Fixed Point Theory and Applications
a SpringerOpen Journal

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Simple projection algorithm for a countable family of weak relatively nonexpansive mappings and applications

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Abstract

Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}: C \to C$ be a countable family of weak relatively nonexpansive mappings such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in C by the following algorithm:

$$\begin{cases} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, T_n x_n) = \phi(z, x_n)\}, \quad n = 0, 1, 2, 3, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases}$$

Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$.

MSC: 47H05; 47H09; 47H10

Keywords: relatively nonexpansive mapping; weak relatively nonexpansive mapping; generalized projection; projection algorithm; monotone hybrid algorithm

1 Introduction

Let E be a real Banach space with the dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. The duality mapping J has the following properties: (1) if E is smooth, then J is single-valued; (2) if E is strictly convex, then J is one-to-one; (3) if E is reflexive, then J is surjective; (4) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E; (5) if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E and J is singe-valued and also one-to-one (see [1–4]).

Let E be a smooth Banach space with the dual E^*. The functional $\phi : E \times E \to R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,$$

for all $x, y \in E$.

Let C be a closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p in C is said to be an asymptotic fixed point of T [5] if C contains a sequence $\{x_n\}$ which converges weakly to p such that the strong $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and relatively nonexpansive if $F(T) = \widehat{F}(T)$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [1, 6–9].

Three classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one was introduced in 1953 by Mann [10] and is well known as Mann's iteration process defined as follows:

$$\begin{cases} x_0 & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, & n \geq 0, \end{cases} \tag{1.1}$$

where the sequence $\{\alpha_n\}$ is chosen in $[0,1]$. Fourteen years later, Halpern [11] proposed the new innovation iteration process which resembled Mann's iteration (1.1). It is defined by

$$\begin{cases} x_0 & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, & n \geq 0, \end{cases} \tag{1.2}$$

where the element $u \in C$ is fixed. Seven years later, Ishikawa [2] enlarged and improved Mann's iteration (1.1) to the new iteration method, which is often cited as Ishikawa's iteration process and defined recursively by

$$\begin{cases} x_0 & \text{chosen arbitrarily,} \\ y_n = \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Ty_n, & n \geq 0, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$.

In both Hilbert space [11–13] and uniformly smooth Banach space [14–16] the iteration process (1.2) has been proved to be strongly convergent if the sequence $\{\alpha_n\}$ satisfies the following conditions:

(i) $\alpha_n \to 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

By the restriction of condition (ii), it is widely believed that Halpern's iteration process (1.2) has slow convergence though the rate of convergence has not been determined. Halpern [11] proved that conditions (i) and (ii) are necessary in the strong convergence of (1.2) for a nonexpansive mapping T on a closed convex subset C of a Hilbert space H. Moreover, Wittmann [13] showed that (1.2) converges strongly to $P_{F(T)}u$ when $\{\alpha_n\}$ satisfies (i), (ii) and (iii), where $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

Both iteration processes (1.1) and (1.3) have only weak convergence in a general Banach space (see [17] for more details). As a matter of fact, the process (1.1) may fail to converge, while the process (1.3) can still converge for a Lipschitz pseudo-contractive mapping in a Hilbert space [18]. For example, Reich [19] proved that if E is a uniformly convex Banach space with the Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However, we note that Mann's iteration process (1.1) has only weak convergence even in a Hilbert space [17].

Some attempts to modify the Mann iteration method so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [20] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where C is a closed convex subset of H, P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(T)}(x_0)$, where $F(T)$ denotes the fixed point set of T.

The ideas to generalize the process (1.4) from a Hilbert space to a Banach space have recently been made. By using available properties on a uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [9] presented their ideas as the following method for a single relatively nonexpansive mapping T in a Banach space E:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JTx_n), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.5}$$

They proved the following convergence theorem.

Theorem MT *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n < 1$ and $\limsup_{n\to\infty}\alpha_n < 1$. Suppose that $\{x_n\}$ is given by (1.6), where J is the duality mapping on E. If $F(T)$ is nonempty, then $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$, where $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.*

In 2007, Plubtieng and Ungchittrakool [21] proposed the following hybrid algorithms for two relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

Theorem SK1 *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let T, S be two relatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x_0) \end{cases} \tag{1.6}$$

with the following restrictions:

(i) $0 \le \alpha_n < 1$, $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $0 \le \beta_n^{(1)}, \beta_n^{(1)}, \beta_n^{(3)} \le 1$, $\lim_{n\to\infty} \beta_n^{(1)} = 0$, $\liminf_{n\to\infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Theorem SK2 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T, S be two relatively nonexpansive mappings from C into itself with $F := F(T) \cap F(S)$ is nonempty. Let a sequence $\{x_n\}$ be defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ y_n = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)Jz_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx_0\rangle)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n}(x_0) \end{cases} \tag{1.7}$$

with the following restrictions:

(i) $0 < \alpha_n < 1$, $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $0 \le \beta_n^{(1)}, \beta_n^{(1)}, \beta_n^{(3)} \le 1$, $\lim_{n\to\infty} \beta_n^{(1)} = 0$, $\liminf_{n\to\infty} \beta_n^{(2)} \beta_n^{(3)} > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

In 2010, Su, Xu and Zhang [22] proposed the following hybrid algorithms for two countable families of weak relatively nonexpansive mappings in a Banach space and proved the following convergence theorems.

Theorem SKZ *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\}$, $\{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $F := (\bigcap_{n=0}^{\infty} F(T_n)) \cap$*

$(\bigcap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm*:

$$\begin{cases} x_0 \in C \quad \text{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ C_0 = \{z \in C : \phi(z, y_0) \leq \phi(z, x_0)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0) \end{cases} \tag{1.8}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

(iii) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0, 1)$.

Then $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection from* C *onto* F.

Unfortunately, in recent years, many hybrid algorithms have been very complex, so these complex algorithms are not applicable or are very difficult in applications. Naturally, we hope to obtain some simple and practical algorithms. The purpose of this article is to present a simple projection algorithm for a countable family of weak relatively nonexpansive mappings and to prove strong convergence theorems in Banach spaces.

In addition, we shall give an example which is a countable family of weak relatively nonexpansive mappings, but not a countable family of relatively nonexpansive mappings.

2 Preliminaries

Let E be a smooth Banach space with the dual E^*. The functional $\phi : E \times E \to R$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \tag{2.1}$$

for all $x, y \in E$. Observe that in a Hilbert space H, (2.1) reduces to $\phi(x, y) = \|x - y\|^2$, $x, y \in H$.

Recall that if C is a nonempty, closed and convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This is true only when H is a real Hilbert space. In this connection, Alber [23] has recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection in Hilbert spaces. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y \in C} \phi(y, x). \tag{2.2}$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J. In a Hilbert space, $\Pi_C = P_C$. It is obvious from the definition of the functional ϕ that

$$\left(\|x\| - \|y\|\right)^2 \leq \phi(y,x) \leq \left(\|y\|^2 + \|x\|^2\right) \tag{2.3}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) - 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for all $x, y \in E$. See [24] for more details.

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Remark 2.1 If E is a reflexive strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(x,y) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(x,y) = 0$ then $x = y$. From (2.3), we have $\|x\| = \|y\|$. This implies $\langle x, Jy\rangle = \|x\|^2 = \|Jy\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, then we have $x = y$; see [13, 16, 25] for more details.

In this paper, we give the definitions of a countable family of relatively nonexpansive mappings and a countable family of weak relatively nonexpansive mappings which are generalizations of a relatively nonexpansive mapping and a weak relatively nonexpansive mapping respectively. We also give an example which is a countable family of weak relatively nonexpansive mappings, but not a countable family of relatively nonexpansive mappings.

Let C be a closed convex subset of E, and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into itself. We denote by F the set of common fixed points of $\{T_n\}_{n=0}^{\infty}$. That is $F = \bigcap_{n=0}^{\infty} F(T_n)$, where $F(T_n)$ denotes the set of fixed points of T_n for all $n \geq 0$. A point p in C is said to be an *asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|T_n x_n - x_n\| = 0$. The set of asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. A point p in C is said to be a *strong asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|T_n x_n - x_n\| = 0$. The set of strong asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted by $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 2.2 The countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be *a countable family of relatively nonexpansive mappings* if the following conditions are satisfied:

(1) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;
(2) $\phi(u, T_n x) \leq \phi(u,x)$, $\forall u \in F(T_n)$, $x \in C$, $n \geq 0$;
(3) $\widehat{F}(\{T_n\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(T_n)$.

Definition 2.3 The countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be *a countable family of weak relatively nonexpansive mappings* if the following conditions are satisfied:

(1) $F(\{T_n\}_{n=0}^{\infty})$ is nonempty;
(2) $\phi(u, T_n x) \leq \phi(u,x)$, $\forall u \in F(T_n)$, $x \in C$, $n \geq 0$;
(3) $\widetilde{F}(\{T_n\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(T_n)$.

Definition 2.4 [21] The mapping T is said to be *a relatively nonexpansive mapping* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T)$, $x \in C$;
(3) $\widetilde{F}(T) = F(T)$.

Definition 2.5 The mapping T is said to be *a weak relatively nonexpansive mapping* if the following conditions are satisfied:
(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \leq \phi(u, x)$, $\forall u \in F(T)$, $x \in C$;
(3) $\widetilde{F}(T) = F(T)$.

Definition 2.4 (Definition 2.5) is a special form of Definition 2.2 (Definition 2.3) as $T_n \equiv T$ for all $n \geq 0$.

The hybrid algorithms for a fixed point of relatively nonexpansive mappings and applications have been studied by many authors; see, for example, [1, 6, 7, 18, 26, 27]. In recent years, the definition of a weak relatively nonexpansive mapping has been presented and studied by many authors [7, 18, 25, 27], but they have not given an example of a mapping which is weak relatively nonexpansive, but not relatively nonexpansive.

In the next section, we shall give an example which is a countable family of weak relatively nonexpansive mappings, but not a countable family of relatively nonexpansive mappings.

We need the following lemmas for the proof of our main results.

Lemma 2.6 [24] *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Lemma 2.7 [23, 24, 26] *Let C be a nonempty closed convex subset of a smooth real Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0 \quad \textit{for all } y \in C.$$

Lemma 2.8 [23, 24, 26] *Let E be a reflexive, strictly convex and smooth real Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_c x) + \phi(\Pi_c x, x) \leq \phi(y, x) \quad \textit{for all } y \in C.$$

Lemma 2.9 [27] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leq r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda \|x\|^2 + \mu \|y\|^2 + \gamma \|z\|^2 - \lambda \mu g\bigl(\|x - y\|\bigr) \tag{2.5}$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

It is easy to prove the following result.

Lemma 2.10 *Let E be a strictly convex and smooth real Banach space, let C be a closed convex subset of E, and let T be a weak relatively nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

3 Main results

Firstly, we give an example which is a countable family of weak relatively nonexpansive mappings, but not a countable family of relatively nonexpansive mappings in the Banach space l^2.

Example 1 Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots), \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n \ldots) \in l^2.$$

It is well known that l^2 is a Hilbert space, so $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned}
&x_0 = (1, 0, 0, 0, \ldots), \\
&x_1 = (1, 1, 0, 0, \ldots), \\
&x_2 = (1, 0, 1, 0, 0, \ldots), \\
&x_3 = (1, 0, 0, 1, 0, 0, \ldots), \\
&\ldots \\
&x_n = (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots), \\
&\ldots,
\end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1, \end{cases}$$

for all $n \geq 1$.

Define a countable family of mappings $T_n : E \to E$ as follows:

$$T_n(x) = \begin{cases} \frac{n}{n+1} x_n & \text{if } x = x_n, \\ -x & \text{if } x \neq x_n, \end{cases}$$

for all $n \geq 0$.

Conclusion 3.1 *$\{x_n\}$ converges weakly to x_0.*

Proof For any $f = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k, \ldots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \to 0,$$

as $n \to \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

Conclusion 3.2 *$\{x_n\}$ is not a Cauchy sequence, so, it does not converge strongly to any element of l^2.*

Proof In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

Conclusion 3.3 *T_n has a unique fixed point 0, that is, $F(T_n) = \{0\}$ for all $n \geq 0$.*

Proof The conclusion is obvious. □

Conclusion 3.4 *x_0 is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$.*

Proof Since $\{x_n\}$ converges weakly to x_0 and

$$\|T_n x_n - x_n\| = \left\| \frac{n}{n+1} x_n - x_n \right\| = \frac{1}{n+1} \|x_n\| \to 0$$

as $n \to \infty$, so, x_0 is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$. □

Conclusion 3.5 *$\{T_n\}_{n=0}^{\infty}$ has a unique strong asymptotic fixed point 0, so, $\bigcap_{n=0}^{\infty} F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^{\infty})$.*

Proof In fact, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, from Conclusion 3.2, there exists a sufficiently large natural number N such that $z_n \neq x_m$ for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. □

Conclusion 3.6 *$\{T_n\}_{n=0}^{\infty}$ is a countable family of weak relatively nonexpansive mappings.*

Proof Since $E = L^2$ is a Hilbert space, for any $n \geq 0$, we have

$$\begin{aligned}\phi(0, T_n x) &= \|0 - T_n x\|^2 = \|T_n x\|^2 \\ &\leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall x \in E.\end{aligned}$$

From Conclusion 3.5, we have $\bigcap_{n=0}^{\infty} F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^{\infty})$, then $\{T_n\}_{n=0}^{\infty}$ is a countable family of weak relatively nonexpansive mappings. □

Conclusion 3.7 *$\{T_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mappings.*

Proof From Conclusions 3.3 and 3.4, we have $\bigcap_{n=0}^{\infty} F(T_n) \neq \widehat{F}(\{T_n\}_{n=0}^{\infty})$, so, $\{T_n\}_{n=0}^{\infty}$ is not a countable family of relatively nonexpansive mappings. □

Secondly, we give another example which is a weak relatively nonexpansive mapping, but not a relatively nonexpansive mapping in the Banach space l^2.

Example 2 Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots), \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n \ldots) \in l^2.$$

It is well known that l^2 is a Hilbert space, so $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned}
&x_0 = (1, 0, 0, 0, \ldots), \\
&x_1 = (1, 1, 0, 0, \ldots), \\
&x_2 = (1, 0, 1, 0, 0, \ldots), \\
&x_3 = (1, 0, 0, 1, 0, 0, \ldots), \\
&\cdots \\
&x_n = (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots), \\
&\cdots,
\end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1 & \text{if } k = 1, n+1, \\ 0 & \text{if } k \neq 1, k \neq n+1, \end{cases}$$

for all $n \geq 1$. Define the mapping $T : E \to E$ as follows

$$T(x) = \begin{cases} \frac{n}{n+1} x_n & \text{if } x = x_n \ (\exists n \geq 1), \\ -x & \text{if } x \neq x_n \ (\forall n \geq 1). \end{cases}$$

Conclusion 3.8 $\{x_n\}$ *converges weakly to* x_0.

Proof For any $f = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_k, \ldots) \in l^2 = (l^2)^*$, we have

$$f(x_n - x_0) = \langle f, x_n - x_0 \rangle = \sum_{k=2}^{\infty} \zeta_k \xi_{n,k} = \zeta_{n+1} \to 0,$$

as $n \to \infty$. That is, $\{x_n\}$ converges weakly to x_0. □

Conclusion 3.9 *$\{x_n\}$ is not a Cauchy sequence, so, it does not converge strongly to any element of l^2.*

Proof In fact, we have $\|x_n - x_m\| = \sqrt{2}$ for any $n \neq m$. Then $\{x_n\}$ is not a Cauchy sequence. □

Conclusion 3.10 *T has a unique fixed point 0, that is, $F(T) = \{0\}$.*

Proof The conclusion is obvious. □

Conclusion 3.11 *x_0 is an asymptotic fixed point of T.*

Proof Since $\{x_n\}$ converges weakly to x_0 and

$$\|Tx_n - x_n\| = \left\| \frac{n}{n+1} x_n - x_n \right\| = \frac{1}{n+1} \|x_n\| \to 0$$

as $n \to \infty$, then x_0 is an asymptotic fixed point of T. □

Conclusion 3.12 *T has a unique strong asymptotic fixed point 0, so, $F(T) = \widetilde{F}(T)$.*

Proof In fact, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - Tz_n\| \to 0$ as $n \to \infty$, from Conclusion 3.9, there exists a sufficiently large natural number N such that $z_n \neq x_m$, for any $n, m > N$. Then $Tz_n = -z_n$ for $n > N$, it follows from $\|z_n - Tz_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$. □

Conclusion 3.13 *T is a weak relatively nonexpansive mapping.*

Proof Since $E = L^2$ is a Hilbert space, we have

$$\begin{aligned} \phi(0, Tx) &= \|0 - Tx\|^2 = \|Tx\|^2 \\ &\leq \|x\|^2 = \|x - 0\|^2 = \phi(0, x), \quad \forall x \in E. \end{aligned}$$

From Conclusion 3.12, we have $F(T) = \widetilde{F}(T)$, then T is a weak relatively nonexpansive mapping. □

Conclusion 3.14 *T is not a relatively nonexpansive mapping.*

Proof From Conclusions 3.10 and 3.11, we have $F(T) \neq \widehat{F}(T)$, so, T is not a relatively nonexpansive mapping. □

Next, we prove our convergence theorems as follows.

Theorem 3.15 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\} : C \to C$ be a countable family of weak relatively nonexpansive mappings such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For any given gauss $x_0 \in C$,* *define a sequence* $\{x_n\}$ *in* C *by the following algorithm*:

$$\begin{cases} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, T_n x_n) \leq \phi(z, x_n)\}, \quad n = 0, 1, 2, 3, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.1}$$

Then $\{x_n\}$ *converges strongly to* $q = \Pi_F x_0$.

Proof Firstly, C_n is closed and convex. Since T is a closed hemi-relatively nonexpansive mapping, then $F(T) \subseteq C_n$, $n = 0, 1, 2, 3, \ldots$.

Since $x_n = \Pi_{C_n} x_0$ and $C_n \subset C_{n-1}$, then we get

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \text{for all } n \geq 0. \tag{3.2}$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. On the other hand, by Lemma 2.8 we have

$$\begin{aligned} \phi(x_n, x_0) &= \phi(\Pi_{C_n} x_0, x_0) \\ &\leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0), \end{aligned}$$

for all $p \in F(T) \subset C_n$ and for all $n \geq 0$. Therefore, $\phi(x_n, x_0)$ is also bounded. This together with (3.2) implies that the limit of $\{\phi(x_n, x_0)\}$ exists. Put

$$\lim_{n\to\infty} \phi(x_n, x_0) = d. \tag{3.3}$$

From Lemma 2.8, we have, for any positive integer m, that

$$\begin{aligned} \phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{C_n} x_0) \\ &\leq \phi(x_{n+m}, x_0) - \phi(\Pi_{C_n} x_0, x_0) \\ &= \phi(x_{n+m}, x_0) - \phi(x_{n+1}, x_0), \end{aligned}$$

for all $n \geq 0$. This together with (3.3) implies that

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0$$

holds, uniformly for all m. By using Lemma 2.6, we get that

$$\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$$

holds, uniformly for all m. Then $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$.

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \subset C_{n+1} \subset C_n$, then

$$\phi(x_{n+1}, T_n x_n) \leq \phi(x_{n+1}, x_n), \quad n = 0, 1, 2, 3, \ldots.$$

By using Lemma 2.6, we have $\|x_n - T_n x_n\| \to 0$; therefore, $p \in F(T)$.

Finally, we prove that $p = \Pi_F x_0$. From Lemma 2.8, we have

$$\phi(p, \Pi_F x_0) + \phi(\Pi_F x_0, x_0) \le \phi(p, x_0).$$

On the other hand, since $x_{n+1} = \Pi_{C_{n+1}} x_0$ and $F \subset C_n$ for all n, also from Lemma 2.8, we have

$$\phi(\Pi_F x_0, x_{n+1}) + \phi(x_{n+1}, x_0) \le \phi(\Pi_F x_0, x_0). \tag{3.4}$$

By the definition of $\phi(x,y)$, we know that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_0) = \phi(p, x_0). \tag{3.5}$$

Combining (3.4) and (3.5), we know that $\phi(p, x_0) = \phi(\Pi_F x_0, x_0)$. Therefore, it follows from the uniqueness of $\Pi_F x_0$ that $p = \Pi_F x_0$. This completes the proof. □

Theorem 3.16 *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let $\{T_n\} : C \to C$ be a countable family of weak relatively nonexpansive mappings such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. For any given gauss $x_0 \in C$, define a sequence $\{x_n\}$ in C by the following algorithm*:

$$\begin{cases} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, T_n x_n) = \phi(z, x_n)\}, \quad n = 0,1,2,3,\ldots, \\ x_{n+1} = \Pi_{C_{n+1}} x_0. \end{cases} \tag{3.6}$$

Then $\{x_n\}$ converges strongly to $q = \Pi_F x_0$.

Proof Let $\{x_n\}$ be defined by (3.1). We claim that

$$x_{n+1} \in \bigl\{z \in C_n : \phi(z, T_n x_n) = \phi(z, x_n)\bigr\}, \quad n = 0,1,2,3,\ldots.$$

Therefore,

$$x_{n+1} = \Pi_{\{z\in C_n:\phi(z,T_n x_n)=\phi(z,x_n)\}} x_0.$$

If not, there exists x_{n+1} such that

$$\phi(x_{n+1}, T_n x_n) < \phi(x_{n+1}, x_n).$$

We define

$$z(t) = (1-t)x_{n+1} + t x_0 \in C, \quad t \in [0,1].$$

Observe that $z(0) = x_{n+1}$. Since $\phi(\cdot, T_n x_n)$, $\phi(\cdot, x_n)$ are continuous, then there exists $t_0 \in (0,1)$ such that

$$\phi\bigl(z(t_0), T_n x_n\bigr) < \phi\bigl(z(t_0), x_n\bigr),$$

that is, $z(t_0) \in C_{n+1}$. On the other hand, we have

$$\begin{aligned}
\phi\big(z(t_0), x_0\big) &= \phi\big((1-t_0)x_{n+1} + t_0x_0, x_0\big) \\
&= \big\|(1-t_0)x_{n+1} + t_0x_0\big\|^2 \\
&\quad - 2\big\langle(1-t_0)x_{n+1} + t_0x_0, Jx_0\big\rangle + \|x_0\|^2 \\
&\le (1-t_0)\|x_{n+1}\|^2 + t_0\|x_0\|^2 \\
&\quad - 2(1-t_0)\langle x_{n+1}, Jx_0\rangle - 2t_0\langle x_0, Jx_0\rangle + \|x_0\|^2 \\
&= (1-t_0)\|x_{n+1}\|^2 + t_0\|x_0\|^2 \\
&\quad - 2(1-t_0)\langle x_{n+1}, Jx_0\rangle - 2t_0\|x_0\|^2 + \|x_0\|^2 \\
&= (1-t_0)\|x_{n+1}\|^2 + (1-t_0)\|x_0\|^2 \\
&\quad - 2(1-t_0)\langle x_{n+1}, Jx_0\rangle \\
&= (1-t_0)\phi(x_{n+1}, x_0) < \phi(x_{n+1}, x_0).
\end{aligned}$$

This is a contradiction to $x_{n+1} = \Pi_{C_{n+1}}x_0$ and $z(t_0) \in C_{n+1}$. This completes the proof. □

4 Applications

Now, we apply Theorem 3.15 to prove a strong convergence theorem concerning maximal monotone operators in a Banach space E.

Let A be a multi-valued operator from E to E^* with the domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \{z \in E : z \in D(A)\}$. An operator A is said to be monotone if

$$\langle x_1 - x_2, y_1 - y_2\rangle \ge 0$$

for each $x_1, x_2 \in D(A)$ and $y_1 \in Ax_1, y_2 \in Ax_2$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone operator. We know that if A is a maximal monotone operator, then $A^{-1}0$ is closed and convex. The following result is also well known.

Theorem 4.1 (Rockafellar [28]) *Let E be a reflexive, strictly convex and smooth Banach space, and let A be a monotone operator from E to E^*. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a reflexive, strictly convex and smooth Banach space, and let A be a maximal monotone operator from E to E^*. Using Theorem 4.1 and strict convexity of E, we obtain that for every $r > 0$ and $x \in E$, there exists a unique x_r such that

$$Jx \in Jx_r + rAx_r.$$

Then we can define a single valued mapping $J_r : E \to D(A)$ by $J_r = (J + rA)^{-1}J$ and such a J_r is called the resolvent of A. We know that $A^{-1} = F(J_r)$ for all $r > 0$, see [4, 15] for more details. Using Theorem 3.15, we can consider the problem of strong convergence concerning maximal monotone operators in a Banach space. Such a problem has been also studied in [4, 5, 15, 20, 22, 24, 29–36].

Theorem 4.2 *Let E be a uniformly convex and uniformly smooth real Banach space, let A be a maximal monotone operators from E to E^* such that $A^{-1}0 \neq \emptyset$, let J_r be the resolvent of A, where $r > 0$. For any given gauss $x_0 \in C_0 = C$, define a sequence $\{x_n\}$ in C by the following algorithm*:

$$\begin{cases} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, J_{r_n} x_n) = \phi(z, x_n)\}, \quad n = 0, 1, 2, 3, \ldots, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

with the condition, $r_n > 0$, $\liminf_{n\to\infty} r_n > 0$. *Then $\{x_n\}$ converges strongly to $q = \Pi_{A^{-1}0} x_0$.*

Proof We only need to prove that $\{J_{r_n}\}_{n=0}^{\infty}$, is a countable family of weak relatively nonexpansive mappings.

Firstly, we have $\bigcap_{n=0}^{\infty} F(J_{r_n}) = A^{-1}0 \neq \emptyset$. Secondly, from the monotonicity of A, we have

$$\begin{aligned} \phi(p, J_{r_n} w) &= \|p\|^2 - 2\langle p, JJ_{r_n} w\rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 + 2\langle p, Jw - JJ_{r_n} w - Jw\rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 + 2\langle p, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 - 2\langle J_{r_n} w - p - J_{r_n} w, Jw - JJ_{r_n} w - Jw\rangle \\ &\quad - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 - 2\langle J_{r_n} w - p, Jw - JJ_{r_n} w - Jw\rangle \\ &\quad + 2\langle J_{r_n} w, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\ &\le \|p\|^2 + 2\langle J_{r_n} w, Jw - JJ_{r_n} w\rangle - 2\langle p, Jw\rangle + \|J_{r_n} w\|^2 \\ &= \|p\|^2 - 2\langle p, Jw\rangle + \|w\|^2 - \|J_{r_n} w\|^2 \\ &\quad + 2\langle J_{r_n} w, Jw\rangle - \|w\|^2 \\ &= \phi(p, w) - \phi(J_{r_n} w, w) \\ &\le \phi(p, w) \end{aligned}$$

for all $n \ge 0$. Thirdly, we prove the set of strong asymptotic fixed points $\widetilde{F}(\{J_{r_n}\}_{n=0}^{\infty}) = \bigcap_{n=0}^{\infty} F(J_{r_n}) = A^{-1}0$.

We first show that $\widetilde{F}(\{J_{r_n}\}_{n=0}^{\infty}) \subset A^{-1}0$. Let $p \in \widetilde{F}(\{J_{r_n}\}_{n=0}^{\infty})$, then there exists $\{z_n\} \subset E$ such that $z_n \to p$ and $\lim_{n\to\infty} \|z_n - J_{r_n} z_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we obtain

$$\frac{1}{r_n}(Jz_n - JJ_{r_n} z_n) \to 0.$$

It follows from

$$\frac{1}{r_n}(Jz_n - JJ_{r_n} z_n) \in AJ_{r_n} z_n$$

and the monotonicity of A that

$$\left\langle w - J_{r_n} z_n, w^* - \frac{1}{r_n}(Jz_n - JJ_{r_n} z_n)\right\rangle \geq 0$$

for all $w \in D(A)$ and $w^* \in Aw$. Letting $n \to \infty$, we have $\langle w - p, w^* \rangle \geq 0$ for all $w \in D(A)$ and $w^* \in Aw$. Therefore, from the maximality of A, we obtain $p \in A^{-1}0$. On the other hand, we know that $F(J_{r_n}) = A^{-1}0$, $F(J_{r_n}) \subset \widetilde{F}(J_{r_n})$ for all $n \geq 0$; therefore, $A^{-1}0 = \bigcap_{n=0}^{\infty} F(J_{r_n}) = \widetilde{F}(\bigcap_{n=0}^{\infty} J_{r_n})$. From above three conclusions, we have proved $\{J_{r_n}\}_{n=0}^{\infty}$ is a countable family of weak relatively nonexpansive mappings. By using Theorem 3.16, we can conclude that $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0} x_0$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the writing of the present article. All authors read and approved the final manuscript.

Acknowledgements
This project is supported by the National Natural Science Foundation of China under Grant (11071279).

Received: 14 April 2012 Accepted: 29 October 2012 Published: 13 November 2012

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doi:10.1186/1687-1812-2012-205
Cite this article as: Zhang et al.: Simple projection algorithm for a countable family of weak relatively nonexpansive mappings and applications. *Fixed Point Theory and Applications* 2012 **2012**:205.