equiv 0$, then problem (1.1) reduces to the minimized problem, which is to find $x \in C$ (see, *e.g.*, [18–28]) such that

$$\varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of (1.7) is denoted by $\mathrm{Argmin}(\varphi)$.

Let $A, B : C \to H$ be two mappings. Ceng *et al.* [2] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.8}$$

which is called a general system of variational inequalities where $\lambda > 0$ and $\mu > 0$ are two constants. In particular, if $A = B$, $x^* = y^*$, then problem (1.8) reduces to the classical variational inequality problem (1.6).

In order to find the common element of the solutions of problem (1.8) and the set of fixed points of one nonexpansive mapping S, Ceng *et al.* [2] studied the following algorithm: fix $u \in C$, $x_0 \in C$, and

$$\begin{cases} z_n = T_{\lambda_n}^{(\Theta,\varphi)}(x_n - \lambda_n F x_n), \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad n \geq 0. \end{cases} \tag{1.9}$$

Under appropriate conditions, they obtained one strong convergence theorem.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{A_i\}_{i=1}^N : C \to H$ be a family of mappings. Cai and Bu [1] considered the following problem of finding $(x_1^*, x_2^*, \dots, x_N^*) \in C \times C \times \cdots \times C$ such that

$$\begin{cases} \langle \lambda_N A_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_{N-1} A_{N-1} x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0, \quad \forall x \in C, \\ \vdots \\ \langle \lambda_2 A_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \lambda_1 A_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C. \end{cases} \tag{1.10}$$

And (1.10) can be rewritten as

$$\begin{cases} \langle x_1^* - (I - \lambda_N A_N) x_N^*, x - x_1^* \rangle \geq 0, \quad \forall x \in C, \\ \langle x_N^* - (I - \lambda_{N-1} A_{N-1}) x_{N-1}^*, x - x_N^* \rangle \geq 0, \quad \forall x \in C, \\ \vdots \\ \langle x_3^* - (I - \lambda_2 A_2) x_2^*, x - x_3^* \rangle \geq 0, \quad \forall x \in C, \\ \langle x_2^* - (I - \lambda_1 A_1) x_1^*, x - x_2^* \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.11}$$

which is called a more general system of variational inequalities in Hilbert spaces, where $\lambda_i > 0$ for all $i \in \{1, 2, \dots, N\}$. The set of solutions to (1.10) is denoted by Ω. In particular, if $N = 2$, $A_1 = B$, $A_2 = A$, $\lambda_1 = \mu$, $\lambda_2 = \lambda$, $x_1^* = x^*$, $x_2^* = y^*$, then problem (1.10) reduces to problem (1.8).

In order to find a common element of the solutions of problem (1.10) and the common fixed points of a family of strictly pseudocontractive mappings, Cai and Bu [1] studied the following algorithm: pick any $x_0 \in H$, set $C_1 = C$, $x_1 = P_{C_1} x_0$, and

$$\begin{cases} u_n = T_{r_{M,n}}^{(F_M, \varphi_M)} (I - r_{M,n} B_M) T_{r_{M-1,n}}^{(F_{M-1}, \varphi_{M-1})} \\ \qquad \times (I - r_{M-1,n} B_{M-1}) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} (I - r_{1,n} B_1) x_n, \\ y_n = P_C (I - \lambda_N A_N) P_C (I - \lambda_{N-1} A_{N-1}) \cdots P_C (I - \lambda_2 A_2) P_C (I - \lambda_1 A_1) u_n, \\ z_n = \alpha_n y_n + (1 - \alpha_n) S_n y_n, \\ C_{n+1} = \{ z \in C_n : \| z_n - z \| \leq \| x_n - z \| \}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 1. \end{cases} \tag{1.12}$$

Under suitable conditions, they also obtained one strong convergence theorem.

In this paper, motivated and inspired by the above facts, we study a new iterative algorithm by the relaxed extragradient-like method for finding a common element of the set of solutions of generalized mixed equilibrium problems, the set of solutions of a more general system of variational inequalities for finite inverse strongly monotone mappings and the set of solutions of a fixed point problem of a strictly pseudocontractive mapping in a Hilbert space. Then we prove strong convergence of the scheme to a common element of the three above described sets.

2 Preliminaries

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for any $x,y \in C$;

(A3) F is weakly upper semicontinuous, *i.e.*, for each $x,y,z \in C$,

$$\limsup_{t\to 0} F\bigl(x + t(z-x), y\bigr) \leq F(x,y);$$

(A4) $F(x,\cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) For each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C\setminus D_x$,

$$F(z,y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0;$$

(B2) C is a bounded set.

Let H be a real Hilbert space. It is well known that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle \tag{2.1}$$

and

$$\|x\|^2 - \|y\|^2 \leq \|x-y\|\bigl(\|x\| + \|y\|\bigr) \tag{2.2}$$

for all $x,y \in H$.

Definition 2.1 Let C be a nonempty closed convex subset of a real Hilbert space H.

(1) A mapping $T: C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x-y\|, \quad \forall x,y \in C;$$

(2) A mapping $T: C \to H$ is said to be L-Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \forall x,y \in C;$$

(3) A mapping $T: C \to C$ is said to be k-strictly pseudocontractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\bigl\|(I-T)x - (I-T)y\bigr\|^2, \quad \forall x,y \in C. \tag{2.3}$$

It is obvious that $k = 0$, then the mapping T is nonexpansive;

(4) A mapping $T: C \to H$ is said to be monotone if

$$\langle Tx - Ty, x-y\rangle \geq 0, \quad \forall x,y \in C;$$

(5) A mapping $T : C \to H$ is said to be α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

It is obvious that any α-inverse-strongly monotone mapping T is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Definition 2.2 $P_C : H \to C$ is called a metric projection if for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

In order to prove our main results in the next section, we recall some lemmas.

Lemma 2.1 [30] *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a k-strictly pseudocontractive mapping, then the following results hold*:

(1) *equation* (2.3) *is equivalent to*

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-k}{2}\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C; \tag{2.4}$$

(2) *T is Lipschitz continuous with a constant $\frac{1+k}{1-k}$, i.e.,*

$$\|Tx - Ty\| \leq \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C; \tag{2.5}$$

(3) (*Demi-closed principle*) *$I - T$ is demi-closed on C, that is,*

$$\text{if } x_n \rightharpoonup x^* \in C \quad \text{and} \quad (I - T)x_n \to 0, \quad \text{then } x^* = Tx^*.$$

Lemma 2.2 [1] *Let C be a nonempty closed convex subset of H and let $T : C \to H$ be an α-inverse-strongly monotone mapping, then for all $x, y \in C$ and $\lambda > 0$, we have*

$$\begin{aligned}\big\|(I - \lambda T)x - (I - \lambda T)y\big\|^2 &= \big\|(x - y) - \lambda(Tx - Ty)\big\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle Tx - Ty, x - y\rangle + \lambda^2\|Tx - Ty\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Tx - Ty\|^2.\end{aligned}$$

So, if $0 < \lambda \leq 2\alpha$, then $I - \lambda T$ is a nonexpansive mapping from C to H.

Lemma 2.3 *Let C be a nonempty closed convex subset of H and let $P_C : H \to C$ be a metric projection, then*

(1) $\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle$, $\forall x, y \in H$;
(2) *moreover, P_C is a nonexpansive mapping, i.e.*, $\|P_C x - P_C y\| \leq \|x - y\|$, $\forall x, y \in H$;
(3) $\langle x - P_C x, y - P_C x\rangle \leq 0$, $\forall x \in H$, $y \in C$;
(4) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$, $\forall x \in H$, $y \in C$.

Lemma 2.4 [4] *Let C be a nonempty closed convex subset of H. Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4) *and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to C$ as follows*:

$$T_r^{(F,\varphi)}(x) = \left\{ z \in C : F(z,y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) *for each $x \in H$, $T_r^{(F,\varphi)}(x) \neq \emptyset$ and $T_r^{(F,\varphi)}$ is single-valued*;

(2) *$T_r^{(F,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r^{(F,\varphi)}(x) - T_r^{(F,\varphi)}(y) \right\|^2 \leq \left\langle T_r^{(F,\varphi)}(x) - T_r^{(F,\varphi)}(y), x - y \right\rangle;$$

(3) $\mathcal{F}(T_r^{(F,\varphi)}) = \mathrm{MEP}(F,\varphi)$;

(4) $\mathrm{MEP}(F,\varphi)$ *is closed and convex.*

Lemma 2.5 [3] *Let C be a nonempty closed convex subset of H. Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4), *$B : C \to H$ is a continuous monotone mapping and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $K_r^{(F,\varphi)} : H \to C$ as follows*:

$$K_r^{(F,\varphi)}(x) = \left\{ z \in C : F(z,y) + \varphi(y) - \varphi(z) + \langle Bx, y - z\rangle + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) *for each $x \in H$, $K_r^{(F,\varphi)}(x) \neq \emptyset$ and $K_r^{(F,\varphi)}$ is single-valued*;

(2) *$K_r^{(F,\varphi)}$ is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| K_r^{(F,\varphi)}(x) - K_r^{(F,\varphi)}(y) \right\|^2 \leq \left\langle K_r^{(F,\varphi)}(x) - K_r^{(F,\varphi)}(y), x - y \right\rangle;$$

(3) $\mathcal{F}(K_r^{(F,\varphi)}) = \mathrm{GMEP}(F,\varphi,B)$;

(4) $\mathrm{GMEP}(F,\varphi,B)$ *is closed and convex.*

Lemma 2.6 *Let C be a nonempty closed convex subset of H. Let $\{F_k\}_{k=1}^M$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *let $\{\varphi_k\}_{k=1}^M$ be a family of lower semicontinuous functions from C into $\mathbb{R}$, and let $\{B_k\}_{k=1}^M$ be a family of β_k-inverse-strongly monotone mappings from C into H. For F_k and φ_k, $k = 1, 2, \ldots, M$, assume that either* (B1) *or* (B2) *holds. Let $T : C \to H$ be a mapping defined by*

$$T(x) = T_{r_M}^{(F_M,\varphi_M)}(I - r_M B_M) T_{r_{M-1}}^{(F_{M-1},\varphi_{M-1})}(I - r_{M-1}B_{M-1}) \cdots T_{r_1}^{(F_1,\varphi_1)}(I - r_1 B_1)x, \quad \forall x \in C.$$

Putting $\Theta^0 = I$, where I is an identity mapping,

$$\Theta^k = T_{r_k}^{(F_k,\varphi_k)}(I - r_k B_k) T_{r_{k-1}}^{(F_{k-1},\varphi_{k-1})}(I - r_{k-1}B_{k-1}) \cdots T_{r_1}^{(F_1,\varphi_1)}(I - r_1 B_1), \quad k = 1, 2, \ldots, M.$$

If $x \in \bigcap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, B_k)$ and $0 < r_k \leq 2\beta_k$, $k = 1, 2, \ldots, M$, then

(1) $\Theta^k x = x$, $k = 1, 2, \ldots, M$;

(2) *T is nonexpansive.*

Proof (1) Since $\{B_k\}_{k=1}^M$ is a family of β_k-inverse-strongly monotone mappings from C into H, so they are continuous monotone mappings. Observe that

$$\begin{aligned} &T_{r_k}^{(F_k,\varphi_k)}(I - r_k B_k)x \\ &\quad = \left\{ z \in C : F_k(z,y) + \varphi_k(y) - \varphi_k(z) + \frac{1}{r_k}\langle y - z, z - (I - r_k B_k)x\rangle \geq 0, \forall y \in C \right\} \\ &\quad = \left\{ z \in C : F_k(z,y) + \varphi_k(y) - \varphi_k(z) + \langle B_k x, y - z\rangle + \frac{1}{r_k}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \\ &\quad = K_{r_k}^{(F_k,\varphi_k)}(x). \end{aligned}$$

By Lemma 2.5, we know that if $x \in \bigcap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, B_k)$ then x is the fixed point of the mapping $K_{r_k}^{(F_k,\varphi_k)}$, $k = 1, 2, \ldots, M$, so we have

$$x = K_{r_k}^{(F_k,\varphi_k)}(x) = T_{r_k}^{(F_k,\varphi_k)}(I - r_k B_k)x, \tag{2.6}$$

which implies that x is a fixed point of the mapping $T_{r_k}^{(F_k,\varphi_k)}(I - r_k B_k)$. Therefore we get

$$\Theta^k x = x, \quad k = 1, 2, \ldots, M.$$

(2) Since $T_r^{(F,\varphi)}$ is firmly nonexpansive, then it is obvious that $T_r^{(F,\varphi)}$ is nonexpansive. And from Lemma 2.2, we have

$$\begin{aligned} \|T(x) - T(y)\| &= \|\Theta^M x - \Theta^M y\| \\ &= \|T_{r_M}^{(F_M,\varphi_M)}(I - r_M B_M)\Theta^{M-1}x - T_{r_M}^{(F_M,\varphi_M)}(I - r_M B_M)\Theta^{M-1}y\| \\ &\leq \|(I - r_M B_M)\Theta^{M-1}x - (I - r_M B_M)\Theta^{M-1}y\| \\ &\leq \|\Theta^{M-1}x - \Theta^{M-1}y\| \leq \cdots \leq \|\Theta^0 x - \Theta^0 y\| \\ &= \|x - y\|, \end{aligned}$$

which implies T is nonexpansive. □

Lemma 2.7 [1] *Let C be a nonempty closed convex subset of H. Let A_i be α_i-inverse-strongly monotone from C into H, respectively, where $i \in \{1, 2, \ldots, N\}$. Let $G : C \to C$ be a mapping defined by*

$$G(x) = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)x, \quad \forall x \in C.$$

If $0 < \lambda_i \leq 2\alpha_i$, $i = 1, 2, \ldots, N$, then G is nonexpansive.

Proof Put $\Omega^i = P_C(I - \lambda_i A_i)P_C(I - \lambda_{i-1}A_{i-1}) \cdots P_C(I - \lambda_1 A_1)$, $i = 1, 2, \ldots, N$, and $\Omega^0 = I$, where I is an identity mapping. Since P_C is nonexpansive and from Lemma 2.2, we have

$$\begin{aligned} \|G(x) - G(y)\| &= \|\Omega^N x - \Omega^N y\| \\ &= \|P_C(I - \lambda_N A_N)\Omega^{N-1}x - P_C(I - \lambda_N A_N)\Omega^{N-1}y\| \end{aligned}$$

$$\begin{aligned}&\le \left\|(I-\lambda_N A_N)\Omega^{N-1}x-(I-\lambda_N A_N)\Omega^{N-1}y\right\|\\&\le \left\|\Omega^{N-1}x-\Omega^{N-1}y\right\|\le\cdots\le\left\|\Omega^0x-\Omega^0y\right\|\\&\le\|x-y\|,\end{aligned}$$

which implies G is nonexpansive. □

Lemma 2.8 *Let C be a nonempty closed convex subset of H. Let $A_i: C\to H$ be a nonlinear mapping, where $i=1,2,\ldots,N$. For given $x_i^*\in C$, $i=1,2,\ldots,N$, $(x_1^*,x_2^*,\ldots,x_N^*)$ is a solution of problem* (1.10) *if and only if*

$$x_1^*=P_C(I-\lambda_N A_N)x_N^*, x_i^*=P_C(I-\lambda_{i-1}A_{i-1})x_{i-1}^*,\quad i=2,3,\ldots,N,\tag{2.7}$$

that is,

$$x_1^*=P_C(I-\lambda_N A_N)P_C(I-\lambda_{N-1}A_{N-1})\cdots P_C(I-\lambda_2A_2)P_C(I-\lambda_1A_1)x_1^*.$$

Proof ($\Longleftarrow$) From Lemma 2.3(3), it is obvious that (2.7) is the solution of problem (1.10).
($\Longrightarrow$) Since

$$\begin{aligned}&\left\langle\lambda_N A_N x_N^*+x_1^*-x_N^*,x-x_1^*\right\rangle\ge0,\quad\forall x\in C\\&\quad\Rightarrow\quad\left\langle x_1^*-(I-\lambda_N A_N)x_N^*,x-x_1^*\right\rangle\ge0,\quad\forall x\in C\\&\quad\Rightarrow\quad\left\langle(I-\lambda_N A_N)x_N^*-x_1^*,(I-\lambda_N A_N)x_N^*-x_1^*-(I-\lambda_N A_N)x_N^*+x\right\rangle\le0,\quad\forall x\in C\\&\quad\Rightarrow\quad\left\|(I-\lambda_N A_N)x_N^*-x_1^*\right\|^2\le\left\langle(I-\lambda_N A_N)x_N^*-x_1^*,(I-\lambda_N A_N)x_N^*-x\right\rangle,\quad\forall x\in C\\&\quad\Rightarrow\quad\left\|(I-\lambda_N A_N)x_N^*-x_1^*\right\|\le\left\|(I-\lambda_N A_N)x_N^*-x\right\|,\quad\forall x\in C\\&\quad\Rightarrow\quad x_1^*=P_C(I-\lambda_N A_N)x_N^*.\end{aligned}$$

Similarly, we get

$$x_i^*=P_C(I-\lambda_{i-1}A_{i-1})x_{i-1}^*,\quad i=2,3,\ldots,N.$$

Therefore we have

$$x_1^*=P_C(I-\lambda_N A_N)P_C(I-\lambda_{N-1}A_{N-1})\cdots P_C(I-\lambda_2A_2)P_C(I-\lambda_1A_1)x_1^*,$$

which completes the proof. □

From Lemma 2.8, we know that $x_1^*=G(x_1^*)$, that is, x_1^* is a fixed point of the mapping G, where G is defined by Lemma 2.7. Moreover, if we find the fixed point x_1^*, it is easy to solve the other points by (2.7).

Lemma 2.9 [31] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$. Suppose that $x_{n+1}=\beta_n x_n+(1-\beta_n)y_n$ for all integers $n\ge0$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le0$. Then $\lim_{n\to\infty}\|y_n-x_n\|=0$.*

Lemma 2.10 [30] *Let $T : C \to C$ be a nonexpansive mapping with $\mathcal{F}(T) \neq \emptyset$. If $x_n \rightharpoonup x^*$ and $(I - T)x_n \to y^*$, then $(I - T)x^* = y^*$.*

Lemma 2.11 [30] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1 - \gamma_n)\alpha_n + \delta_n, \quad \forall n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

3 Main results

In this section, we state and verify our main results. We have the following theorem.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{F_k\}_{k=1}^M$ be a family of bifunctions from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *let $\{\varphi_k\}_{k=1}^M : C \to \mathbb{R}$ be a family of lower semicontinuous and convex functions and let $\{B_k\}_{k=1}^M$ be a family of β_k-inverse-strongly monotone mappings from C into H. Let A_i be α_i-inverse-strongly monotone from C into H, respectively, where $i \in \{1, 2, \ldots, N\}$. Let S be a δ-strict pseudocontractive mapping from C into itself such that $\mathcal{F} = [\bigcap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, B_k)] \cap \mathcal{F}(G) \cap \mathcal{F}(S) \neq \emptyset$, where G is defined by Lemma* 2.7. *For F_k and φ_k, $k = 1, 2, \ldots, M$, assume that either* (B1) *or* (B2) *holds. Pick any $x_0 \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = T_{r_{M,n}}^{(F_M,\varphi_M)}(I - r_{M,n}B_M)T_{r_{M-1,n}}^{(F_{M-1},\varphi_{M-1})} \\ \qquad \times (I - r_{M-1,n}B_{M-1}) \cdots T_{r_{1,n}}^{(F_1,\varphi_1)}(I - r_{1,n}B_1)x_n, \\ y_n = P_C(I - \lambda_N A_N)P_C(I - \lambda_{N-1}A_{N-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1)z_n, \\ x_{n+1} = a_n x_0 + b_n x_n + c_n y_n + d_n S y_n, \quad \forall n \ge 0, \end{cases} \tag{3.1}$$

where $\lambda_i \in (0, 2\alpha_i)$, $i = 1, 2, \ldots, N$, $\delta \in (0,1)$. $\{a_n\}, \{b_n\}, \{c_n\}, \{d_n\} \subset [0,1]$ satisfy the following conditions:
(i) $a_n + b_n + c_n + d_n = 1$ *and* $(c_n + d_n)\delta \le c_n$ *for all* $n \ge 0$;
(ii) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=0}^{\infty} a_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$ *and* $\liminf_{n\to\infty} d_n > 0$;
(iv) $\lim_{n\to\infty}(\frac{c_{n+1}}{1-b_{n+1}} - \frac{c_n}{1-b_n}) = 0$;
(v) $0 < \liminf_{n\to\infty} r_{k,n} \le \limsup_{n\to\infty} r_{k,n} < 2\beta_k$, $k = 1, 2, \ldots, M$.
Then $\{x_n\} \subset C$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof Putting

$$\Theta_n^k = T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}B_k)T_{r_{k-1,n}}^{(F_{k-1},\varphi_{k-1})}(I - r_{k-1,n}B_{k-1}) \cdots T_{r_{1,n}}^{(F_1,\varphi_1)}(I - r_{1,n}B_1), \\ \forall k \in \{1, \ldots, M\}, n \in \mathbb{N},$$

and

$$\Omega^i = P_C(I - \lambda_i A_i)P_C(I - \lambda_{i-1}A_{i-1}) \cdots P_C(I - \lambda_2 A_2)P_C(I - \lambda_1 A_1), \quad \forall i \in \{1, 2, \ldots, N\},$$

$\Theta_n^0 = \Omega^0 = I$, where I is the identity mapping on H. Then we have that $z_n = \Theta_n^M x_n$ and $y_n = \Omega^N z_n$. From Lemma 2.6 and Lemma 2.7, it can be seen easily that Θ_n^k and Ω^i are nonexpansive, where $k \in \{1,2,\dots,M\}$, $i \in \{1,2,\dots,N\}$. We divide the proof into six steps.

Step 1. Firstly, we show that $\{x_n\}$ is bounded.

Indeed, take $p \in \mathcal{F}$ arbitrarily. Since $p = \Theta_n^k p = Sp$, $\forall k \in \{1,2,\dots,M\}$, $\forall n \in \mathbb{N}$. By Lemma 2.6, we have

$$\|z_n - p\| = \left\|\Theta_n^M x_n - \Theta_n^M p\right\| \le \|x_n - p\|. \tag{3.2}$$

It follows from Lemma 2.7 and (3.2) that

$$\|y_n - p\| = \left\|\Omega^N z_n - \Omega^N p\right\| \le \|z_n - p\| \le \|x_n - p\|. \tag{3.3}$$

Furthermore, from (3.1), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|a_n x_0 + b_n x_n + c_n y_n + d_n S y_n - p\| \\ &= \left\|a_n(x_0 - p) + b_n(x_n - p) + c_n(y_n - p) + d_n(Sy_n - p)\right\| \\ &\le a_n\|x_0 - p\| + b_n\|x_n - p\| + \left\|c_n(y_n - p) + d_n(Sy_n - p)\right\|. \end{aligned} \tag{3.4}$$

Since $(c_n + d_n)\delta \le c_n$, (2.3) and (2.4), we have

$$\begin{aligned} &\left\|c_n(y_n - p) + d_n(Sy_n - p)\right\|^2 \\ &\quad = c_n^2\|y_n - p\|^2 + d_n^2\|Sy_n - p\|^2 + 2c_n d_n\langle Sy_n - p, y_n - p\rangle \\ &\quad \le c_n^2\|y_n - p\|^2 + d_n^2\left[\|y_n - p\|^2 + \delta\|y_n - Sy_n\|^2\right] \\ &\qquad + 2c_n d_n\left[\|y_n - p\|^2 - \frac{1-\delta}{2}\|y_n - Sy_n\|^2\right] \\ &\quad = (c_n + d_n)^2\|y_n - p\|^2 + \left[d_n^2\delta - (1-\delta)c_n d_n\right]\|y_n - Sy_n\|^2 \\ &\quad = (c_n + d_n)^2\|y_n - p\|^2 + d_n\left[(c_n + d_n)\delta - c_n\right]\|y_n - Sy_n\|^2 \\ &\quad \le (c_n + d_n)^2\|y_n - p\|^2, \end{aligned}$$

which implies that

$$\left\|c_n(y_n - p) + d_n(Sy_n - p)\right\| \le (c_n + d_n)\|y_n - p\|. \tag{3.5}$$

From (3.2)-(3.5) it follows that

$$\begin{aligned} \|x_{n+1} - p\| &\le a_n\|x_0 - p\| + b_n\|x_n - p\| + \left\|c_n(y_n - p) + d_n(Sy_n - p)\right\| \\ &\le a_n\|x_0 - p\| + b_n\|x_n - p\| + (c_n + d_n)\|y_n - p\| \\ &\le a_n\|x_0 - p\| + b_n\|x_n - p\| + (c_n + d_n)\|x_n - p\| \\ &= a_n\|x_0 - p\| + (1 - a_n)\|x_n - p\|. \end{aligned}$$

So, we have

$$\|x_{n+1}-p\| \le \max\bigl\{\|x_0-p\|,\|x_n-p\|\bigr\}, \quad \forall n\ge 0.$$

By induction, we obtain that

$$\|x_n-p\| \le \|x_0-p\|, \quad \forall n\ge 0.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{z_n\}$, $\{y_n\}$, $\{Sy_n\}$ are bounded.

Step 2. Next, we prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Indeed, define $x_{n+1}=b_nx_n+(1-b_n)w_n$ for all $n\ge 0$. It follows that

$$\begin{aligned} w_{n+1}-w_n &= \frac{x_{n+2}-b_{n+1}x_{n+1}}{1-b_{n+1}}-\frac{x_{n+1}-b_nx_n}{1-b_n} \\ &= \frac{a_{n+1}x_0+c_{n+1}y_{n+1}+d_{n+1}Sy_{n+1}}{1-b_{n+1}}-\frac{a_nx_0+c_ny_n+d_nSy_n}{1-b_n} \\ &= \frac{a_{n+1}x_0}{1-b_{n+1}}-\frac{a_nx_0}{1-b_n}+\frac{c_{n+1}(y_{n+1}-y_n)+d_{n+1}(Sy_{n+1}-Sy_n)}{1-b_{n+1}} \\ &\quad +\left(\frac{c_{n+1}}{1-b_{n+1}}-\frac{c_n}{1-b_n}\right)y_n+\left(\frac{d_{n+1}}{1-b_{n+1}}-\frac{d_n}{1-b_n}\right)Sy_n. \end{aligned} \tag{3.6}$$

Observe that

$$\begin{aligned} &\bigl\|c_{n+1}(y_{n+1}-y_n)+d_{n+1}(Sy_{n+1}-Sy_n)\bigr\|^2 \\ &\quad = c_{n+1}^2\|y_{n+1}-y_n\|^2+d_{n+1}^2\|Sy_{n+1}-Sy_n\|^2+2c_{n+1}d_{n+1}\langle Sy_{n+1}-Sy_n,y_{n+1}-y_n\rangle \\ &\quad \le c_{n+1}^2\|y_{n+1}-y_n\|^2+d_{n+1}^2\bigl[\|y_{n+1}-y_n\|^2+\delta\bigl\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\bigr\|^2\bigr] \\ &\qquad +2c_{n+1}d_{n+1}\left[\|y_{n+1}-y_n\|^2-\frac{1-\delta}{2}\bigl\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\bigr\|^2\right] \\ &\quad = (c_{n+1}+d_{n+1})^2\|y_{n+1}-y_n\|^2+\bigl[d_{n+1}^2\delta-(1-\delta)c_{n+1}d_{n+1}\bigr]\bigl\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\bigr\|^2 \\ &\quad = (c_{n+1}+d_{n+1})^2\|y_{n+1}-y_n\|^2+d_{n+1}\bigl[(c_{n+1}+d_{n+1})\delta-c_{n+1}\bigr] \\ &\qquad \times\bigl\|(y_{n+1}-Sy_{n+1})-(y_n-Sy_n)\bigr\|^2 \\ &\quad \le (c_{n+1}+d_{n+1})^2\|y_{n+1}-y_n\|^2, \end{aligned}$$

which implies that

$$\bigl\|c_{n+1}(y_{n+1}-y_n)+d_{n+1}(Sy_{n+1}-Sy_n)\bigr\| \le (c_{n+1}+d_{n+1})\|y_{n+1}-y_n\|. \tag{3.7}$$

Since

$$\|z_{n+1}-z_n\| = \bigl\|\Theta_n^M x_{n+1}-\Theta_n^M x_n\bigr\| \le \|x_{n+1}-x_n\|, \tag{3.8}$$

then we have

$$\|y_{n+1}-y_n\| = \bigl\|\Omega^N z_{n+1}-\Omega^N z_n\bigr\| \le \|z_{n+1}-z_n\| \le \|x_{n+1}-x_n\|. \tag{3.9}$$

Hence it follows from (3.6), (3.7), (3.9) and $\frac{c_{n+1}+d_{n+1}}{1-b_{n+1}} = \frac{1-a_{n+1}-b_{n+1}}{1-b_{n+1}} < 1$ that

$$\begin{aligned}\|w_{n+1}-w_n\| &\le \left(\frac{a_{n+1}}{1-b_{n+1}}+\frac{a_n}{1-b_n}\right)\|x_0\| + \frac{\|c_{n+1}(y_{n+1}-y_n)+d_{n+1}(Sy_{n+1}-Sy_n)\|}{1-b_{n+1}} \\ &\quad + \left|\frac{c_{n+1}}{1-b_{n+1}}-\frac{c_n}{1-b_n}\right|\|y_n\| + \left|\frac{d_{n+1}}{1-b_{n+1}}-\frac{d_n}{1-b_n}\right|\|Sy_n\| \\ &\le \left(\frac{a_{n+1}}{1-b_{n+1}}+\frac{a_n}{1-b_n}\right)\|x_0\| + \frac{c_{n+1}+d_{n+1}}{1-b_{n+1}}\|x_{n+1}-x_n\| \\ &\quad + \left|\frac{c_{n+1}}{1-b_{n+1}}-\frac{c_n}{1-b_n}\right|\|y_n\| + \left|\frac{a_{n+1}+c_{n+1}}{1-b_{n+1}}-\frac{a_n+c_n}{1-b_n}\right|\|Sy_n\| \\ &\le \frac{a_{n+1}}{1-b_{n+1}}\big(\|x_0\|+\|Sy_n\|\big) + \frac{a_n}{1-b_n}\big(\|x_0\|+\|Sy_n\|\big) + \|x_{n+1}-x_n\| \\ &\quad + \left|\frac{c_{n+1}}{1-b_{n+1}}-\frac{c_n}{1-b_n}\right|\big(\|y_n\|+\|Sy_n\|\big). \end{aligned} \tag{3.10}$$

Consequently, it follows from (3.10), conditions (ii), (iv) and $\{y_n\}$, $\{Sy_n\}$ are bounded that

$$\begin{aligned}&\limsup_{n\to\infty}\big(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|\big) \\ &\quad \le \limsup_{n\to\infty}\left\{\frac{a_{n+1}}{1-b_{n+1}}\big(\|x_0\|+\|Sy_n\|\big)+\frac{a_n}{1-b_n}\big(\|x_0\|+\|Sy_n\|\big)\right. \\ &\quad\quad \left. + \left|\frac{c_{n+1}}{1-b_{n+1}}-\frac{c_n}{1-b_n}\right|\big(\|y_n\|+\|Sy_n\|\big)\right\} = 0.\end{aligned}$$

Hence, by Lemma 2.9, we get $\lim_{n\to\infty}\|w_n-x_n\| = 0$. Thus, from condition (iii), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-b_n)\|w_n-x_n\| = 0. \tag{3.11}$$

Step 3. We show that $\lim_{n\to\infty}\|B_k\Theta_n^{k-1}x_n - B_kp\| = 0$, $k = 1,2,\ldots,M$ and $\lim_{n\to\infty}\|A_i\Omega^{i-1}\times z_n - A_i\Omega^{i-1}p\| = 0$, $i = 1,2,\ldots,N$.

It follows from Lemma 2.6 that

$$\begin{aligned}\|z_n-p\|^2 &= \big\|\Theta_n^M x_n - \Theta_n^M p\big\|^2 \le \big\|\Theta_n^k x_n - \Theta_n^k p\big\|^2 \\ &= \big\|T_{r_{k,n}}^{(F_k,\varphi_k)}(I-r_{k,n}B_k)\Theta_n^{k-1}x_n - T_{r_{k,n}}^{(F_k,\varphi_k)}(I-r_{k,n}B_k)\Theta_n^{k-1}p\big\|^2 \\ &\le \big\|(I-r_{k,n}B_k)\Theta_n^{k-1}x_n - (I-r_{k,n}B_k)\Theta_n^{k-1}p\big\|^2 \\ &\le \big\|\Theta_n^{k-1}x_n - p\big\|^2 + r_{k,n}(r_{k,n}-2\beta_k)\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\|^2 \\ &\le \|x_n-p\|^2 + r_{k,n}(r_{k,n}-2\beta_k)\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\|^2.\end{aligned} \tag{3.12}$$

By Lemma 2.3 and Lemma 2.2, we have

$$\begin{aligned}\|y_n-p\|^2 &= \big\|P_C(I-\lambda_NA_N)\Omega^{N-1}z_n - P_C(I-\lambda_NA_N)\Omega^{N-1}p\big\|^2 \\ &\le \big\|(I-\lambda_NA_N)\Omega^{N-1}z_n - (I-\lambda_NA_N)\Omega^{N-1}p\big\|^2 \\ &\le \big\|\Omega^{N-1}z_n - \Omega^{N-1}p\big\|^2 + \lambda_N(\lambda_N-2\alpha_N)\big\|A_N\Omega^{N-1}z_n - A_N\Omega^{N-1}p\big\|^2.\end{aligned}$$

By induction, we get

$$\|y_n - p\|^2 \le \|z_n - p\|^2 + \sum_{i=1}^{N} \lambda_i(\lambda_i - 2\alpha_i)\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|^2. \tag{3.13}$$

From condition (i) and (3.5), we get

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\langle a_n(x_0 - p) + b_n(x_n - p) + c_n(y_n - p) + d_n(Sy_n - p), x_{n+1} - p\big\rangle \\
&\quad = a_n\langle x_0 - p, x_{n+1} - p\rangle + b_n\langle x_n - p, x_{n+1} - p\rangle + \big\langle c_n(y_n - p) + d_n(Sy_n - p), x_{n+1} - p\big\rangle \\
&\quad \le a_n\langle x_0 - p, x_{n+1} - p\rangle + b_n\|x_n - p\|\|x_{n+1} - p\| \\
&\qquad + \big\|c_n(y_n - p) + d_n(Sy_n - p)\big\|\|x_{n+1} - p\| \\
&\quad \le a_n\langle x_0 - p, x_{n+1} - p\rangle + b_n\|x_n - p\|\|x_{n+1} - p\| + (c_n + d_n)\|y_n - p\|\|x_{n+1} - p\| \\
&\quad \le a_n\langle x_0 - p, x_{n+1} - p\rangle + \frac{b_n}{2}\big(\|x_n - p\|^2 + \|x_{n+1} - p\|^2\big) \\
&\qquad + \frac{c_n + d_n}{2}\big(\|y_n - p\|^2 + \|x_{n+1} - p\|^2\big),
\end{aligned}$$

that is,

$$\|x_{n+1} - p\|^2 \le \frac{2a_n}{1 + a_n}\langle x_0 - p, x_{n+1} - p\rangle + \frac{b_n}{1 + a_n}\|x_n - p\|^2 + \frac{c_n + d_n}{1 + a_n}\|y_n - p\|^2. \tag{3.14}$$

So, in terms of (3.12) and (3.13), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \frac{2a_n}{1 + a_n}\|x_0 - p\|\|x_{n+1} - p\| + \frac{b_n}{1 + a_n}\|x_n - p\|^2 \\
&\quad + \frac{c_n + d_n}{1 + a_n}\left\{\|z_n - p\|^2 + \sum_{i=1}^{N}\lambda_i(\lambda_i - 2\alpha_i)\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|^2\right\} \\
&\le \frac{2a_n}{1 + a_n}\|x_0 - p\|\|x_{n+1} - p\| + \frac{b_n}{1 + a_n}\|x_n - p\|^2 \\
&\quad + \frac{c_n + d_n}{1 + a_n}\left\{\|x_n - p\|^2 + r_{k,n}(r_{k,n} - 2\beta_k)\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\|^2\right. \\
&\quad \left. + \sum_{i=1}^{N}\lambda_i(\lambda_i - 2\alpha_i)\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|^2\right\} \\
&\le \frac{2a_n}{1 + a_n}\|x_0 - p\|\|x_{n+1} - p\| + \frac{1 - a_n}{1 + a_n}\|x_n - p\|^2 \\
&\quad + \frac{c_n + d_n}{1 + a_n}\left\{r_{k,n}(r_{k,n} - 2\beta_k)\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\|^2\right. \\
&\quad \left. + \sum_{i=1}^{N}\lambda_i(\lambda_i - 2\alpha_i)\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|^2\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned} &r_{k,n}(2\beta_k - r_{k,n})\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\|^2 + \sum_{i=1}^{N}\lambda_i(2\alpha_i - \lambda_i)\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|^2 \\ &\quad\le \frac{2a_n}{c_n+d_n}\|x_0-p\|\|x_{n+1}-p\| + \frac{1-a_n}{c_n+d_n}\big(\|x_n-p\|^2 - \|x_{n+1}-p\|^2\big) \\ &\quad\le \frac{2a_n}{c_n+d_n}\|x_0-p\|\|x_{n+1}-p\| + \frac{1-a_n}{c_n+d_n}\|x_{n+1}-x_n\|\big(\|x_{n+1}-p\| + \|x_n-p\|\big). \end{aligned}$$

Since $\lim_{n\to\infty} a_n = 0$, $0 < \liminf_{n\to\infty} r_{k,n} \le \limsup_{n\to\infty} r_{k,n} < 2\beta_k$, $k = 1,2,\ldots,M$, $\lambda_i \in (0,2\alpha_i)$, $i = 1,2,\ldots,N$, $\delta \in (0,1)$, $\liminf_{n\to\infty}(c_n + d_n) > 0$ and $\{x_n\}$ is bounded, we have

$$\lim_{n\to\infty}\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\| = 0, \quad k = 1,2,\ldots,M, \tag{3.15}$$

and

$$\lim_{n\to\infty}\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\| = 0, \quad i = 1,2,\ldots,N. \tag{3.16}$$

Step 4. We prove that $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$.

Indeed, utilizing firmly nonexpansive of $T_{r_k}^{(F_k,\varphi_k)}$ and Lemma 2.2, we have

$$\begin{aligned} \big\|\Theta_n^kx_n - \Theta_n^kp\big\|^2 &= \big\|T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}B_k)\Theta_n^{k-1}x_n - T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}B_k)p\big\|^2 \\ &\le \big\langle (I - r_{k,n}B_k)\Theta_n^{k-1}x_n - (I - r_{k,n}B_k)p, \Theta_n^kx_n - p\big\rangle \\ &= \frac{1}{2}\Big(\big\|(I - r_{k,n}B_k)\Theta_n^{k-1}x_n - (I - r_{k,n}B_k)p\big\|^2 + \big\|\Theta_n^kx_n - p\big\|^2 \\ &\quad - \big\|(I - r_{k,n}B_k)\Theta_n^{k-1}x_n - (I - r_{k,n}B_k)p - \big(\Theta_n^kx_n - p\big)\big\|^2\Big) \\ &\le \frac{1}{2}\Big(\big\|\Theta_n^{k-1}x_n - p\big\|^2 + \big\|\Theta_n^kx_n - p\big\|^2 \\ &\quad - \big\|\Theta_n^{k-1}x_n - \Theta_n^kx_n - r_{k,n}\big(B_k\Theta_n^{k-1}x_n - B_kp\big)\big\|^2\Big), \end{aligned}$$

which implies

$$\begin{aligned} \big\|\Theta_n^kx_n - p\big\|^2 &\le \big\|\Theta_n^{k-1}x_n - p\big\|^2 - \big\|\Theta_n^{k-1}x_n - \Theta_n^kx_n - r_{k,n}\big(B_k\Theta_n^{k-1}x_n - B_kp\big)\big\|^2 \\ &= \big\|\Theta_n^{k-1}x_n - p\big\|^2 - \big\|\Theta_n^{k-1}x_n - \Theta_n^kx_n\big\|^2 - r_{k,n}^2\big\|B_k\Theta_n^{k-1}x_n - B_kp\big\|^2 \\ &\quad + 2r_{k,n}\big\langle\Theta_n^{k-1}x_n - \Theta_n^kx_n, B_k\Theta_n^{k-1}x_n - B_kp\big\rangle \\ &\le \big\|\Theta_n^{k-1}x_n - p\big\|^2 - \big\|\Theta_n^{k-1}x_n - \Theta_n^kx_n\big\|^2 \\ &\quad + 2r_{k,n}\big\langle\Theta_n^{k-1}x_n - \Theta_n^kx_n, B_k\Theta_n^{k-1}x_n - B_kp\big\rangle. \end{aligned} \tag{3.17}$$

From (3.14), (3.3), Lemma 2.6 and (3.17), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \frac{2a_n}{1+a_n}\langle x_0 - p, x_{n+1} - p\rangle + \frac{b_n}{1+a_n}\|x_n - p\|^2 + \frac{c_n+d_n}{1+a_n}\|z_n - p\|^2 \\ &\le \frac{2a_n}{1+a_n}\langle x_0 - p, x_{n+1} - p\rangle + \frac{b_n}{1+a_n}\|x_n - p\|^2 + \frac{c_n+d_n}{1+a_n}\big\|\Theta_n^kx_n - \Theta_n^kp\big\| \end{aligned}$$

$$\begin{aligned}
&\le \frac{2a_n}{1+a_n}\langle x_0 - p, x_{n+1} - p\rangle + \frac{b_n}{1+a_n}\|x_n - p\|^2 \\
&\quad + \frac{c_n + d_n}{1+a_n}\Big[\big\|\Theta_n^{k-1}x_n - p\big\|^2 - \big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|^2 \\
&\quad + 2r_{k,n}\big\langle\Theta_n^{k-1}x_n - \Theta_n^k x_n, B_k\Theta_n^{k-1}x_n - B_k p\big\rangle\Big] \\
&\le \frac{2a_n}{1+a_n}\langle x_0 - p, x_{n+1} - p\rangle + \frac{b_n}{1+a_n}\|x_n - p\|^2 \\
&\quad + \frac{c_n + d_n}{1+a_n}\Big[\|x_n - p\|^2 - \big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|^2 \\
&\quad + 2r_{k,n}\big\langle\Theta_n^{k-1}x_n - \Theta_n^k x_n, B_k\Theta_n^{k-1}x_n - B_k p\big\rangle\Big] \\
&\le \frac{2a_n}{1+a_n}\|x_0 - p\|\|x_{n+1} - p\| + \frac{b_n}{1+a_n}\|x_n - p\|^2 \\
&\quad + \frac{c_n + d_n}{1+a_n}\Big[\|x_n - p\|^2 - \big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|^2 \\
&\quad + 2r_{k,n}\big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|\big\|B_k\Theta_n^{k-1}x_n - B_k p\big\|\Big].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{c_n + d_n}{1+a_n}\big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|^2 \\
&\quad \le \frac{2a_n}{1+a_n}\|x_0 - p\|\|x_{n+1} - p\| + \frac{1-a_n}{1+a_n}\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad\quad + \frac{2r_{k,n}(c_n + d_n)}{1+a_n}\big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|\big\|B_k\Theta_n^{k-1}x_n - B_k p\big\| \\
&\quad \le \frac{2a_n}{1+a_n}\|x_0 - p\|\|x_{n+1} - p\| + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad\quad + \frac{2r_{k,n}(c_n + d_n)}{1+a_n}\big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|\big\|B_k\Theta_n^{k-1}x_n - B_k p\big\| \\
&\quad \le 2a_n\|x_0 - p\|\|x_{n+1} - p\| + \|x_{n+1} - x_n\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) \\
&\quad\quad + \frac{2r_{k,n}(c_n + d_n)}{1+a_n}\big\|\Theta_n^{k-1}x_n - \Theta_n^k x_n\big\|\big\|B_k\Theta_n^{k-1}x_n - B_k p\big\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}\frac{c_n+d_n}{1+a_n} > 0$, $a_n \to 0$, $\|x_{n+1} - x_n\| \to 0$ and $\|B_k\Theta_n^{k-1}x_n - B_k p\| \to 0$, we conclude that

$$\lim_{n\to\infty}\big\|\Theta_n^k x_n - \Theta_n^{k-1}x_n\big\| = 0, \quad k = 1, 2, \ldots, M. \tag{3.18}$$

Therefore we get

$$\begin{aligned}
\|x_n - z_n\| &= \big\|\Theta_n^0 x_n - \Theta_n^M x_n\big\| \\
&\le \big\|\Theta_n^0 x_n - \Theta_n^1 x_n\big\| + \big\|\Theta_n^1 x_n - \Theta_n^2 x_n\big\| + \cdots + \big\|\Theta_n^{M-1}x_n - \Theta_n^M x_n\big\| \to 0 \\
&\quad \text{as } n \to \infty.
\end{aligned} \tag{3.19}$$

From Lemma 2.3(1), we obtain

$$\begin{aligned}
\left\|\Omega^N z_n - \Omega^N p\right\|^2 &= \left\|P_C(I-\lambda_N A_N)\Omega^{N-1}z_n - P_C(I-\lambda_N A_N)\Omega^{N-1}p\right\|^2 \\
&\le \left\langle (I-\lambda_N A_N)\Omega^{N-1}z_n - (I-\lambda_N A_N)\Omega^{N-1}p, \Omega^N z_n - \Omega^N p\right\rangle \\
&= \frac{1}{2}\Big(\left\|(I-\lambda_N A_N)\Omega^{N-1}z_n - (I-\lambda_N A_N)\Omega^{N-1}p\right\|^2 + \left\|\Omega^N z_n - \Omega^N p\right\|^2 \\
&\quad - \left\|(I-\lambda_N A_N)\Omega^{N-1}z_n - (I-\lambda_N A_N)\Omega^{N-1}p - \left(\Omega^N z_n - \Omega^N p\right)\right\|^2\Big) \\
&\le \frac{1}{2}\Big(\left\|\Omega^{N-1}z_n - \Omega^{N-1}p\right\|^2 + \left\|\Omega^N z_n - \Omega^N p\right\|^2 \\
&\quad - \left\|\Omega^{N-1}z_n - \Omega^N z_n + \Omega^N p - \Omega^{N-1}p \right. \\
&\quad \left. - \lambda_N\left(A_N\Omega^{N-1}z_n - A_N\Omega^{N-1}p\right)\right\|^2\Big),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\left\|\Omega^N z_n - \Omega^N p\right\|^2 \\
&\quad \le \left\|\Omega^{N-1}z_n - \Omega^{N-1}p\right\|^2 \\
&\qquad - \left\|\Omega^{N-1}z_n - \Omega^N z_n + \Omega^N p - \Omega^{N-1}p - \lambda_N\left(A_N\Omega^{N-1}z_n - A_N\Omega^{N-1}p\right)\right\|^2 \\
&\quad = \left\|\Omega^{N-1}z_n - \Omega^{N-1}p\right\|^2 \\
&\qquad - \left\|\Omega^{N-1}z_n - \Omega^N z_n + \Omega^N p - \Omega^{N-1}p\right\|^2 - \lambda_N^2\left\|A_N\Omega^{N-1}z_n - A_N\Omega^{N-1}p)\right\|^2 \\
&\qquad + 2\lambda_N\left\langle\Omega^{N-1}z_n - \Omega^N z_n + \Omega^N p - \Omega^{N-1}p, A_N\Omega^{N-1}z_n - A_N\Omega^{N-1}p\right\rangle \\
&\quad \le \left\|\Omega^{N-1}z_n - \Omega^{N-1}p\right\|^2 - \left\|\Omega^{N-1}z_n - \Omega^N z_n + \Omega^N p - \Omega^{N-1}p\right\|^2 \\
&\qquad + 2\lambda_N\left\|\Omega^{N-1}z_n - \Omega^N z_n + \Omega^N p - \Omega^{N-1}p\right\|\left\|A_N\Omega^{N-1}z_n - A_N\Omega^{N-1}p\right\|. \qquad (3.20)
\end{aligned}$$

By induction, we have

$$\begin{aligned}
\left\|\Omega^N z_n - \Omega^N p\right\|^2 &\le \|z_n - p\|^2 - \sum_{i=1}^N\left\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\right\|^2 \\
&\quad + 2\sum_{i=1}^N\lambda_i\left\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\right\|\left\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\right\| \\
&\le \|x_n - p\|^2 - \sum_{i=1}^N\left\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{N-1}p\right\|^2 \\
&\quad + 2\sum_{i=1}^N\lambda_i\left\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\right\|\left\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\right\|,
\end{aligned}$$

that is,

$$\begin{aligned}
\|y_n - p\|^2 &\le \|x_n - p\|^2 - \sum_{i=1}^N\left\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\right\|^2 \\
&\quad + 2\sum_{i=1}^N\lambda_i\left\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\right\|\left\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\right\|. \qquad (3.21)
\end{aligned}$$

From (3.14) and (3.21),

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& \frac{2a_n}{1+a_n}\langle x_0-p, x_{n+1}-p\rangle + \frac{b_n}{1+a_n}\|x_n-p\|^2 \\
&+ \frac{c_n+d_n}{1+a_n}\Bigg[\|x_n-p\|^2 - \sum_{i=1}^{N}\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|^2 \\
&+ 2\sum_{i=1}^{N}\lambda_i\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|\Bigg] \\
\le{}& \frac{2a_n}{1+a_n}\|x_0-p\|\|x_{n+1}-p\| + \frac{b_n}{1+a_n}\|x_n-p\|^2 \\
&+ \frac{c_n+d_n}{1+a_n}\Bigg[\|x_n-p\|^2 - \sum_{i=1}^{N}\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|^2 \\
&+ 2\sum_{i=1}^{N}\lambda_i\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|\Bigg].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{c_n+d_n}{1+a_n}\sum_{i=1}^{N}\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|^2 \\
&\quad\le \frac{2a_n}{1+a_n}\|x_0-p\|\|x_{n+1}-p\| + \frac{1-a_n}{1+a_n}\|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\qquad + \frac{2(c_n+d_n)}{1+a_n}\sum_{i=1}^{N}\lambda_i\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\| \\
&\quad\le \frac{2a_n}{1+a_n}\|x_0-p\|\|x_{n+1}-p\| + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\qquad + \frac{2(c_n+d_n)}{1+a_n}\sum_{i=1}^{N}\lambda_i\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\| \\
&\quad\le \frac{2a_n}{1+a_n}\|x_0-p\|\|x_{n+1}-p\| + \|x_{n+1}-x_n\|\big(\|x_{n+1}-p\| + \|x_n-p\|\big) \\
&\qquad + \frac{2(c_n+d_n)}{1+a_n}\sum_{i=1}^{N}\lambda_i\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\| \\
&\quad\le 2a_n\|x_0-p\|\|x_{n+1}-p\| + \|x_{n+1}-x_n\|\big(\|x_{n+1}-p\| + \|x_n-p\|\big) \\
&\qquad + \frac{2(c_n+d_n)}{1+a_n}\sum_{i=1}^{N}\lambda_i\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\|\big\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\big\|.
\end{aligned}$$

Since $\liminf_{n\to\infty}\frac{c_n+d_n}{1+a_n} > 0$, $a_n \to 0$, $\|x_{n+1}-x_n\| \to 0$ and $\|A_i\Omega^{i-1}z_n - A_i\Omega^{i-1}p\| \to 0$, we conclude that

$$\lim_{n\to\infty}\sum_{i=1}^{N}\big\|\Omega^{i-1}z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1}p\big\| = 0. \tag{3.22}$$

Therefore, we get

$$\begin{aligned}\|z_n - y_n\| &= \left\|\Omega^0 z_n - \Omega^N z_n\right\| \\ &\le \sum_{i=1}^{N}\left\|\Omega^{i-1} z_n - \Omega^i z_n + \Omega^i p - \Omega^{i-1} p\right\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.23}$$

Thus from (3.19) and (3.23), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.24}$$

Observe that

$$\begin{aligned} d_n\|Sy_n - x_n\| &= \|d_n Sy_n - d_n x_n\| \\ &= \|x_{n+1} - a_n x_0 - b_n x_n - c_n y_n - d_n x_n\| \\ &= \left\|x_{n+1} - x_n + (1 - b_n - d_n)x_n - c_n y_n - a_n x_0\right\| \\ &= \left\|x_{n+1} - x_n + a_n(x_n - x_0) + c_n(x_n - y_n)\right\| \\ &\le \|x_{n+1} - x_n\| + a_n\|x_n - x_0\| + c_n\|x_n - y_n\|. \end{aligned}$$

Since $\liminf_{n\to\infty} d_n > 0$, $\|x_{n+1} - x_n\| \to 0$, $a_n \to 0$ and $\|x_n - y_n\| \to 0$, we have

$$\lim_{n\to\infty} \|Sy_n - x_n\| = 0. \tag{3.25}$$

From (3.24) and (3.25), we conclude that

$$\lim_{n\to\infty} \|Sy_n - y_n\| = 0. \tag{3.26}$$

Step 5. In this step, we prove that $\limsup_{n\to\infty}\langle x_0 - \overline{x}, x_n - \overline{x}\rangle \le 0$, where $\overline{x} = P_{\mathcal{F}} x_0$. Indeed, take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_0 - \overline{x}, x_n - \overline{x}\rangle = \lim_{n\to\infty}\langle x_0 - \overline{x}, x_{n_i} - \overline{x}\rangle.$$

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ which converges weakly to x^*. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup x^*$. From (3.18) and (3.24), we have $\Theta_{n_i}^k x_{n_i} \rightharpoonup x^*$, $y_{n_i} \rightharpoonup x^*$, where $k \in \{1, 2, \dots, M\}$. From (3.26) and Lemma 2.1, we have $x^* = Sx^*$ that is $x^* \in \mathcal{F}(S)$. Utilizing Lemma 2.7, we known that G is nonexpansive. And from (3.23), we obtain

$$\|y_{n_i} - Gy_{n_i}\| = \|Gz_{n_i} - Gy_{n_i}\| \le \|z_{n_i} - y_{n_i}\| \to 0 \quad \text{as } i \to \infty.$$

According to Lemma 2.10, we obtain $(I - G)x^* = 0$, that is, $x^* \in \mathcal{F}(G)$.

Next we prove that $x^* \in \bigcap_{k=1}^{M} \mathrm{GMEP}(F_k, \varphi_k, B_k)$. Since

$$\Theta_n^k x_n = T_{r_{k,n}}^{(F_k,\varphi_k)}(I - r_{k,n}B_k)\Theta_n^{k-1} x_n, \quad n \ge 1, k \in \{1, 2, \dots, M\}.$$

For all $y \in C$, we have

$$F_k\left(\Theta_n^k x_n, y\right) + \varphi_k(y) - \varphi_k\left(\Theta_n^k x_n\right) + \left\langle B_k \Theta_n^{k-1} x_n, y - \Theta_n^k x_n\right\rangle + \frac{1}{r_{k,n}}\left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n\right\rangle \geq 0.$$

By (A2), we have

$$\varphi_k(y) - \varphi_k\left(\Theta_n^k x_n\right) + \left\langle B_k \Theta_n^{k-1} x_n, y - \Theta_n^k x_n\right\rangle + \frac{1}{r_{k,n}}\left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n\right\rangle \geq F_k\left(y, \Theta_n^k x_n\right).$$

Replacing n by n_i in the above inequality, we have

$$\varphi_k(y) - \varphi_k\left(\Theta_{n_i}^k x_{n_i}\right) + \left\langle B_k \Theta_{n_i}^{k-1} x_{n_i}, y - \Theta_{n_i}^k x_{n_i}\right\rangle + \frac{1}{r_{k,n_i}}\left\langle y - \Theta_{n_i}^k x_{n_i}, \Theta_{n_i}^k x_{n_i} - \Theta_{n_i}^{k-1} x_{n_i}\right\rangle \geq F_k\left(y, \Theta_{n_i}^k x_{n_i}\right).$$

Let $z_t = ty + (1-t)x^*$ for all $t \in (0,1]$ and $y \in C$. This implies that $z_t \in C$. Then we have

$$\begin{aligned}
&\left\langle z_t - \Theta_{n_i}^k x_{n_i}, B_k z_t\right\rangle \\
&\quad\geq \left\langle z_t - \Theta_{n_i}^k x_{n_i}, B_k z_t\right\rangle + \varphi_k\left(\Theta_{n_i}^k x_{n_i}\right) - \varphi_k(z_t) \\
&\qquad - \left\langle B_k \Theta_{n_i}^{k-1} x_{n_i}, z_t - \Theta_{n_i}^k x_{n_i}\right\rangle - \frac{1}{r_{k,n_i}}\left\langle z_t - \Theta_{n_i}^k x_{n_i}, \Theta_{n_i}^k x_{n_i} - \Theta_{n_i}^{k-1} x_{n_i}\right\rangle + F_k\left(z_t, \Theta_{n_i}^k\right) \\
&\quad= \varphi_k\left(\Theta_{n_i}^k x_{n_i}\right) - \varphi_k(z_t) + \left\langle z_t - \Theta_{n_i}^k x_{n_i}, B_k z_t - B_k \Theta_{n_i}^k x_{n_i}\right\rangle \\
&\qquad + \left\langle z_t - \Theta_{n_i}^k x_{n_i}, B_k \Theta_{n_i}^k x_{n_i} - B_k \Theta_{n_i}^{k-1} x_{n_i}\right\rangle \\
&\qquad - \left\langle z_t - \Theta_{n_i}^k x_{n_i}, \frac{\Theta_{n_i}^k x_{n_i} - \Theta_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}}\right\rangle + F_k\left(z_t, \Theta_{n_i}^k x_{n_i}\right).
\end{aligned}$$

By (3.18) we have $\|B_k \Theta_{n_i}^k x_{n_i} - B_k \Theta_{n_i}^{k-1} x_{n_i}\| \to 0$ as $i \to \infty$. Furthermore, by the monotonicity of B_k, we obtain $\langle z_t - \Theta_{n_i}^k x_{n_i}, B_k z_t - B_k \Theta_{n_i}^k x_{n_i}\rangle \geq 0$. Then from (A4), the lower semicontinuity of φ and

$$\frac{\Theta_{n_i}^k x_{n_i} - \Theta_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}} \to 0, \quad \Theta_{n_i}^k x_{n_i} \rightharpoonup x^*,$$

we obtain that

$$\left\langle z_t - x^*, B_k z_t\right\rangle \geq \varphi_k\left(x^*\right) - \varphi_k(z_t) + F_k\left(z_t, x^*\right). \tag{3.27}$$

Using (A1), (A4) and (3.27), we have

$$\begin{aligned}
0 &= F_k(z_t, z_t) + \varphi_k(z_t) - \varphi_k(z_t) \\
&\leq tF_k(z_t, y) + (1-t)F_k\left(z_t, x^*\right) + t\varphi_k(y) + (1-t)\varphi_k\left(x^*\right) - t\varphi_k(z_t) - (1-t)\varphi_k(z_t)
\end{aligned}$$

$$\begin{aligned}
&= t\bigl[F_k(z_t,y) + \varphi_k(y) - \varphi_k(z_t)\bigr] + (1-t)\bigl[F_k\bigl(z_t,x^*\bigr) + \varphi_k\bigl(x^*\bigr) - \varphi_k(z_t)\bigr] \\
&\le t\bigl[F_k(z_t,y) + \varphi_k(y) - \varphi_k(z_t)\bigr] + (1-t)\bigl\langle z_t - x^*, B_k z_t\bigr\rangle \\
&= t\bigl[F_k(z_t,y) + \varphi_k(y) - \varphi_k(z_t)\bigr] + (1-t)t\bigl\langle y - x^*, B_k z_t\bigr\rangle,
\end{aligned}$$

and hence

$$0 \le F_k(z_t,y) + \varphi_k(y) - \varphi_k(z_t) + (1-t)\bigl\langle y - x^*, B_k z_t\bigr\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \le F_k\bigl(x^*,y\bigr) + \varphi_k(y) - \varphi_k\bigl(x^*\bigr) + \bigl\langle y - x^*, B_k x^*\bigr\rangle.$$

This implies that $x^* \in \mathrm{GMEP}(F_k,\varphi_k,B_k)$. Hence $x^* \in \bigcap_{k=1}^M \mathrm{GMEP}(F_k,\varphi_k,B_k)$. Therefore,

$$x^* \in \mathcal{F} = \left[\bigcap_{k=1}^M \mathrm{GMEP}(F_k,\varphi_k,B_k)\right] \cap \mathcal{F}(G) \cap \mathcal{F}(S).$$

This together with the property of metric projection implies that

$$\limsup_{n\to\infty}\langle x_0 - \overline{x}, x_n - \overline{x}\rangle = \lim_{n\to\infty}\langle x_0 - \overline{x}, x_{n_i} - \overline{x}\rangle = \bigl\langle x_0 - \overline{x}, x^* - \overline{x}\bigr\rangle \le 0. \tag{3.28}$$

Step 6. Finally, we can easily show that $x_n \to \overline{x}$ as $n \to \infty$.

Indeed, from (3.14) and (3.3), we have

$$\begin{aligned}
\|x_{n+1} - \overline{x}\|^2 &\le \frac{2a_n}{1+a_n}\langle x_0 - \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{b_n}{1+a_n}\|x_n - \overline{x}\|^2 + \frac{c_n + d_n}{1+a_n}\|x_n - \overline{x}\|^2 \\
&= \frac{2a_n}{1+a_n}\langle x_0 - \overline{x}, x_{n+1} - \overline{x}\rangle + \left(1 - \frac{2a_n}{1+a_n}\right)\|x_n - \overline{x}\|^2.
\end{aligned}$$

It is clear that $\sum_{n=1}^\infty \frac{2a_n}{1+a_n} = \infty$. Hence, applying (3.28) and Lemma 2.11, we obtain immediately that $x_n \to \overline{x}$ as $n \to \infty$. This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements

The project is supported by the National Natural Science Foundation of China (Grant Nos. 11071041, 11201074) and Fujian Natural Science Foundation (Grant No. 2013J01003).

Received: 20 August 2012 Accepted: 29 April 2013 Published: 15 May 2013

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doi:10.1186/1687-1812-2013-126
Cite this article as: Ke and Ma: **A new relaxed extragradient-like algorithm for approaching common solutions of generalized mixed equilibrium problems, a more general system of variational inequalities and a fixed point problem.** *Fixed Point Theory and Applications* 2013 **2013**:126.