Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theory for cyclic generalized contractions in partial metric spaces

Ravi P Agarwal[1,3], Maryam A Alghamdi[2] and Naseer Shahzad[3*]

* Correspondence: nshahzad@kau.edu.sa
[3]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

In this article, we give some fixed point theorems for mappings satisfying cyclical generalized contractive conditions in complete partial metric spaces.

1 Introduction

The well known Banach's fixed point theorem asserts that: If (X, d) is a complete metric space and $f : X \to X$ is a mapping such that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$ and some $\lambda \in [0,1)$, then f has a unique fixed point in X. Kannan [1] extended Banach's fixed point theorem to the class of maps $f : X \to X$ satisfying the following contractive condition:

$$d(f(x), f(y)) \leq \lambda[d(x, f(x)) + d(y, f(y))]$$

for all $x, y \in X$ and some $\lambda \in (0,1/2)$. Reich [2] generalized both results using the contractive condition:

$$d(f(x), f(y)) \leq \alpha d(x, y) + \beta d(x, f(x)) + \gamma d(y, f(y))$$

for each $x, y \in X$, where α, β, γ are nonnegative real numbers statisfying $\alpha + \beta + \gamma < 1$.

Matkowski [3] used the following contractive condition:

$$d(f(x), f(y)) \leq \varphi(d(x, y))$$

for all $x, y \in X$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$.

In 1994, Matthews [4] introduced the notion of a partial metric space and obtained a generalization of Banach's fixed point theorem for partial metric spaces. Recently, Altun et al. [5] (see also Altun and Sadarangani [6]) gave some generalized versions of the fixed point theorem of Matthews [4]. Di Bari and Vetro [7] obtained some results concerning cyclic mappings in the framework of partial metric spaces. We recall below the definition of partial metric space and some of its properties (see [4,5,8,9]).

Definition 1 *A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z, \in X$:*

p_1 $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
p_2 $p(x, x) \le p(x, y)$,
p_3 $p(x, y) = p(y, x)$,
p_4 $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) where X is a nonempty set and p is a partial metric on X. The function $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}_+$ defines a partial metric on $\mathbb{R}_+$. Other interesting examples of partial metric spaces can be found in [4,10,11]. It is known [8] that each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}_+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

defines a metric on X (see [12]).

Let (X, p) be a partial metric space.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to a point $x \in X$ [4,5,8] if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence [4,5,8] if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete [4,5,8] if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

It is evident that every closed subset of a complete partial metric space is complete.

Lemma 2 [4,5,8]*Let (X, p) be a partial metric space.*

(1)
$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(2)
A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Definition 3 [13]*Let X be a nonempty set, m a positive integer and $f : X \to X$ an operator. By definition, $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to f if*

(i) X_i, $i = 1,\ldots, m$ are nonempty sets;

(ii) $f(X_1) \subset X_2, \ldots, f(X_{m-1}) \subset X_m, f(X_m) \subset X_1$.

Definition 4 [13]*A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies:*

(i) ϕ is monotone increasing, i.e., $t_1 \le t_2$ implies $\phi(t_1) \le \phi(t_2)$, for any $t_1, t_2 \in \mathbb{R}_+$;

(ii) $(\phi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$ for all $t \in \mathbb{R}_+$.

Definition 5 [13]*A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a (c)-comparison function if it satisfies:*

(i) ϕ is monotone increasing;

(ii) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\varphi^{k+1}(t) \leq \alpha\varphi^k(t) + v_k,$$

for $k \geq k_0$ and any $t \in \mathbb{R}_+$.

Lemma 6 [13]*If $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a (c)-comparison function, then the following hold:*

(i) ϕ is a comparison function;

(ii) $\phi(t) < t$, for any $t \in \mathbb{R}_+$;

(iii) ϕ is continuous at 0;

(iv) the series $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for any $t \in \mathbb{R}_+$.

In [13], Păcurar and Rus discussed fixed point theorey for cyclic ϕ- contractions in metric spaces and in [14], Karapinar obtained a fixed point theorem for cyclic weak ϕ-contraction mappings still in metric spaces.

In this article, we prove some fixed point theorems for generalized contractions defined on cyclic representation in the setting of partial metric spaces.

2 Main results

Definition 7 *Let (X,p) be a partial metric space. A mapping $f : X \to X$ is called a ϕ-contraction if there exists a comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$p(f(x), f(y)) \leq \varphi(p(x, y))$$

for all $x, y \in X$.

Definition 8 *Let (X, p) be a partial metric space, m a positive integer, $A_1,\ldots, A_m$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. An operator $f : Y \to Y$ is called a cyclic ϕ-contraction if*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y w.r.t f;

(ii) There exists a (c)-comparison function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$p(f(x), f(y)) \leq \varphi(p(x, y)) \tag{2.1}$$

for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$.

Theorem 9 *Let (X, p) be a complete partial metric space, m a positive integer, $A_1,\ldots, A_m$ closed nonempty subsets of X, $Y = \bigcup_{i=1}^{m} A_i$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$a (c)-comparison function and $f :Y \to Y$ an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y w.r.t f;

(ii) f is a cyclic ϕ-contraction.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $\{x_n\}$ converges to x^* for any initial point $x_0 \in Y$.

Proof. Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$, and set

$$x_n = f(x_{n-1}), \quad n \geq 1.$$

For any $n \geq 0$ there is $i_n \in \{i, ..., m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Then by (2.1) we have

$$p(x_n, x_{n+1}) = p(f(x_{n-1}), f(x_n)) \leq \varphi(p(x_{n-1}, x_n)).$$

Since ϕ is monotone increasing, we get by induction that

$$p(x_n, x_{n+1}) \leq \varphi^n(p(x_0, x_1)). \tag{2.2}$$

By definition of ϕ, thus letting $n \to \infty$ in (2.2), we obtain that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

On the other hand, since

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \text{ and } p(x_{n+1}, x_{n+1}) \leq p(x_n, x_{n+1}),$$

then from (2.2) we have

$$p(x_n, x_n) \leq \varphi^n(p(x_0, x_1)) \text{ and } p(x_{n+1}, x_{n+1}) \leq \varphi^n(p(x_0, x_1)). \tag{2.3}$$

Thus, we have

$$p^s(x_n, x_{n+1}) \quad \leq \quad 4\varphi^n(p(x_0, x_1)).$$

Since ϕ is a (c)-comparison function, from Lemma 6, it follows that

$$\lim_{n\to\infty} p^s(x_n, x_{n+1}) = 0.$$

So for $k \geq 1$, we have

$$\begin{aligned} p^s(x_n, x_{n+k}) &\leq p^s(x_n, x_{n+1}) + \cdots + p^s(x_{n+k-1}, x_{n+k}) \\ &\leq 4 \sum_{m=n}^{n+k-1} \varphi^m(p(x_0, x_1)). \end{aligned}$$

Again since ϕ is a (c)-comparison function, by Lemma 6, it follows that

$$\sum_{m=0}^{\infty} \varphi^m(p(x_0, x_1)) < \infty.$$

This implies that $\{x_n\}$ is a Cauchy sequence in the metric subspace (Y, p^s). Since Y is closed, the subspace (Y, p) is complete. Then from Lemma 2, we have that (Y, p^s) is complete. Let

$$\lim_{n\to\infty} p^s(x_n, y) = 0.$$

Now Lemma 2 further implies that

$$p(y, y) = \lim_{n\to\infty} p(x_n, y) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.4}$$

Therefore, since $\{x_n\}$ is a Cauchy sequence in the metric space (Y, p^s), it implies that $\lim_{n,m\to\infty} p^s(x_n, x_m) = 0$. Also from (2.3) we have $\lim_{n\to\infty} p(x_n, x_n) = 0$, and using the definition of p^s we obtain $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Consequently, from (2.4) we have

$$p(y, y) = \lim_{n\to\infty} p(x_n, y) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

As a result, $\{x_n\}$ is a Cauchy sequence in the complete partial metric subspace (Y, p), and it is convergent to a point $y \in Y$.

On the other hand, the sequence $\{x_n\}$ has an infinite number of terms in each A_i, $i = 1,...,m$. Since (Y, p) is complete, in each A_i, $i = 1,..., m$, we can construct a subsequence of $\{x_n\}$ which converges to y. Since A_i, $i = 1,..., m$ are closed, we see that

$$y \in \bigcap_{i=1}^{m} A_i; \text{ i.e.,}$$

$\bigcap_{i=1}^{m} A_i \neq \emptyset$. Now we can consider the restriction

$$f|_{\bigcap_{i=1}^{m} A_i} : \quad \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i,$$

which satisfies the conditions of Theorem 1 in [5,6], since $\bigcap_{i=1}^{m} A_i$ is also closed and complete. Thus $f|_{\bigcap_{i=1}^{m} A_i}$ has a unique fixed point, say $x^* \in \bigcap_{i=1}^{m} A_i$. We claim that for any initial value $x \in Y$, we get the same limit point $x^* \in \bigcap_{i=1}^{m} A_i$. Indeed, for $x \in Y = \bigcup_{i=1}^{m} A_i$, by repeating the above process, the corresponding iterative sequence yields that $f|_{\bigcap_{i=1}^{m} A_i}$ has a unique fixed point, say $z \in \bigcap_{i=1}^{m} A_i$. Regarding that $x^*, z \in \bigcap_{i=1}^{m} A_i$, we have $x^*\ z \in A_i$ for all i, hence $p(x^*, z)$ and $p(f(x^*), f(z))$ are well defined. Due to (2.1), we have

$$p(x^*, z) = p(f(x^*), f(z)) \leq \varphi(p(x^*, z)),$$

which is a contradiction. Thus, x^* is a unique fixed point of f for any initial value $x \in Y$.

To prove that the Picard iteration converges to x^* for any initial point $x \in Y$. Let $x \in Y = \bigcup_{i=1}^{m} A_i$. There exists $i_0 \in \{1,..., m\}$ such that $x \in A_{i_0}$. As $x^* \in \bigcap_{i=1}^{m} A_i$ it follows that $x^* \in A_{i_0+1}$ as well. Then we obtain:

$$p(f(x), f(x^*)) \leq \varphi(p(x, x^*)).$$

By induction, it follows that:

$$p(f^n(x), x^*) \leq \varphi^n(p(x, x^*)), \quad n \geq 0.$$

Since

$$p(x^*, x^*) \leq p(f^n(x), x^*),$$

we have

$$p(x^*, x^*) \leq \varphi^n(p(x, x^*)).$$

Now letting $n \to \infty$, and supposing $x \neq x^*$, we have

$$p(x^*, x^*) = \lim_{n\to\infty} p(f^n(x), x^*) = 0,$$

i.e., the Picard iteration converges to the unique fixed point of f for any initial point $x \in Y$.

Theorem 10 *Let $f: Y \to Y$ as in Theorem 9. Then*

$$\sum_{n=0}^{\infty} p(f^n(x), f^{n+1}(x)) < \infty,$$

for any $x \in Y$, i.e., f is a good Picard operator.

Proof. Let $x = x_0 \in Y$. Then

$$p(f^n(x_0), f^{n+1}(x_0)) = p(x_n, x_{n+1}) \leq \varphi^n(p(x_0, x_1)).$$

for all $n \in \mathbb{N}$ Thus, by Lemma 6, we have

$$\sum_{n=0}^{\infty} p(f^n(x_0), f^{n+1}(x_0)) \leq \sum_{n=0}^{\infty} \varphi^n(p(x_0, x_1)) < \infty,$$

since $p(x_0, x_1) > 0$. So, f is a good Picard operator.

Theorem 11 *Let $f: Y \to Y$ as in Theorem 9. Then*

$$\sum_{n=0}^{\infty} p(f^n(x), x^*) < \infty,$$

for any $x \in Y$, i.e., f is a special Picard operator.

Proof. Since

$$p(f^n(x), x^*) \leq \varphi^n(p(x, x^*)), \quad n \geq 0$$

holds for any $x \in Y$, by Lemma 6, we have

$$\sum_{n=0}^{\infty} p(f^n(x), x^*) \leq \sum_{n=0}^{\infty} \varphi^n(p(x, x^*)) < \infty.$$

This shows that f is a special Picard operator.

Theorem 12 (*Reich type*). *Let (X, p) be a complete partial metric space, m a positive integer, $A_1,\ldots,A_m$ closed nonempty subsets of X, $Y = \bigcup_{i=1}^{m} A_i$, and $f: Y \to Y$ an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y w.r.t f;

(ii) for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$, *we* have

$$p(f(x), f(y)) \leq \alpha p(x, y) + \beta p(x, f(x)) + \gamma p(y, f(y)), \tag{2.5}$$

where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$ and the Picard iteration $\{x_n\}$ converges to x^* for any initial point $x_0 \in Y$ if $\alpha + 2\beta + 2\gamma < 1$.

Proof. Let $x_0 \in Y = \bigcup_{i=1}^{m} A_i$, and set

$$x_n = f(x_{n-1}), \quad n \geq 1.$$

For any $n \geq 0$ there is $i_n \in \{i,..., m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_{n+1}}$. Then by (2.5) we have

$$\begin{aligned} p(x_n, x_{n+1}) &= p(f(x_{n-1}), f(x_n)) \\ &\leq \alpha p(x_{n-1}, x_n) + \beta p(x_{n-1}, f(x_{n-1})) + \gamma p(x_n, f(x_n)) \\ &= \alpha p(x_{n-1}, x_n) + \beta p(x_{n-1}, x_n) + \gamma p(x_n, x_{n+1}) \\ &= (\alpha + \beta) p(x_{n-1}, x_n) + \gamma p(x_n, x_{n+1}), \end{aligned}$$

which implies

$$p(x_n, x_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma} p(x_{n-1}, x_n).$$

Therefore,

$$p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1), \tag{2.6}$$

where

$$\lambda = \frac{\alpha + \beta}{1 - \gamma}.$$

It is clear that $\lambda \in [0,1)$, thus letting $n \to \infty$ in (2.6), we obtain that

$$\lim_{n \to \infty} p(x_n, x_{n+1}) = 0.$$

On the other hand, since

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \text{ and } p(x_{n+1}, x_{n+1}) \leq p(x_n, x_{n+1}),$$

from (2.6) we have

$$p(x_n, x_n) \leq \lambda^n p(x_0, x_1) \text{ and } p(x_{n+1}, x_{n+1}) \leq \lambda^n p(x_0, x_1). \tag{2.7}$$

Hence,

$$p^s(x_n, x_{n+1}) \leq 4\lambda^n p(x_0, x_1).$$

This implies that

$$\lim_{n \to \infty} p^s(x_n, x_{n+1}) = 0.$$

Now, for $k \geq 1$, we have

$$\begin{aligned} p^s(x_n, x_{n+k}) &\leq p^s(x_n, x_{n+1}) + \cdots + p^s(x_{n+k-1}, x_{n+k}) \\ &\leq 4\lambda^n p(x_0, x_1) + \cdots + 4\lambda^{n+k-1} p(x_0, x_1) \\ &\leq 4 \frac{\lambda^n}{1 - \lambda} p(x_0, x_1). \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence in the metric subspace (Y, p^s). Since Y is closed, the subspace (Y, p) is complete and so from Lemma 2, we have that (Y, p^s) is complete. So the sequence $\{x_n\}$ is convergent in the metric subspace (Y, p^s). Let

$$\lim_{n\to\infty} p^s(x_n, y) = 0.$$

Again from Lemma 2, we get

$$p(y, y) = \lim_{n\to\infty} p(x_n, y) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.8}$$

As in the proof of Theorem 9, from (2.8) we have

$$p(y, y) = \lim_{n\to\infty} p(x_n, y) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

This shows that $\{x_n\}$ is a Cauchy sequence in the complete partial metric subspace (Y, p), and it is convergent to a point $y \in Y$.

On the other hand, the sequence $\{x_n\}$ has an infinite number of terms in each A_i, $i = 1,\ldots,m$. Since (Y, p) is complete, in each A_i, $i = 1,\ldots, m$, we can construct a subsequence of $\{x_n\}$ which converges to y. Since each A_i, $i = 1,\ldots, m$ is closed, it follows that

$$y \in \bigcap_{i=1}^{m} A_i;\ i.e.,$$

$\bigcap_{i=1}^{m} A_i \neq \emptyset$. Now we can consider the restriction

$$f|_{\bigcap_{i=1}^{m} A_i} : \quad \bigcap_{i=1}^{m} A_i \to \bigcap_{i=1}^{m} A_i,$$

which satisfies the conditions of Corollary 4 in [5], as $\bigcap_{i=1}^{m} A_i$ is also closed and complete. Thus, $f|_{\bigcap_{i=1}^{m} A_i}$ has a unique fixed point, say $x^* \in \bigcap_{i=1}^{m} A_i$. We claim that for any initial value $x \in Y$, we get the same limit point $x^* \in \bigcap_{i=1}^{m} A_i$. In fact, for $x \in Y = \bigcup_{i=1}^{m} A_i$, by repeating the above process, the corresponding iterative sequence yields that $f|_{\bigcap_{i=1}^{m} A_i}$ has a unique fixed point, say $z \in \bigcap_{i=1}^{m} A_i$. Since $x^*, z \in \bigcap_{i=1}^{m} A_i$, we have $x^*, z \in A_i$ for all i, hence $p(x^*,z)$, and $p(f(x^*), f(z))$ are well defined. Due to (2.5),

$$\begin{aligned} p(x^*, z) &= p(f(x^*), f(z)) \\ &\leq \alpha p(x^*, z) + \beta p(x^*, f(x^*)) + \gamma p(z, f(z)) \\ &\leq \alpha p(x^*, z) + \beta p(x^*, z) + \gamma p(x^*, z), \end{aligned}$$

which is a contradiction. Thus, x^* is the unique fixed point of f for any initial value $x \in Y$.

To prove that the Picard iteration converges to x^* for any initial point $x \in Y$. Let $x \in Y = \bigcup_{i=1}^{m} A_i$. There exists $i_0 \in \{1,\ldots, m\}$ such that $x \in A_{i_0}$. As $x^* \in \bigcap_{i=1}^{m} A_i$ it follows that $x^* \in A_{i_0+1}$ as well. Then we obtain:

$$\begin{aligned} p(f(x), f(x^*)) &\leq \alpha p(x, x^*) + \beta p(x, f(x)) + \gamma p(x^*, f(x^*)) \\ &\leq \alpha p(x, x^*) + \beta [p(x, x^*) + p(x^*, f(x)) - p(x^*, x^*)] \\ &\quad + \gamma [p(x^*, f(x)) + p(f(x), f(x^*)) - p(f(x), f(x))] \\ &\leq \alpha p(x, x^*) + \beta [p(x, x^*) + p(x^*, f(x))] \\ &\quad + \gamma [p(x^*, f(x)) + p(f(x), f(x^*))], \end{aligned}$$

which implies

$$p(f(x), f(x^*)) \leq \frac{\alpha + \beta}{1 - \beta - 2\gamma} p(x, x^*).$$

Let

$$\lambda_1 = \frac{\alpha + \beta}{1 - \beta - 2\gamma},$$

and suppose that $\alpha + 2\beta + 2\gamma < 1$. Then, by induction, it follows that:

$$p(f^n(x), x^*) \leq \lambda_1^n p(x, x^*).$$

Since

$$p(x^*, x^*) \leq p(f^n(x), x^*),$$

we have

$$p(x^*, x^*) \leq \lambda_1^n p(x, x^*).$$

Now letting $n \to \infty$, and supposing $x \neq x^*$, we have

$$p(x^*, x^*) = \lim_{n \to \infty} p(f^n(x), x^*) = 0$$

i.e., the Picard iteration converges to the unique fixed point of f for any initial point $x \in Y$ provided $\alpha + 2\beta + 2\gamma < 1$.

Corollary 13 (*Banach type*). *Let* (X, p) *be a complete partial metric space, m a positive integer,* $A_1,\dots, A_m$ *closed nonempty subsets of* $X, Y = \bigcup_{i=1}^{m} A_i$, *and* $f: Y \to Y$ *an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y w.r.t f;
(ii) for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$, we have

$$p(f(x), f(y)) \leq \alpha p(x, y), \quad 0 \leq \alpha < 1.$$

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$.

Corollary 14 (*Kannan type*). *Let* (X, p) *be a complete partial metric space, m a positive integer,* $A_1,\dots, A_m$ *closed nonempty subsets of* $X, Y = \bigcup_{i=1}^{m} A_i$, *and* $f: Y \to Y$ *an operator. Assume that*

(i) $\bigcup_{i=1}^{m} A_i$ is a cyclic representation of Y w.r.t f;
(ii) for any $x \in A_i$, $y \in A_{i+1}$, where $A_{m+1} = A_1$, we have

$$p(f(x), f(y)) \leq \beta p(x, f(x)) + \gamma p(y, f(y)),$$

where $\beta, \gamma \geq 0$ with $\beta + \gamma < \frac{1}{2}$.

Then f has a unique fixed point $x^* \in \bigcap_{i=1}^{m} A_i$.

Theorem 15 *Let $f : Y \to Y$ as in Theorem 12. Then*

$$\sum_{n=0}^{\infty} p(f^n(x), f^{n+1}(x)) < \infty,$$

for any $x \in Y$, i.e., f is a good Picard operator.

Proof. Let $x = x_0 \in Y$. Then, as in the proof of Theorem 12,

$$p(f^n(x_0), f^{n+1}(x_0)) = p(x_n, x_{n+1}) \le \lambda^n p(x_0, x_1)$$

for all $n \in \mathbb{N}$. So, we have

$$\sum_{n=0}^{\infty} p(f^n(x_0), f^{n+1}(x_0)) \le \sum_{n=0}^{\infty} \lambda^n p(x_0, x_1) < \infty,$$

since $\lambda \in [0,1)$. Thus, f is a good Picard operator.

Theorem 16 *Let $f : Y \to Y$ as in Theorem 12. If $\alpha + 2\beta + 2\gamma < 1$, then*

$$\sum_{n=0}^{\infty} p(f^n(x), x^*) < \infty,$$

for any $x \in Y$, i.e., f is a special Picard operator.

Proof. As in the proof of Theorem 12, we have

$$p(f^n(x), x^*) \le \lambda_1^n p(x, x^*)$$

holds for any $x \in Y$, where $\lambda_1 = \dfrac{\alpha + \beta}{1 - \beta - 2\gamma}$. Hence, if $\alpha + 2\beta + 2\gamma < 1$, we have

$$\sum_{n=0}^{\infty} p(f^n(x), x^*) \le \sum_{n=0}^{\infty} \lambda_1^n p(x, x^*) < \infty.$$

This shows that f is a special Picard operator.

Acknowledgements
The second and third authors would like to thank the Deanship of Scientific Research (DSR) at the King Abdulaziz University, Jeddah for supporting this work through research project No. 1432/363/31.

Author details
[1]Department of Mathematics, Texas A&M University-Kingsville 700 University Blvd. Kingsville, TX 78363-8202, USA [2]Department of Mathematics, King Abdulaziz University, Sciences Faculty for Girls, P.O. Box 4087, Jeddah 21491, Saudi Arabia [3]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21859, Saudi Arabia

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 16 September 2011 Accepted: 15 March 2012 Published: 15 March 2012

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doi:10.1186/1687-1812-2012-40
Cite this article as: Agarwal *et al.*: Fixed point theory for cyclic generalized contractions in partial metric spaces. *Fixed Point Theory and Applications* 2012 **2012**:40.