Chen and Chen *Fixed Point Theory and Applications* 2012, **2012**:79
http://www.fixedpointtheoryandapplications.com/content/2012/1/79

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Periodic points for the weak contraction mappings in complete generalized metric spaces

Chi-Ming Chen[1*] and Chao-Hung Chen[2]

* Correspondence: ming@mail.nhcue.edu.tw
[1]Department of Applied Mathematics, National Hsinchu University of Education, Hsin-Chu, Taiwan
Full list of author information is available at the end of the article

Abstract

In this article, we introduce the notions of (φ - φ)-weak contraction mappings and (ψ - φ)-weak contraction mappings in complete generalized metric spaces and prove two theorems which assure the existence of a periodic point for these two types of weak contraction.

Mathematical Subject Classification: 47H10; 54C60; 54H25; 55M20.

Keywords: Periodic point, Meir-Keeler function, (φ - φ)-weak contraction mapping, (ψ - φ)-weak contraction mapping

1 Introduction and preliminaries

Let (X, d) be a metric space, D a subset of X and $f : D \to X$ be a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y).$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. In 1969, Boyd and Wong [2] introduced the notion of φ-contraction. A mapping $f : X \to X$ on a metric space is called φ-contraction if there exists an upper semi-continuous function $\varphi : [0, \infty) \to [0, \infty)$ such that

$$d(fx, fy) \leq \phi(d(x, y)) \quad \text{for all} \quad x, y \in X.$$

Generalization of the above Banach contraction principle has been a heavily investigated research branch. (see, e.g., [3,4]).

In 2000, Branciari [5] introduced the following notion of a generalized metric space where the triangle inequality of a metric space had been replaced by an inequality involing three terms instead of two. Later, many authors worked on this interesting space (e.g. [6-11]).

Let (X, d) be a generalized metric space. For $\gamma > 0$ and $x \in X$, we define

$$B_\gamma(x) := \{y \in X | d(x, y) < \gamma\}.$$

Branciari [5] also claimed that $\{B_\gamma(x): \gamma > 0, x \in X\}$ is a basis for a topology on X, d is continuous in each of the coordinates and a generalized metric space is a Hausdorff space. We recall some definitions of a generalized metric space, as follows:

Definition 1 [5]*Let X be a nonempty set and* $d : X \times X \to [0, \infty)$ *be a mapping such that for all* $x, y \in X$ *and for all distinct point* $u, v \in X$ *each of them different from* × *and y, one has*

(i) $d(x, y) = 0$ *if and only if* $\times = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ *(rectangular inequality).*

Then (X, d) *is called a generalized metric space (or shortly g.m.s).*

We present an example to show that not every generalized metric on a set X is a metric on X.

Example 1 *Let* $X = \{t, 2t, 3t, 4t, 5t\}$ *with* $t > 0$ *is a constant, and we define* $d : X \times X \to [0, \infty)$ *by*

(1) $d(x, x) = 0$, *for all* $\times \in X$;
(2) $d(x, y) = d(y, x)$, *for all* $x, y \in X$;
(3) $d(t, 2t) = 3\gamma$;
(4) $d(t, 3t) = d(2t, 3t) = \gamma$;
(5) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2\gamma$;
(6) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = (4t, 5t) = \frac{3}{2}\gamma$,

where $\gamma > 0$ *is a constant. Then* (X, d) *be a generalized metric space, but it is not a metric space, because*

$$d(t, 2t) = 3\gamma > d(t, 3t) + d(3t, 2t) = 2\gamma.$$

Definition 2 [5]*Let* (X, d) *be a g.m.s,* $\{x_n\}$ *be a sequence in X and* $x \in X$. *We say that* $\{x_n\}$ *is g.m.s convergent to* × *if and only if* $d(x_n, x) \to 0$ *as* $n \to \infty$. *We denote by* $x_n \to x$ *as* $n \to \infty$.

Definition 3 [5]*Let* (X, d) *be a g.m.s,* $\{x_n\}$ *be a sequence in X and* $x \in X$. *We say that* $\{x_n\}$ *is g.m.s Cauchy sequence if and only if for each* $\varepsilon > 0$, *there exists* $n_0 \in \mathbb{N}$*such that* $d(x_m, x_n) < \varepsilon$ *for all* $n > m > n_0$.

Definition 4 [5]*Let* (X, d) *be a g.m.s. Then X is called complete g.m.s if every g.m.s Cauchy sequence is g.m.s convergent in X.*

In this article, we also recall the notion of Meir-Keeler function (see [12]). A function $\varphi : [0, \infty) \to [0, \infty)$ is said to be a Meir-Keeler function if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in [0, \infty)$ with $\eta \leq t < \eta + \delta$, we have $\varphi(t) < \eta$. Generalization of the above function has been a heavily investigated research branch. Praticularly, in [13,14], the authors proved the existence and uniqueness of fixed points for various Meir-Keeler type contractive functions. In this study, we introduce the below notions of the weaker Meir-Keeler function $\varphi : [0, \infty) \to [0, \infty)$ and stronger Meir-Keeler function $\psi : [0, \infty) \to [0, 1)$.

Definition 5 *We call* $\varphi : [0, \infty) \to [0, \infty)$ *a weaker Meir-Keeler function if the function* φ *satisfies the following condition*

$$\forall \eta > 0 \quad \exists \delta > 0 \quad \forall t \in [0, \infty) \quad \left(\eta \leq t < \delta + \eta \quad \Rightarrow \quad \exists n_0 \in \mathbb{N} \quad \phi(t)^{n_0} < \eta\right).$$

The following provides an example of a weaker Meir-Keeler function which is not a Meir-Keeler function.

Example 2 *Let* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$*be defined by*

$$\phi(t) = \begin{cases} 0, & \text{if } t \le 1, \\ 3t, & \text{if } 1 < t < 3, \\ 1, & \text{if } t \ge 3. \end{cases}$$

Then φ *is a weaker Meir-Keeler function which is not a Meir-Keeler function.*

Definition 6 *We call* $\psi : [0, \infty) \to [0, 1)$ *a stronger Meir-Keeler function if the function* ψ *satisfies the following condition*

$$\forall \eta > 0 \ \exists \delta > 0 \ \exists \gamma_\eta \in [0,1) \quad \forall t \in [0,\infty) \quad \left(\eta \le t < \delta + \eta \quad \Rightarrow \quad \psi(t) < \gamma_\eta\right).$$

The following provides an example of a stronger Meir-Keeler function.

Example 3 *Let* $\psi : \mathbb{R}^+ \to [0,1)$*be defined by*

$$\psi\left(d\left(x,y\right)\right) = \frac{2t}{3t+1}.$$

Then ψ *is a stronger Meir-Keeler function.*

The following provides an example of a Meir-Keeler function which is not a stronger Meir-Keeler function.

Example 4 *Let* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$*be defined by*

$$\varphi(t) = \begin{cases} t - 1, & \text{if } t > 1; \\ 0, & \text{if } t \le 1. \end{cases}$$

Then φ *is a Meir-Keeler function which is not a stronger Meir-Keeler function.*

2 Main results

In the sequel, we let the function $\varphi : [0, \infty) \to [0, \infty)$ satisfies the following conditions:

(φ_1) $\varphi : [0, \infty) \to [0, \infty)$ is a weaker Meir-Keeler function;

(φ_2) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;

(φ_3) for all $t \in (0, \infty)$, $\{\phi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;

(φ_4) for $t_n \in [0, \infty)$, we have that

(a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \varphi(t_n) < \gamma$, and

(b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \varphi(t_n) = 0$.

Let the function $\psi : [0, \infty) \to [0, 1)$ satisfies the following conditions:

(ψ_1) $\psi : [0, \infty) \to [0, 1)$ is a stronger Meir-Keeler function;

(ψ_2) $\psi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$.

And, we let the function φ : $[0, \infty) \to [0, \infty)$ satisfies the following conditions:

(φ$_1$) for all $t \in (0, \infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty}$ φ$(t_n) = 0$;

(φ$_2$) φ$(t) > 0$ for $t > 0$ and φ$(0) = 0$;

(φ$_3$) φ is subadditive, that is, for every $\mu_1, \mu_2 \in [0, \infty)$, φ$(\mu_1 + \mu_2) \le$ φ(μ_1) + φ(μ_2).

Using the functions φ and φ, we first introduce the notion of the (φ-φ)-weak contraction mapping and prove a theorem which assures the existence of a periodic point for the (φ-φ)-weak contraction mapping.

Definition 7 *Let* (X, d) *be a g.m.s, and let* $f : X \to X$ *be a function satisfying*

$$\varphi\left(d\left(fx, fy\right)\right) \le \phi\left(\varphi\left(d(x, y\right)\right) \qquad (1)$$

for all $x, y \in X$. *Then f is said to be a* $(\varphi$ - $\phi)$*-weak contraction mapping.*

Theorem 1 *Let* (X, d) *be a Hausdorff and complete g.m.s, and let f be a* $(\varphi$ - $\phi)$*-weak contraction mapping. Then f has a periodic point* μ *in X, that is, there exists* $\mu \in X$ *such that* $\mu = f^p\mu$ for some $p \in \mathbb{N}$.

Proof. Given x_0 and define a sequence $\{x_n\}$ in X by

$$x_{n+1} = f x_n \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}.$$

Step 1. We shall prove that

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+1}\right)\right) = 0, \qquad (2)$$

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+2}\right)\right) = 0. \qquad (3)$$

Using the inequality (1), we have that for each $n \in \mathbb{N}$

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+1}\right)\right) &= \varphi\left(d\left(f x_{n-1}, f x_n\right)\right)\\ &\le \phi\left(\varphi\left(d(x_{n-1}, x_n\right)\right),\end{aligned}$$

and so

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+1}\right)\right) &\le \phi\left(\varphi\left(d\left(x_{n-1}, x_n\right)\right)\right)\\ &\le \phi\left(\phi\left(\varphi\left(d(x_{n-2}, x_{n-1}\right)\right)\right) = \phi^2\left(\varphi\left(d(x_{n-2}, x_{n-1}\right)\right)\right)\\ &\le \cdots\cdots\\ &\le \phi^n\left(\varphi\left(d\left(x_0, x_1\right)\right)\right).\end{aligned}$$

Since $\{\phi^n(\varphi(d(x_0, x_1)))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler function φ, corresponding to η use, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \le \phi(d(x_0, x_1)) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\phi^{n_0}(\varphi(d(x_0, x_1))) < \eta$. Since $\lim_{n\to\infty} \varphi^n(\phi(d(x_0, x_1))) = \eta$, there exists $p_0 \in \mathbb{N}$ such that $\eta \le \varphi^p(\phi(d(x_0, x_1))) < \delta + \eta$, for all $p \ge p_0$. Thus, we conclude that $\phi^{p_0+n_0}(\varphi(d(x_0, x_1))) < \eta$. So we get a contradiction. Therefore $\lim_{n\to\infty} \phi^n(\phi(d(x_0, x_1))) = 0$, that is,

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+1}\right)\right) = 0.$$

Using the inequality (1), we also have that for each $n \in \mathbb{N}$

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+2}\right)\right) &= \varphi\left(d\left(f x_{n-1}, f x_{n+1}\right)\right)\\ &\le \phi\left(\varphi\left(d(x_{n-1}, x_{n+1}\right)\right),\end{aligned}$$

and so

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+2}\right)\right) &\le \phi\left(\varphi\left(d\left(x_{n-1}, x_{n+1}\right)\right)\right)\\ &\le \phi\left(\phi\left(\varphi\left(d(x_{n-2}, x_n\right)\right)\right) = \phi^2\left(\varphi\left(d(x_{n-2}, x_n\right)\right)\right)\\ &\le \cdots\cdots\\ &\le \phi^n\left(\varphi\left(d\left(x_0, x_1\right)\right)\right).\end{aligned}$$

Since $\{\varphi^n(d(x_0, x_2))\}_{n\in\mathbb{N}}$ is decreasing, by the same proof process, we also conclude

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+2}\right)\right) = 0.$$

Next, we claim that $\{x_n\}$ is *g.m.s* Cauchy. We claim that the following result holds:

Step 2. Claim that $\lim_{n\to\infty}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) = 0$, that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$ then $\varphi(d(x_p, x_q)) < \varepsilon$.

Suppose the above statement is false. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$\varphi\left(d\left(x_{q_n}, x_{p_n}\right)\right) \geq \varepsilon.$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that it the smallest integer with $p_n > q_n \geq n$ and $\varphi\left(d\left(x_{q_n}, x_{p_n}\right)\right) \geq \varepsilon$. Therefore $\varphi\left(d\left(x_{q_n}, x_{p_n-1}\right)\right) < \varepsilon$. By the rectangular inequality and (2), (3), we have

$$\begin{aligned}\varepsilon &\leq \varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right)\\ &\leq \varphi\left(d(x_{p_n}, x_{p_n-2}) + d\left(x_{p_n-2}, x_{p_n-1}\right) + d\left(x_{p_n-1}, x_{q_n}\right)\right)\\ &\leq \varphi\left(d\left(x_{p_n}, x_{p_n-2}\right)\right) + \varphi\left(d\left(x_{p_n-2}, x_{p_n-1}\right)\right) + \varepsilon.\end{aligned}$$

Letting $n \to \infty$. Then we get

$$\lim_{n\to\infty} \varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) = \varepsilon.$$

On the other hand, we have

$$\begin{aligned}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) &\leq \varphi\left(d\left(x_{p_n}, x_{p_{n-1}}\right) + d\left(x_{p_{n-1}}, x_{q_{n-1}}\right) + d\left(x_{q_{n-1}}, x_{q_n}\right)\right)\\ &\leq \varphi\left(d\left(x_{p_n}, x_{p_n-1}\right)\right) + \varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right) + \varphi\left(d\left(x_{q_n-1}, x_{q_n}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right) &\leq \varphi\left(d\left(x_{p_n-1}, x_{p_n}\right) + d\left(x_{p_n}, x_{q_n}\right) + d\left(x_{q_n}, x_{q_n-1}\right)\right)\\ &\leq \varphi\left(d\left(x_{p_n-1}, x_{p_n}\right)\right) + \varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) + \varphi\left(d\left(x_{q_n}, x_{q_n-1}\right)\right).\end{aligned}$$

Letting $n \to \infty$. Then we get

$$\lim_{n\to\infty} \varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right) = \varepsilon.$$

Using the inequality (1), we have

$$\begin{aligned}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) &= \varphi\left(d\left(f x_{p_n-1}, f x_{q_n-1}\right)\right)\\ &\leq \phi\left(\varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right)\right),\end{aligned}$$

Letting $n \to \infty$, by the definitions of the functions φ and ϕ, we have

$$\varepsilon \leq \lim_{n\to\infty} \phi\left(\varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right)\right) < \varepsilon.$$

So we get a contradiction. Therefore $\lim_{n\to\infty}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) = 0$, by the condition (ϕ_1), we have $\lim_{n\to\infty} d\left(x_{p_n}, x_{q_n}\right) = 0$. Therefore $\{x_n\}$ is *g.m.s* Cauchy.

Step 3. We claim that f has a periodic point in X.

Suppose, on contrary, f has no periodic point. Then $\{x_n\}$ is a sequence of distinct points, that is, $x_p \neq x_q$ for all $p, q \in \mathbb{N}$ with $p \neq q$. By step 2, since X is complete *g.m.s*, there exists $v \in X$ such that $x_n \to v$. Using the inequality (1), we have

$$\varphi\left(d\left(fx_n, fv\right)\right) \le \phi\left(\varphi\left(d\left(x_n, v\right)\right)\right)$$

Letting $n \to \infty$, we have

$$\varphi\left(d\left(fx_n, fv\right)\right) \to 0, \quad \text{as} \quad n \to \infty,$$

by the condition (φ_1), we get

$$d\left(fx_n, fv\right) \to 0, \quad \text{as} \quad n \to \infty,$$

that is,

$$x_{n+1} = fx_n \to fv, \quad \text{as} \quad n \to \infty.$$

As (X, d) is Hausdorff, we have $v = fv$, a contradiction with our assumption that f has no periodic point. Therefore, there exists $v \in X$ such that$v = f^p(v)$ for some $p \in \mathbb{N}$. So f has a periodic point in X. □

Using the functions ψ and φ, we next introduce the notion of the (ψ-φ)-weak contraction mapping and prove a theorem which assures the existence of a periodic point for the (ψ-φ)-weak contraction mapping.

Definition 8 *Let (X, d) be a g.m.s, and let $f : X \to X$ be a function satisfying*

$$\varphi\left(d\left(fx, fy\right)\right) \le \psi\left(\varphi\left(d(x, y)\right)\right) \cdot \varphi(d(x, y) \tag{4}$$

for all $x, y \in X$. Then f is said to be a (ψ - φ)-weak contraction mapping.

Theorem 2 *Let (X, d) be a Hausdorff and complete g.m.s, and let f be a (ψ - φ)-weak contraction mapping. Then f has a periodic point μ in X.*

Proof. Given x_0 and define a sequence $\{x_n\}$ in X by

$$x_{n+1} = fx_n \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}.$$

Step 1. We shall prove that

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+1}\right)\right) = 0, \tag{5}$$

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+2}\right)\right) = 0. \tag{6}$$

Taking into account (4) and the definition of stronger Meir-Keeler function ψ, we have that for each $n \in \mathbb{N}$

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+1}\right)\right) &= \varphi\left(d\left(fx_{n-1}, fx_n\right)\right)\\ &\le \psi\left(\varphi\left(d(x_{n-1}, x_n)\right)\right) \cdot \varphi\left(d(x_{n-1}, x_n\right)\\ &< \varphi\left(d(x_{n-1}, x_n\right).\end{aligned}$$

Thus the sequence $\{\varphi(d(x_n, x_{n+1}))\}$ is descreasing and bounded below and hence it is con-vergent. Let $\lim_{n \to \infty} \varphi(d(x_n, x_{n+1})) = \eta \ge 0$. Then there exists $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \in \mathbb{N}$ with $n \ge n_0$

$$\eta \le \varphi\left(d\left(x_n, x_{n+1}\right)\right) < \eta + \delta. \tag{7}$$

Taking into account (7) and the definition of stronger Meir-Keeler function ψ, corresponding to η use, there exists $\gamma_\eta \in [0, 1)$ such that

$$\psi\left(\varphi\left(d\left(x_n, x_{n+1}\right)\right)\right) < \gamma_n \quad \text{for all} \quad n \ge n_0.$$

Thus, we can deduce that for each $n \in \mathbb{N}$ with $n \geq n_0 + 1$

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+1}\right)\right) &= \varphi\left(d\left(fx_{n-1}, fx_n\right)\right)\\ &\leq \psi\left(\varphi\left(d(x_{n-1}, x_n)\right)\right) \cdot \varphi\left(d(x_{n-1}, x_n\right)\\ &< \gamma_\eta \cdot \varphi\left(d\left(x_{n-1}, x_n\right)\right),\end{aligned}$$

and so

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+1}\right)\right) &\leq \gamma_\eta \cdot \varphi\left(d\left(x_{n-1}, x_n\right)\right)\\ &\leq \gamma_\eta^2 \cdot \varphi\left(d\left(x_{n-2}, x_{n_0-1}\right)\right)\\ &\leq \cdots\\ &\leq \gamma_\eta^{n-n_0} \cdot \varphi\left(d\left(x_{n_0}, x_{n_0+1}\right)\right).\end{aligned}$$

Since $\gamma_\eta \in [0, 1)$, we get

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+1}\right)\right) = 0.$$

Taking into account (4) and the definition of stronger Meir-Keeler function ψ, we have that for each $n \in \mathbb{N}$

$$\begin{aligned}\varphi\left(d\left(x_n, x_{n+2}\right)\right) &= \varphi\left(d\left(fx_{n-1}, fx_{n+1}\right)\right)\\ &\leq \psi\left(\varphi\left(d(x_{n-1}, x_{n+1})\right)\right) \cdot \varphi\left(d(x_{n-1}, x_{n+1}\right)\\ &< \varphi\left(d(x_{n-1}, x_{n+1}\right).\end{aligned}$$

Thus the sequence $\{\varphi(d(x_n, x_{n+2}))\}$ is descreasing and bounded below and hence it is convergent. By the same proof process, we also conclude

$$\lim_{n\to\infty} \varphi\left(d\left(x_n, x_{n+2}\right)\right) = 0.$$

Next, we claim that $\{x_n\}$ is *g.m.s* Cauchy.

Step 2. Claim that $\lim_{n\to\infty}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) = 0$, that is, for every $\varepsilon > 0$, corresponding to above n_0 use, there exists $n \in \mathbb{N}$ with $n \geq n_0 + 1$ such that if $p, q \geq n$ then $\varphi(d(x_p, x_q)) < \varepsilon$.

Suppose the above statement is false. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n \geq n_0 + 1$ satisfying

$$\varphi\left(d\left(x_{q_n}, x_{p_n}\right)\right) \geq \varepsilon.$$

Following from Theorem 1, we have that

$$\lim_{n\to\infty} \varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) = \varepsilon.$$

and

$$\lim_{n\to\infty} \varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right) = \varepsilon.$$

Using the inequality (4), we have

$$\begin{aligned}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) &= \varphi\left(d\left(fx_{p_n-1}, fx_{q_n-1}\right)\right))\\ &\leq \psi\left(\varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right)\right) \cdot \varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right)\\ &< \gamma_\eta \cdot \varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right),\end{aligned}$$

Letting $n \to \infty$, by the definitions of the functions ψ and φ, we have

$$\varepsilon < \lim_{n\to\infty} \gamma_\eta \cdot \varphi\left(d\left(x_{p_n-1}, x_{q_n-1}\right)\right) < \gamma_\eta \cdot \varepsilon < \varepsilon.$$

So we get a contradiction. Therefore $\lim_{n\to\infty}\varphi\left(d\left(x_{p_n}, x_{q_n}\right)\right) = 0$, by the condition ($\varphi_1$), we have $\lim_{n\to\infty} d\left(x_{p_n}, x_{q_n}\right) = 0$. Therefore $\{x_n\}$ is *g.m.s* Cauchy.

Step 3. We claim that *f* has a periodic point in *X*.

Suppose, on contrary, *f* has no periodic point. Then $\{x_n\}$ is a sequence of distinct points, that is, $x_p \neq x_q$ for all $p, q \in \mathbb{N}$ with $p \neq q$. By step 2, since *X* is complete *g.m.s*, there exists $v \in X$ such that $x_n \to v$. Using the inequality (4), we have

$$\varphi\left(d\left(fx_n, fv\right)\right) \leq \psi\left(\varphi\left(d\left(x_n, v\right)\right)\right) \cdot \varphi\left(d\left(x_n, v\right)\right)$$

Letting $n \to \infty$, we have

$$\varphi\left(d\left(fx_n, fv\right)\right) \to 0, \quad \text{as} \quad n \to \infty,$$

by the condition (φ_1), we get

$$d\left(fx_n, fv\right) \to 0, \quad \text{as} \quad n \to \infty,$$

that is,

$$x_{n+1} = fx_n \to fv, \quad \text{as} \quad n \to \infty.$$

As (*X*, *d*) is Hausdorff, we have $v = fv$, a contradiction with our assumption that *f* has no periodic point. Therefore, there exists $v \in X$ such that $v = f^p(v)$ for some $p \in \mathbb{N}$. So *f* has a periodic point in *X*. □

In conclusion, by using the new concepts of (φ-φ)-weak contraction mappings and (ψ - φ)-weak contraction mappings, we obtain two theorems (Theorems 1 and 2) which assure the existence of a periodic point for these two types of weak contraction in complete generalized metric spaces. Our results generalize or improve many recent fixed point theorems in the literature.

Acknowledgements
The authors would like to thank referee(s) for many useful comments and suggestions for the improvement of the paper.

Author details
[1]Department of Applied Mathematics, National Hsinchu University of Education, Hsin-Chu, Taiwan [2]Department of Applied Mathematics, Chung Yuan Christian University, Chungli City, Taiwan

Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 25 December 2011 Accepted: 9 May 2012 Published: 9 May 2012

References

1. Banach, S: Sur les operations dans les ensembles abstraits et leur application aux equations integerales. Fund Math. **3**, 133–181 (1922)
2. Boyd, DW, Wong, SW: On nonlinear contractions. Proc Am Math Soc. **20**, 45864 (1969)
3. Aydi, H, Karapinar, E, Shatnawi, W: Coupled fixed point results for ((ψ - φ)-weakly contractive condition in ordered partial metric spaces. Comput Math Appl. **62**(12):4449–4460 (2011). doi:10.1016/j.camwa.2011.10.021
4. Karapinar, E: Weak φ-contraction on partial metric spaces and existence of fixed points in partially ordered sets. Math Aeterna. **1**(4):237–244 (2011)

5. Branciari, A: A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces. Publ Math Debrecen. **57**, 31–37 (2000)
6. Azam, A, Arshad, M: Kannan fixed point theorem on generalized metric spaces. J Nonlinear Sci Appl. **1**(1):45–48 (2008)
7. Das, P: A fixed point theorem on a class of generalized metric spaces. Korea J Math Sci. **9**, 29–33 (2002)
8. Mihet, D: On Kannan fixed point principle in generalized metric spaces. J Nonlinear Sci Appl. **2**(2):92–96 (2009)
9. Samet, B: A fixed point theorem in a generalized metric space for mappings satisfying a contractive condition of integral type. Int J Math Anal. **26**(3):1265–1271 (2009)
10. Samet, B: Disscussion on: a fixed point theorem of Banach-Caccioppli type on a class of generalized metric spaces. Publ Math Debrecen. **76**(4):493–494 (2010)
11. Lakzian, H, Samet, B: Fixed points for (ψ, φ)-weakly contractive mappings in general-ized metric spaces. Appl Math Lett.**25**(5):902–906
12. Meir, A, Keeler, E: A theorem on contraction mappings. J Math Anal Appl. **28**, 326–329 (1969). doi:10.1016/0022-247X(69)90031-6
13. Anthony Eldred, A, Veeramani, P: Existence and convergence of best proximity points. J Math Anal Appl. **323**, 1001–1006 (2006). doi:10.1016/j.jmaa.2005.10.081
14. De la Sen, M: Linking contractive self-mappings and cyclic Meir-Keeler contractions with Kannan self-mappings. Fixed Point Theory Appl **2010**, 23 (2010). Article ID 572057 doi:10.1155/2010/572057

doi:10.1186/1687-1812-2012-79
Cite this article as: Chen and Chen: **Periodic points for the weak contraction mappings in complete generalized metric spaces.** *Fixed Point Theory and Applications* 2012 **2012**:79.