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Coupled coincidence point theorems for generalized nonlinear contraction in partially ordered metric spaces

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Abstract

This paper is concerned with mixed g-monotone mappings in partially ordered metric spaces. We establish several coupled coincidence and coupled common fixed point theorems, which generalize and complement some known results. Especially, our main results complement some recent results due to Lakshmikantham and Ćirić. Two examples are given to illustrate the usability of our results.

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1 Introduction

The existence of fixed points for monotone mappings in partially ordered metric spaces was initialed in [1], and such problems have been of great interest for many mathematicians (see, e.g, [2-6] and references therein).

The existence of coupled fixed points for mixed monotone mappings in partially ordered metric spaces was firstly studied by Bhaskar and Lakshmikantham [7], where some applications to periodic boundary value problems are studied. Since then, several authors have made contributions on such problems (see, e.g., [8-16]). Especially, Lakshmikantham and Ćirić [13] introduced a new concept of mixed g-monotone mapping:

Definition 1.1. Let $(X, \leq)$ be a partially ordered set, $F: X \times X \to X$ and $g: X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, g(x_1) \leq g(x_2) \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y),$$

and

$$y_1, y_2 \in X, g(y_1) \leq g(y_2) \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Moreover, Lakshmikantham and Ćirić [13] established several coupled coincidence and coupled fixed point theorems for mixed g-monotone mappings in a partially ordered metric space. In [13], one of the key assumption on the mixed g-monotone mapping F is:

$$d(F(x,y),F(u,v)) \le \varphi\left(\frac{d(g(x),g(u)) + d(g(y),g(v))}{2}\right) \tag{1.1}$$

for all $x, y, u, v \in X$ with $g(x) \le g(u)$ and $g(y) \ge g(v)$, where $\phi : [0, +\infty) \to [0, +\infty)$ satisfies $\phi(t)$ <and $\lim_{r\to t+} \varphi(t) < t$ for each $t > 0$.

The aim of this paper is to extend and complement the main results in [13] by replacing the contraction assumption (1.1) by a more general condition (see (A1) in Theorem 1). As one will see, our main results are generalizations and complements of some earlier results (see Examples and Remark 1). For some details see [17,18].

2 Main results

Throughout the rest of this paper, we denote by $\mathbb{N}$ the set of positive integers, and by $(X, \le, d)$ a complete partially ordered metric space, i.e., $\le$ is a partial order on the set X, and d is a complete metric on X. Moreover, we endow the product space $X \times X$ with the following partial order: $(u, v) \le (x, y) \Leftrightarrow x \ge u, y \le v$.

Now, let us present one of our main results.

Theorem 2.1. *Assume that $g: X \to X$ is a continuous mapping, and $F: X \times X \to X$ is a continuous mapping with the mixed g-monotone property on X. Suppose that the following assumptions hold:*

(**A1**) *there exists a non-decreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\lim_{n\to\infty} \phi^n(t) = 0$ for each $t > 0$, and $d(F(x, y), F(u, v)) \le \varphi[M_{F,g}(x, y, u, v)]$ for all $x, y, u, v \in X$ with $gx \ge gu$ and $gy \le gv$, where*

$$M_{F,g}(x,y,u,v) = \max\Big\{d(gx,gu), d(gy,gv), d(gx,F(x,y)), d(gu,F(u,v)), d(gy,F(y,x)), d(gv,F(v,u)), \frac{d(gx,F(u,v)) + d(gu,F(x,y))}{2}, \frac{d(gy,F(v,u)) + d(gv,F(y,x))}{2}\Big\}.$$

(**A2**) *there exist $x_0, y_0 \in X$ such that $gx_0 \le F(x_0, y_0)$ and $F(y_0, x_0) \le gy_0$;*

(**A3**) *$F(X \times X) \subseteq g(X)$, and g and F are commuting, i.e., $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$.*

Then F and g have a coupled coincidence point, i.e., there exist $x_*, y_* \in X$ such that $F(x_*, y_*) = gx_*$ and $F(y_*, x_*) = gy_*$.

Proof. First, we claim that $\varphi(t) < t$ for each $t > 0$. In fact, if $\varphi(t_0) \ge t_0$ for some $t_0 > 0$, then, since φ is non-decreasing, $\varphi^n(t_0) \ge t_0$ for all $\mathbb{N}$, which contradicts with $\lim_{n\to\infty} \phi^n(t_0) = 0$. In addition, it is easy to see that $\varphi(0) = 0$. Since $F(X \times X) \subseteq g(X)$, one can construct two sequences $\{x_n\}$, $\{y_n\}$ in X such that $gx_n = F(x_{n-1}, y_{n-1})$, $gy_n = F(y_{n-1}, x_{n-1})$, $n \in \mathbb{N}$. Observing that F has the mixed g-monotone property on X, by (A2), we get $gx_0 \le gx_1 \le ... \le gx_n \le gx_{n+1} \le ...$ and $... \le gy_{n+1} \le gy_n \le ... \le gy_1 \le gy_0$.

Now, by (A1), we have

$$d(gx_{n+1}, gx_n) = d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \le \phi(M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1})),$$

and

$$d(gy_n, gy_{n+1}) = d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \le \phi(M_{F,g}(y_{n-1}, x_{n-1}, y_n, x_n)),$$

where

$$\begin{aligned}
&M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = M_{F,g}(y_{n-1}, x_{n-1}, y_n, x_n)\\
&= \max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, F(x_n, y_n)), d(gx_{n-1}, F(x_{n-1}, y_{n-1})),\\
&d(gy_n, F(y_n, x_n)), d(gy_{n-1}, F(y_{n-1}, x_{n-1})), \frac{d(gx_n, F(x_{n-1}, y_{n-1})) + d(gx_{n-1}, F(x_n, y_n))}{2},\\
&\left.\frac{d(gy_n, F(y_{n-1}, x_{n-1})) + d(gy_{n-1}, F(y_n, x_n))}{2}\right\}\\
&= \max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}), d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}),\\
&\left.\frac{d(gx_{n-1}, gx_{n+1})}{2}, \frac{d(gy_{n-1}, gy_{n+1})}{2}\right\}.
\end{aligned}$$

Next, let us consider five cases.

Case I. $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = \max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}$.

We have

$$d(gx_{n+1}, gx_n) \le \phi[\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}], \tag{2.1}$$

and

$$d(gy_n, gy_{n+1}) \le \phi[\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}]. \tag{2.2}$$

Case II. $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = d(gx_n, gx_{n+1})$.

We claim that $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = d(gx_n, gx_{n+1}) = 0$. In fact, if $d(gx_n, gx_{n+1}) \ne 0$, then $d(gx_{n+1}, gx_n) \le \varphi[d(gx_{n+1}, gx_n)] < d(gx_{n+1}, gx_n)$, which is contradiction. Since $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = 0$, we also have $d(gy_n, gy_{n+1}) = 0$. Then, it is obvious that (2.1) and (2.2) hold.

Case III. $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = d(gy_n, gy_{n+1})$.

Similar to the proof of Case II, one can also show that (2.1) and (2.2) hold.

Case IV. $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = \frac{d(gx_{n-1}, gx_{n+1})}{2}$.

We also claim that $d(gx_{n-1}, gx_{n+1}) = 0$. In fact, if $d(gx_{n-1}, gx_{n+1}) \ne 0$, then

$$\begin{aligned}
d(gx_{n+1}, gx_n) &\le \phi\left[\frac{d(gx_{n-1}, gx_{n+1})}{2}\right]\\
&< \frac{d(gx_{n-1}, gx_{n+1})}{2}\\
&\le \frac{d(gx_{n-1}, gx_n)}{2} + \frac{d(gx_n, gx_{n+1})}{2},
\end{aligned}$$

which gives that $d(gx_{n+1}, gx_n) < d(gx_{n-1}, gx_n)$. Thus,

$$M_F(x_n, y_n, x_{n-1}, y_{n-1}) = \frac{d(gx_{n-1}, gx_{n+1})}{2} < d(gx_{n-1}, gx_n),$$

which contradicts with the definition of $M_F(x_n, y_n, x_{n-1}, y_{n-1})$. So

$$M_F(x_n, y_n, x_{n-1}, y_{n-1}) = \frac{d(gx_{n-1}, gx_{n+1})}{2} = 0.$$

Thus, $d(gx_n, gx_{n+1}) = d(gy_n, gy_{n+1}) = 0$, which means that (2.1) and (2.2) hold.

Case V. $M_{F,g}(x_n, y_n, x_{n-1}, y_{n-1}) = \frac{d(gy_{n-1}, gy_{n+1})}{2}$.

By using a similar argument to Case IV, one can also show that (2.1) and (2.2) hold. Now, by (2.1) and (2.2), we have for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1})\} \\
&\leq \phi[\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1}))\}] \\
&\leq \phi^n[\max\{d(gx_1, gx_0), d(gy_1, gy_0))\}].
\end{aligned} \tag{2.3}$$

Let $\varepsilon > 0$ be fixed. Without loss of generality, one can assume that

$$\max\{d(gx_1, gx_0), d(gy_1, gy_0))\} \neq 0.$$

In fact, if this is not true, then

$$gx_0 = gx_1 = F(x_0, y_0), \quad gy_0 = gy_1 = F(y_0, x_0),$$

i.e., x_0, y_0 is a coupled coincidence point of F and g. Since $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$, by using (2.3), there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1})\} < \varepsilon - \phi(\varepsilon). \tag{2.4}$$

Next, let us prove that for all $n > N$,

$$\max\{d(gx_{n+p}, gx_n), d(gy_n, gy_{n+p})\} \leq \varepsilon, \forall p \in \mathbb{N}, \tag{2.5}$$

and

$$\max\{d(gx_{n+p-1}, gx_{n+1}), d(gy_{n+1}, gy_{n+p-1})\} \leq \phi(\varepsilon), \forall p \geq 3. \tag{2.6}$$

For $p = 1$, it follows directly from (2.4) that (2.5) holds. For $p = 2$, (2.5) follows from

$$\begin{aligned}
&\max\{d(gx_{n+2}, gx_n), d(gy_{n+2}, gy_n)\} \\
&\leq \max\{d(gx_{n+2}, gx_{n+1}), d(gy_{n+2}, gy_{n+1})\} + \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \\
&\leq \phi[\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}] + \varepsilon - \phi(\varepsilon) \\
&\leq \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon,
\end{aligned}$$

where (2.3) and (2.4) are used. Let us show that (2.5) and (2.6) hold for $p = 3$. Firstly, by (2.3) and (2.4), we have

$$\max\{d(gx_{n+2}, gx_{n+1}), d(gy_{n+2}, gy_{n+1})\} \leq \phi[\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}] \leq \phi(\varepsilon),$$

which means that (2.6) holds for $p = 3$. Secondly, by (A1), we have

$$\max\{d(gx_{n+3}, gx_{n+1}), d(gy_{n+1}, gy_{n+3})\} \leq \phi[z_n], \tag{2.7}$$

where

$$\begin{aligned}
z_n = \max\{&d(gx_{n+2}, gx_n), d(gy_n, gy_{n+2}), d(gx_{n+2}, gx_{n+3}), d(gy_{n+2}, gy_{n+3}), d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \\
&\frac{d(gx_{n+2}, gx_{n+1}) + d(gx_n, gx_{n+3})}{2}, \frac{d(gy_{n+2}, gy_{n+1}) + d(gy_n, gy_{n+3})}{2}\bigg\}.
\end{aligned}$$

We claim that $z_n \leq \varepsilon$. In fact, if $z_n = \max\{d(gx_{n+2}, gx_n), d(gy_n, gy_{n+2})\}$, then by (2.5) ($p = 2$), $z_n \leq \varepsilon$; if $z_n = \max\{d(gx_{n+2}, gx_{n+3}), d(gy_{n+2}, gy_{n+3})\}$, then by (2.3) and (2.4), $z_n \leq \varphi^2(\varepsilon) \leq \varepsilon$; if $z_n = \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\}$, then (2.4) gives that $z_n \leq \varepsilon$; if $z_n = \frac{d(gx_{n+2},gx_{n+1})+d(gx_n,gy_{n+3})}{2}$, then by (2.7), there holds $d(gx_{n+3}, gx_{n+1}) \leq \frac{d(gx_{n+2},gx_{n+1})+d(gx_n,gy_{n+3})}{2} \leq \frac{\phi(\varepsilon)+d(gx_n,gy_{n+3})}{2}$, which yields that

$$\begin{aligned} d(gx_n, gx_{n+3}) &\le d(gx_{n+3}, gx_{n+1}) + d(gx_{n+1}, gx_n) \\ &\le \frac{\phi(\varepsilon) + d(gx_n, gx_{n+3})}{2} + \varepsilon - \phi(\varepsilon) \\ &= \varepsilon - \frac{\phi(\varepsilon)}{2} + \frac{d(gx_n, gx_{n+3})}{2}, \end{aligned}$$

and thus, $z_n = \frac{d(gx_{n+2}, gx_{n+1}) + d(gx_n, gx_{n+3})}{2} \le \frac{\phi(\varepsilon)}{2} + \frac{d(gx_n, gx_{n+3})}{2} \le \varepsilon$; if $z_n = \frac{d(gy_{n+2}, gy_{n+1}) + d(gy_n, gy_{n+3})}{2}$, one can similarly show that $z_n \le \varepsilon$. Hence, in all cases, $z_n \le \varepsilon$. Then, by (2.4) and (2.7), we get

$$\begin{aligned} &\max\{d(gx_{n+3}, gx_n), d(gy_n, gy_{n+3})\} \\ &\le \max\{d(gx_{n+3}, gx_{n+1}), d(gy_{n+1}, gy_{n+3})\} + \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1})\} \\ &\le \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon, \end{aligned}$$

i.e., (2.5) holds for $p = 3$.

Now, suppose that (2.5) and (2.6) hold for all $p \le k - 1$. Let us prove that (2.5) and (2.6) hold for $p = k$. By (A1), (2.3), (2.4), (2.5) for $p = k - 2$, $k - 1$ and for $p = k - 1$ we conclude

$$\begin{aligned} &\max\{d(gx_{n+k-1}, gx_{n+1}), d(gy_{n+1}, gy_{n+k-1})\} \\ &\le \phi\Big[\max\Big\{d(gx_{n+k-2}, gx_n), d(gy_n, gy_{n+k-2}), d(gx_{n+k-2}, gx_{n+k-1}), d(gy_{n+k-2}, gy_{n+k-1}), \\ &\quad d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), \frac{d(gx_{n+k-2}, gx_{n+1}) + d(gx_n, gx_{n+k-1})}{2}, \\ &\quad \frac{d(gy_{n+k-2}, gy_{n+1}) + d(gy_n, gy_{n+k-1})}{2}\Big\}\Big] \\ &\le \phi\left[\max\left\{\varepsilon, \phi^{k-2}(\varepsilon), \frac{\phi(\varepsilon) + \varepsilon}{2}\right\}\right] \le \phi(\varepsilon), \end{aligned}$$

i.e., (2.6) holds for $p = k$. In addition, since max $\{d(gx_{n+k}, gx_{n+1}), d(gy_{n+1}, gy_{n+k})\} \le \varphi[w_n]$, where w_n = max $\{d(gx_{n+k-1}, gx_n), d(gy_n, gy_{n+k-1}), d(gx_{n+k-1}, gx_{n+k}), d(gy_{n+k-1}, gy_{n+k}), d(gx_n, gx_{n+1}),$

$$\begin{aligned} w_n = \max\Big\{&d(gx_{n+k-1}, gx_n), d(gy_n, gy_{n+k-1}), d(gx_{n+k-1}, gx_{n+k}), d(gy_{n+k-1}, gy_{n+k}), d(gx_n, gx_{n+1}), \\ &d(gy_n, gy_{n+1}), \frac{d(gx_{n+k-1}, gx_{n+1}) + d(gx_n, gx_{n+k})}{2}, \frac{d(gy_{n+k-1}, gy_{n+1}) + d(gy_n, gy_{n+k})}{2}\Big\}, \end{aligned}$$

by similar proof to that of z_n (see (2.7)), one can show that $w_n \le \varepsilon$. Thus,

$$\begin{aligned} &\max\{d(gx_{n+k}, gx_n), d(gy_n, gy_{n+k})\} \\ &\le \max\{d(gx_{n+k}, gx_{n+1}), d(gy_{n+1}, gy_{n+k})\} + \max\{d(gx_{n+1}, gx_n), d(gy_n, gy_{n+1})\} \\ &\le \phi(\varepsilon) + \varepsilon - \phi(\varepsilon) = \varepsilon, \end{aligned}$$

i.e., (2.6) holds for $p = k$.

Now, we have proved that (2.5) holds for all $p \in \mathbb{N}$, which means that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Then, by the completeness of X, there exist $x_*, y_* \in X$ such that $\lim_{n\to\infty} gx_n = x_*$, $\lim_{n\to\infty} gy_n = y_*$. By (A3), g commutes with F. So $g(gx_{n+1}) = g(F(x_n, y_n)) = F(gx_n, gy_n)$ and $g(gy_{n+1}) = g(F(y_n, x_n)) = F(gy_n, gx_n)$. Letting $n \to \infty$ and noticing that F and g are continuous, we get $g(x_*) = F(x_*, y_*)$, $g(y_*) = F(y_*, x_*)$. □

We demonstrate the use of Theorems 2.1 with the help of the following examples. It will show also that our theorem is more general than some other known coupled fixed point results ([7,13]).

Examples: (**1**) Let $X = \mathbb{R}$ be endowed with usual order, $d(x, y) = |x - y|$, $F(x,y) = \frac{1}{4}x - \frac{1}{5}y$, $gx = \frac{1}{2}x$, $\phi(t) = \frac{9}{10}t$. We have that mappings F, g and φ satisfy all conditions of the Theorem 2.1, but they do not satisfy (1.1). Therefore F and g have a coupled coincidence point. Here (0, 0) is the coupled coincidence point of F and g. Indeed, for $x \geq u$, $y \leq v$ we have

$$\begin{aligned} d(F(x,y), F(u,v)) &= \left|\frac{1}{4}(x-u) + \frac{1}{5}(v-y)\right| \leq \frac{9}{20}\max\{|x-u|, |y-v|\} \\ &= \frac{9}{10}\max\left\{\left|\frac{x}{2} - \frac{u}{2}\right|, \left|\frac{y}{2} - \frac{v}{2}\right|\right\} \\ &= \frac{9}{10}\max\{d(gx, gu), d(gy, gv)\}. \end{aligned}$$

However, if (1.1) is true, then we have

$$\frac{1}{4} = d(0, \frac{1}{4}) = d(F(0,0), F(1,0)) \leq \varphi\left(\frac{d(g0, g1) + d(g0, g0)}{2}\right) = \varphi(\frac{1}{4}) < \frac{1}{4},$$

which is a contradiction. Hence, the existence of a coupled coincidence point of F and g cannot be obtained by the result from [13].

(**2**) If in the previous example we take $F(x,y) = \frac{1}{2}x - \frac{1}{3}y$, $gx = x$, $\phi(t) = \frac{5}{6}t$, then we obtain that mappings F, g and φ satisfy all conditions of the Theorem 2.1, but they do not satisfy the conditions of corresponding Theorem from [7]. Indeed, in this case we have

$$d(F(x,y), F(u,v)) = \left|\frac{1}{2}(x-u) + \frac{1}{3}(v-y)\right| \leq \frac{5}{6}\max\{|x-u|, |y-v|\}.$$

On the other hand, for $x = 1$, $y = u = v = 0$, we obtain

$$d(F(1,0), F(0,0)) = \frac{1}{2} > \frac{k}{2} = \frac{k}{2}(d(1,0) + d(0,0)),$$

for all $k \in [0, 1)$. Also, this example shows that the existence of a coupled coincidence point of F and g cannot be obtained by the result from [7].

In the case that F is not continuous, one can use the following theorem:

Theorem 2.2. *Suppose all the assumptions of Theorem 2.1. except for the continuity of F are satisfied. Moreover, assume that g is monotone under the partial order $\leq$, and X has the following properties:*

*(**a**) if an non-decreasing sequence $\{x_n\}$ converges to X in X, then $x_n \leq x$ for all $n \in \mathbb{N}$;*

*(**b**) if an non-increasing sequence $\{y_n\}$ converges to y in X, then $y \leq y_n$ for all $n \in \mathbb{N}$.*

Then the conclusions of Theorem 2.1 also hold.

Proof. Let $\{x_n\}$, $\{y_n\}$, x_*, y_* be as in Theorem 1. Then $\lim_{n\to\infty} gx_n = x_*$ and $\lim_{n\to\infty} gy_n = y_*$.

It remains to prove that $g(x_*) = F(x_*, y_*)$, $g(y_*) = F(y_*, x_*)$.

By the proof of Theorem 1, we have $gx_0 \leq gx_1 \leq \ldots \leq gx_n \leq gx_{n+1} \leq \ldots$ and $\ldots \leq gy_{n+1} \leq gy_n \leq \ldots \leq gy_1 \leq gy_0$. It follows from the assumptions (a) and (b) that $gx_n \leq x_*$ and $y_* \leq$

gy_n for all $n \in \mathbb{N}$. In addition, without loss of generality, one can assume that g is non-decreasing about the partial order $\leq$. Then $g^2x_n \leq gx_*$ and $gy_* \leq g^2y_n$ for all $n \in \mathbb{N}$, where $g^2z := g\,(gz)$ for all $z \in X$.

Next, using (A1), we obtain

$$d(F(x_*, y_*), g^2x_{n+1}) = d(F(x_*, y_*), F(gx_n, gy_n)) \leq \phi[a_n], \tag{2.8}$$

$$\text{and } d(g^2y_{n+1}, F(y_*, x_*)) = d(F(gy_n, gx_n), F(y_*, x_*)) \leq \phi[b_n], \tag{2.9}$$

where

$$\begin{aligned} a_n = b_n = \max\{&d(gx_*, g^2x_n), d(y_*, g^2y_n), d(gx_*, F(x_*, y_*)), \\ &d(gy_*, F(y_*, x_*)), d(g^2x_n, g^2x_{n+1}), d(g^2y_n, g^2x_{n+1}), \\ &\left.\frac{d(gx_*, g^2x_{n+1}) + d(g^2x_n, F(x_*, y_*))}{2}, \frac{d(gy_*, g^2y_{n+1}) + d(g^2y_n, F(y_*, x_*))}{2}\right\}. \end{aligned}$$

Now, we claim that

$$\max\{d(gx_*, F(x_*, y_*)), d(gy_*, F(y_*, x_*))\} = 0. \tag{2.10}$$

If this not true, then max $\{d\,(gx_*, F\,(x_*, y_*))\,,\, d\,(gy_*, F\,(y_*, x_*))\} > 0$. Since $\lim_{n\to\infty} gx_n = x_*$, $\lim_{n\to\infty} gy_n = y_*$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$a_n = b_n = \max\{d(gx_*, F(x_*, y_*)), d(gy_*, F(y_*, x_*))\}.$$

Combining this with (2.8) and (2.9), we get for all $n > N$,

$$\begin{aligned} &\max\{d(F(x_*, y_*)g^2x_{n+1}), d(g^2y_{n+1}, F(y_*, x_*))\} \\ &\leq \phi[\max\{d(gx_*, F(x_*, y_*)), d(gy_*, F(y_*, x_*))\}]. \end{aligned}$$

Letting $n \to \infty$ it follows that

$$\begin{aligned} &\max\{d(gx_*, F(x_*, y_*)), d(gy_*, F(y_*, x_*))\} \\ &\leq \phi[\max\{d(gx_*, F(x_*, y_*)), d(gy_*, F(y_*, x_*))\}]. \end{aligned}$$

This is a contradiction. So (2.10) holds. Then, it follows that $gx_* = F\,(x_*, y_*)$ and $gy_* = F\,(y_*, x_*)$.

Remark 1. It is easy to see that Theorems 2.1. and 2.2. are generalizations of corresponding results in [7]. In addition, Theorems 2.1. and 2.2. extends some earlier results for non-decreasing mappings in partially ordered metric spaces. For example, let $g = I$ (the identity map), F be non-decreasing under the first argument and be independent of the second argument, one can deduce [2, Theorem 2.2].

In some cases, one can show that the coupled coincidence point is a coupled common fixed point. For example, we have the following result:

Theorem 2.3. *Suppose all the assumptions of Theorem 2.1. (or Theorem 2.2.) are satisfied. Moreover, assume that*

(A4) the $M_{F,g}(x, y, u, v)$ in (A1) equals to

$$\max\left\{d(gx, gu), d(gy, gv), \frac{d(gx, F(u, v)) + d(gu, F(x, y))}{2}, \frac{d(gy, F(v, u)) + d(gv, F(y, x))}{2}\right\};$$

(A5) for every (x_1, x_2), $(y_1, y_2) \in X \times X$, there exists $(z_1, z_2) \in X \times X$ such that (gz_1, gz_2) is comparable to (gx_1, gx_2) and (gy_1, gy_2).

Then F and g have a unique coupled common fixed point, i.e., there exists a unique $(a, b) \in X \times X$ such that $F(a, b) = ga = a$ and $F(b, a) = gb = b$.

Proof. By Theorem 2.1. (or Theorem 2.2.), we know that F and g have a coupled coincidence point, i.e., there exist $x_*, y_* \in X$ such that $F(x_*, y_*) = gx_*$ and $F(y_*, x_*) = gy_*$. Let $(x^*, y^*) \in X \times X$ be also a coupled coincidence point of F and g. First, let us prove that

$$gx_* = gx^*, gy_* = gy^*. \tag{2.11}$$

By (A5), there exists $(u_0, v_0) \in X \times X$ such that (gu_0, gv_0) is comparable to (gx_*, gy_*) and (gx^*, gy^*) . Let

$$gu_n = F(u_{n-1}, v_{n-1}), gv_n = F(v_{n-1}, u_{n-1}), n = 1, 2, \ldots$$

Since F is mixed g-monotone and (gu_0, gv_0) is comparable to (gx_*, gy_*), we claim that $(F(u_0, v_0), F(v_0, u_0))$ is comparable to $(F(x_*, y_*), F(y_*, x_*))$, i.e., (gu_1, gv_1) is comparable to (gx_*, gy_*) . In fact, if

$$(gu_0, gv_0) \le (gx_*, gy_*),$$

i.e.,

$$gu_0 \ge gx_* \text{ and } gv_0 \le gy_*,$$

and thus

$$gu_1 = F(u_0, v_0) \ge F(x_*, y_*) = gx_* \text{ and } gv_1 = F(v_0, u_0) \le F(y_*, x_*) = gy_*,$$

which means that

$$(gu_1, gv_1) \le (gx_*, gy_*);$$

if $(gu_0, gv_0) = (gx_*, gy_*)$, by a similar proof, we can get $(gu_1, gv_1) \ge (gx_*, gy_*)$. In addition, analogously to the above proof, one can also show that (gu_1, gv_1) is comparable to (gx^*, gy^*) . Hence, by induction, one can prove that for each $n \in \mathbb{N}$, (gu_n, gv_n) is comparable to (gx_*, gy_*) and (gx^*, gy^*) .

Now, by (A4), we have

$$\begin{aligned} &\max\{d(gx_*, gu_{n+1}), d(gy_*, gv_{n+1})\} \\ = &\max\{d(F(x_*, y_*), F(u_n, v_n)), d(F(y_*, x_*), F(v_n, u_n))\} \le \phi[c_n], \end{aligned}$$

where

$$c_n = \max\left\{d(gx_*, gu_n), d(gy_*, gv_n), \frac{d(gx_*, gu_{n+1}) + d(gx_*, gu_n)}{2}, \frac{d(gy_*, gv_{n+1}) + d(gy_*, gv_n)}{2}\right\}.$$

We claim that $c_n = \max\{d(gx_*, gu_n), d(gy_*, gv_n)\}$. In fact, if $c_n = \frac{d(gx_*,gu_{n+1})+d(gx_*,gu_n)}{2} > 0$, then $d(gx_*, gu_{n+1}) \le \phi[c_n] < \frac{d(gx_*,gu_{n+1})+d(gx_*,gu_n)}{2}$, which means that $d(gx_*, gu_{n+1}) < d(gx_*, gu_n)$, and thus $c_n = \frac{d(gx_*,gu_{n+1})+d(gx_*,gu_n)}{2} < d(gx_*, gu_n)$. This is a contradiction. In addition, if $c_n = \frac{d(gy_*,gv_{n+1})+d(gy_*,gv_n)}{2} > 0$, one can also show that there is a contradiction. Thus we have

$$\max\{d(gx_*, gu_{n+1}), d(gy_*, gv_{n+1})\} \le \phi[\max\{d(gx_*, gu_n), d(gy_*, gv_n)\}].$$

Then, it follows that

$$\begin{aligned}&\max\{d(gx_*, gu_{n+1}), d(gy_*, gv_{n+1})\}\\ &\le \phi^{n+1}[\max\{d(gx_*, gu_0), d(gy_*, gv_0)\}].\end{aligned}$$

Analogously to the above proof, one can also show that

$$\max\{d(gx^*, gu_{n+1}), d(gy^*, gv_{n+1})\} \le \phi^{n+1}[\max\{d(gx^*, gu_0), d(gy^*, gv_0)\}].$$

Letting $n \to \infty$, we get $gx_* = \lim_{n\to\infty} gu_{n+1} = gx^*$, $gy_* = \lim_{n\to\infty} gv_{n+1} = gy^*$.

Since $F(gx_*, gy_*) = g(gx_*)$ and $F(gy_*, gx_*) = g(gy_*)$, (gx_*, gy_*) is a coupled coincidence point of F and g, thus, by (2.11), we have $g(gx_*) = gx_*$, $g(gy_*) = gy_*$. Let $a = gx_*$ and $b = gy_*$. Then $F(a, b) = F(gx_*, gy_*) = g(gx_*) = ga = a$ and $F(b, a) = F(gy_*, gx_*) = g(gy_*) = gb = b$. It remains to show the uniqueness. Let $(c, d) \in X \times X$ such that $F(c, d) = gc = c$ and $F(d, c) = gd = d$. Since (a, b) and (c, d) are both coupled coincidence points of F and g, by (2.11), we get $ga = gc$, $gb = gd$, and thus $a = c$, $b = d$. This completes the proof.

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Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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