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Strong convergence of a proximal-type algorithm for an occasionally pseudomonotone operator in Banach spaces

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Abstract
It is known that the proximal point algorithm converges weakly to a zero of a maximal monotone operator, but it fails to converge strongly. Then, in (Math. Program. 87:189-202, 2000), Solodov and Svaiter introduced the new proximal-type algorithm to generate a strongly convergent sequence and established a convergence property for the algorithm in Hilbert spaces. Further, Kamimura and Takahashi (SIAM J. Optim. 13:938-945, 2003) extended Solodov and Svaiter's result to more general Banach spaces and obtained strong convergence of a proximal-type algorithm in Banach spaces. In this paper, by introducing the concept of an occasionally pseudomonotone operator, we investigate strong convergence of the proximal point algorithm in Hilbert spaces, and so our results extend the results of Kamimura and Takahashi.
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1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $T : H \to 2^H$ be a maximal monotone operator (or a multifunction) on H. We consider the classical problem:

Find $x \in H$ such that

$$0 \in Tx. \tag{1.1}$$

A wide variety of the problems, such as optimization problems and related fields, min-max problems, complementarity problems, variational inequalities, equilibrium problems and fixed point problems, fall within this general framework. For example, if T is the subdifferential ∂f of a proper lower semicontinuous convex function $f : H \to (-\infty, \infty)$, then T is a maximal monotone operator and the equation $0 \in \partial f(x)$ is reduced to $f(x) = \min\{f(z) : z \in H\}$. One method of solving $0 \in Tx$ is the proximal point algorithm. Let I denote the identity operator on H. Rockafellar's proximal point algorithm generates, for any starting point $x_0 = x \in H$, a sequence $\{x_n\}$ in H by the rule

$$x_{n+1} = (I + r_n T)^{-1} x_n, \quad \forall n \geq 0, \tag{1.2}$$

where $\{r_n\}$ is a sequence of positive real numbers. Note that (1.2) is equivalent to

$$0 \in Tx_{n+1} + \frac{1}{r_n}(x_{n+1} - x_n), \quad \forall n \geq 0.$$

This algorithm was first introduced by Martinet [1] and generally studied by Rockafellar [2] in the framework of Hilbert spaces. Later, many authors studied the convergence of (1.2) in Hilbert spaces (see Agarwal *et al.* [3], Brezis and Lions [4], Cho *et al.* [5], Cholamjiak *et al.* [6], Güler [7], Lions [8], Passty [9], Qin *et al.* [10], Song *et al.* [11], Solodov and Svaiter [12], Wei and Cho [13] and the references therein). Rockafellar [2] proved that, if $T^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to an element of $T^{-1}0$. Further, Rockafellar [2] posed an open question of whether the sequence $\{x_n\}$ generated by (1.2) converges strongly or not. This question was solved by Güler [7], who introduced an example for which the sequence $\{x_n\}$ generated by (1.2) converges weakly, but not strongly.

On the other hand, Kamimura and Takahashi [14, 15], Solodov and Svaiter [16] one decade ago modified the proximal point algorithm to generate a strongly convergent sequence. In 1999, Solodov and Svaiter [16] introduced the following algorithm $\{x_n\}$:

$$\begin{cases} x_0 \in H, \\ 0 = v_n + \frac{1}{r_n}(y_n - x_n), \quad v_n \in Ty_n, \\ H_n = \{z \in E : \langle z - y_n, v_n\rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, x_0 - x_n\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad \forall n \geq 0. \end{cases} \tag{1.3}$$

To explain how the sequence $\{y_n\}$ is generated, we formally state the above algorithm as follows.

Choose any $x_0 \in H$ and $\sigma \in [0,1)$. At iteration n, having x_n, choose $r_n > 0$ and find (y_n, v_n), an inexact solution of $0 = v_n + \frac{1}{r_n}(y_n - x_n)$, $v_n \in Ty_n$ with tolerance σ. Define H_n and W_n as in (1.3). Take $x_{n+1} = P_{H_n \cap W_n} x_0$. Note that at each iteration, there are two subproblems to be solved: find an inexact solution of the proximal point subproblem and find the projection of x_0 onto $H_n \cap W_n$, the intersection of two half-spaces. By a classical result of Minty [17], the proximal subproblem always has an exact solution, which is unique. Notice that computing an approximate solution makes things easier. Hence, this part of the method is well defined. Regarding the projection step, it is easy to prove that $H_n \cap W_n$ is never empty, even when the solution set is empty. Therefore, the whole algorithm is well defined in the sense that it generates an infinite sequence $\{x_n\}$ and an associated sequence of pairs $\{(y_n, v_n)\}$.

In 2003, Kamimura and Takahashi [18] extended Solodov and Svaiter's result to more general Banach spaces like the spaces L^p $(1 < p < \infty)$ by further modifying the proximal-point algorithm (1.2) in the following form in a smooth Banach space E:

$$\begin{cases} x_0 \in E, \\ 0 = v_n + \frac{1}{r_n}(J_2(y_n) - J_2(x_n)), \quad v_n \in Ty_n, \\ H_n = \{z \in E : \langle z - y_n, v_n\rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, J_2(x_0) - J_2(x_n)\rangle \leq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x_0, \quad \forall n \geq 0, \end{cases} \tag{1.4}$$

to generate a strongly convergent sequence. They proved that if $T^{-1}0 \neq \emptyset$ and $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to a point $P_{T^{-1}0}x_0$.

In this paper, by introducing the concept of an occasionally pseudomonotone operator, we investigate strong convergence of the proximal point algorithm in Hilbert spaces, and so our results extend the results of Kamimura and Takahashi.

2 Preliminaries and definitions

Let E be a real Banach space with norm $\|\cdot\|$, and let E^* denote the dual space of E. Let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Let $\{x_n\}$ be a sequence in E. We denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$, respectively.

Definition 2.1 A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T) = \{z \in E : Tz \neq \emptyset\}$ and range $R(T) = \bigcup\{Tz : z \in D(T)\}$ is said to be *monotone* if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for any $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$. A monotone operator T is said to be *maximal* if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone operator.

Definition 2.2 A multivalued operator $T : E \to 2^{E^*}$ with domain $D(T)$ and range $R(T)$ is said to be *pseudomonotone* (see also Karamardian [19]) if $\langle x_1 - x_2, y_2\rangle \geq 0$ implies $\langle x_1 - x_2, y_1\rangle \geq 0$ for any $x_i \in D(T)$ and $y_i \in Tx_i$, $i = 1, 2$.

It is obvious that each monotone operator is pseudomonotone, but the converse is not true.

We now introduce the concept of occasionally pseudomonotone as follows.

Definition 2.3 A multivalued operator $T : E \to 2^{E^*}$ is said to be *occasionally pseudomonotone* if, for any $x_i \in D(T)$, there exist $y_i \in Tx_i$, $i = 1, 2$, such that $\langle x_1 - x_2, y_2\rangle \geq 0$ implies $\langle x_1 - x_2, y_1\rangle \geq 0$.

It is clear that every monotone operator is pseudomonotone and every pseudomonotone operator is occasionally pseudomonotone, but the converse implications need not be true. To this end, we observe the following examples.

Example 2.1 Let $E = \mathbb{R}^3$ and $T : E \to 2^{E^*}$ be a multi-valued operator defined by

$$Tx = \{y = A_r x : r \in \mathbf{R}\}, \quad \forall x \in E,$$

where

$$A_r = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -r & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then for any $x_1 = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)})^T$, $x_2 = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)})^T$ in $\mathbf{R}^3$, if $y_1 = A_r x_1$ and $y_2 = A_r x_2$, then we have

$$\langle x_1 - x_2, y_1 - y_2\rangle = -r\big(x_2^{(1)} - x_2^{(2)}\big)^2.$$

Thus, if $r \leq 0$, then T is monotone. However, if $r > 0$, then T is neither monotone nor pseudomonotone. Indeed, for $x_1 = (0,1,0)$, then we have $y_1 = A_r x_1 = (0,-r,0)$, $x_2 = (0,0,0)$ and $\langle x_1 - x_2, y_2 \rangle = 0 \geq 0$, but $\langle x_1 - x_2, y_1 \rangle = -r < 0$.

Further, we see that T is occasionally pseudomonotone. To effect this, for any $x_1 = (x_1^{(1)}, x_2^{(1)}, x_3^{(1)})^T$ and $x_2 = (x_1^{(2)}, x_2^{(2)}, x_3^{(2)})^T$ in $\mathbf{R}^3$, if $y_i = A_0 x_i$, $i = 1,2$, then we have

$$\langle x_1 - x_2, y_2 \rangle = 0 \geq 0 \quad \Longrightarrow \quad \langle x_1 - x_2, y_1 \rangle = 0 \geq 0.$$

Example 2.2 The rotation operator on $\mathbf{R}^2$ given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is monotone and hence it is pseudomonotone. Thus, it follows that A is also occasionally pseudomonotone.

Maximality of pseudomonotone and occasionally pseudomonotone operators are defined as similar to maximality of a monotone operator. We denote by $L[x_1, x_2]$ the ray passing through x_1, x_2.

A Banach space E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be *uniformly convex* if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$.

It is known that a uniformly convex Banach space is reflexive and strictly convex. The spaces ℓ^1 and L^1 are neither reflexive nor strictly convex. Note also that a reflexive Banach space is not necessarily uniformly convex. For example, consider a finite dimensional Banach space in which the surface of the unit ball has a 'flat' part. We note that such a Banach space is reflexive because of finite dimension. But the 'flat' portion in the surface of the ball makes it nonuniformly convex. It is also well known that a Banach space E is reflexive if and only if every bounded sequence of elements of E contains a weakly convergent sequence.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit (2.1) is attained uniformly for $x, y \in U$.

It is well known that the spaces ℓ^p, L^p and W^m (Sobolev space) ($1 < p < \infty$, m is a positive integer) are uniformly convex and uniformly smooth Banach spaces.

For any $p \in (1, \infty)$, the mapping $J_p : E \to 2^{E^*}$ defined by

$$J_p x = \{f \in E^* : \langle x, f \rangle = \|f\| \cdot \|x\|, \|f\| = \|x\|^{p-1}\}, \quad \forall x \in E,$$

is called the *duality mapping* with the gauge function $\varphi(t) = t^{p-1}$. In particular, for $p = 2$, the duality mapping J_2 with gauge function $\varphi(t) = t$ is called the *normalized duality mapping*.

The following proposition gives some basic properties of the duality mapping.

Proposition 2.1 *Let E be a real Banach space. For $1 < p < \infty$, the duality mapping $J_p : E \to 2^{E^*}$ has the following properties*:

(1) $J_p(x) \neq \phi$ *for all* $x \in E$ *and* $D(J_p) = E$, *where* $D(J_p)$ *denotes the domain of* J_p;
(2) $J_p(x) = \|x\|^{p-2} \cdot J_2 x$ *for all* $x \in E$ *with* $x \neq 0$;
(3) $J_p(\alpha x) = \alpha^{p-1} \cdot J_2 x$ *for all* $\alpha \in [0, \infty)$;
(4) $J_p(-x) = -J_p(x)$;
(5) $\|x\|^p - \|y\|^p \geq p\langle x - y, j\rangle$ *for all* $x, y \in E$ *and* $j \in J_p y$;
(6) *If E is smooth, then* J_p *is norm-to-weak* continuous*;
(7) *If E is uniformly smooth, then* J_p *is uniformly norm-to-norm continuous on each bounded subset of E*;
(8) J_p *is bounded, i.e., for any bounded subset* $A \subset E$, $J_p(A)$ *is a bounded subset in* E^*;
(9) J_p *can be equivalently defined as the subdifferential of the functional* $\psi(x) = p^{-1}\|x\|^p$ (Asplund [20]), *i.e.*,

$$J_p(x) = \partial\psi(x) = \{f \in E^* : \psi(y) - \psi(x) \geq \langle y - x, f\rangle, \forall y \in E\};$$

(10) *E is a uniformly smooth Banach space* (*equivalently,* E^* *is a uniformly convex Banach space*) *if and only if* J_p *is single-valued and uniformly continuous on any bounded subset of E* (*see, for instance, Xu and Roach* [21], *Browder* [22]).

Proposition 2.2 *Let E be a real Banach space, and let* $J_p : E \to 2^{E^*}$, $1 < p < \infty$, *be the duality mapping. Then for any* $x, y \in E$,

$$\|x + y\|^p \leq \|x\|^p + p\langle y, j_p\rangle, \quad \forall j_p \in J_p(x + y).$$

Proof It is a straightforward consequence of the assertion (5) of Proposition 2.1 applied for x and $x + y$. Alternatively, from Proposition 2.1(9), it follows that $J_p(x) = \partial\psi(x)$ (subdifferential of the functional $\psi(x)$), where $\psi(x) = p^{-1}\|x\|^p$. Also, it follows from the definition of the subdifferential of ψ that

$$\psi(x) - \psi(x + y) \geq \langle x - (x + y), j_p\rangle, \quad \forall j_p \in J_p(x + y).$$

Now, substituting $\psi(x)$ by $p^{-1}\|x\|^p$, we have

$$\|x + y\|^p \leq \|x\|^p + p\langle y, j_p\rangle, \quad \forall j_p \in J_p(x + y).$$

This completes the proof. □

Remark 2.1 If E is a uniformly smooth Banach space, it follows from Proposition 2.1(10) that J_p ($1 < p < \infty$) is a single-valued mapping. We now define the functions $\Psi, \phi : E \times E \to \mathbb{R}$ by

$$\Psi(x, y) = \|x\|^p - p\langle x - y, J_p(y)\rangle - \|y\|^p, \quad \forall x, y \in E,$$

and ϕ is the support function satisfying the following condition:

$$\Psi(x, y) = \phi(x, y) + \|x - y\|^p, \quad \forall x, y \in E. \tag{2.2}$$

It is obvious from the definition of Ψ and Proposition 2.1(5) that

$$\Psi(x,y) \geq 0, \quad \forall x,y \in E. \tag{2.3}$$

Also, we see that

$$\begin{aligned}\Psi(x,y) &= \|x\|^p - p\langle x, J_p(y)\rangle + (p-1)\|y\|^p \\ &\geq \|x\|^p - p\|x\|\,\|J_p(y)\| + (p-1)\|y\|^p \\ &= \|x\|^p - p\|x\|\,\|y\|^{p-1} + (p-1)\|y\|^p.\end{aligned} \tag{2.4}$$

In particular, for $p = 2$, we have $\Psi(x,y) \geq (\|x\| - \|y\|)^2$.

Further, we can show the following two propositions.

Proposition 2.3 *Let E be a smooth Banach space, and let $\{y_n\}$, $\{z_n\}$ be two sequences in E. If $\Psi(y_n,z_n) \to 0$, then $y_n - z_n \to 0$.*

Proof It follows from $\Psi(y_n,z_n) \to 0$ that

$$\phi(y_n,z_n) \to 0, \qquad \bigl|\|y_n\| - \|z_n\|\bigr| \leq \|y_n - z_n\| \to 0$$

because of (2.2) and (2.3). Therefore, if $\{z_n\}$ is bounded, then $\{y_n\}$ (and also if $\{y_n\}$ is bounded, then $\{z_n\}$) is also bounded and $y_n - z_n \to 0$. This completes the proof. □

Proposition 2.4 *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that*

$$\Psi(x_0,x) = \inf\{\Psi(z,x) : z \in C\}. \tag{2.5}$$

Proof Since E is reflexive and $\|z_n\| \to \infty$ implies $\Psi(z_n,x) \to \infty$, there exists $x_0 \in C$ such that $\Psi(x_o,x) = \inf\{\Psi(z,x) : z \in C\}$. Since E is strictly convex, $\|\cdot\|^p$ is a strictly convex function, that is,

$$\|\lambda x_1 + (1-\lambda)x_2\|^p < \lambda\|x_1\|^p + (1-\lambda)\|x_2\|^p$$

for all $x_1, x_2 \in E$ with $x_1 \neq x_2$, $1 \leq p < \infty$ and $\lambda \in (0,1)$. Then the function $\Psi(\cdot,y)$ is also strictly convex. Therefore, $x_0 \in C$ is unique. This completes the proof. □

For each nonempty closed convex subset C of a reflexive, convex and smooth Banach space E, we define the mapping R_C of E onto C by $R_C x = x_0$, where x_0 is defined by (2.5). For the case $p = 2$, it is easy to see that the mapping is coincident with the metric projection in the setting of Hilbert spaces. In our discussion, instead of the metric projection, we make use of the mapping R_C. Finally, we prove two results concerning Proposition 2.4 and the mapping R_C. The first one is the usual analogue of a characterization of the metric projection in a Hilbert space.

Proposition 2.5 *Let E be a smooth Banach space, let C be a convex subset of E, let $x \in E$ and $x_0 \in C$. Then*

$$\Psi(x_0, x) = \inf\{\Psi(z, x) : z \in C\} \tag{2.6}$$

if and only if

$$\langle z - x_0, J_p(x_0) - J_p(x)\rangle \geq 0, \quad \forall z \in C. \tag{2.7}$$

Proof First, we show that (2.6) ⇒ (2.7). Let $z \in C$ and $\lambda \in (0, 1)$. It follows from $\Psi(x_0, x) \leq \Psi((1 - \lambda)x_0 + \lambda z, x)$ that

$$\begin{aligned}
0 &\leq \|(1-\lambda)x_0 + \lambda x\|^p - p\langle(1-\lambda)x_0 + \lambda z - x, J_p(x)\rangle \\
&\quad - \|x\|^p - \|x_0\|^p + p\langle x_0 - x, J_p(x)\rangle + \|x\|^p \\
&= \|(1-\lambda)x_0 + \lambda z\|^p - \|x_0\|^p - p\lambda\langle z - x_0, J_p(x)\rangle \\
&\leq p\lambda\langle z - x_0, J_p((1-\lambda)x_0 + \lambda z)\rangle - p\lambda\langle z - x_0, J_p(x)\rangle \\
&= p\lambda\langle z - x_0, J_p((1-\lambda)x_0 + \lambda z) - J_p(x)\rangle,
\end{aligned}$$

which implies

$$\langle z - x_0, J_p((1-\lambda)x_0 + \lambda z) - J_p(x)\rangle \geq 0.$$

Taking $\lambda \downarrow 0$, since J_p is norm-to-weak* continuous, we obtain

$$\langle z - x_0, J_p(x_0) - J_p(x)\rangle \geq 0,$$

which shows (2.7).

Next, we show that (2.7) ⇒ (2.6). For any $z \in C$, we have

$$\begin{aligned}
\Psi(z, x) - \Psi(x_0, x) &= \|z\|^p - p\langle z - x, J_p(x)\rangle - \|x\|^p \\
&\quad - \|x_0\|^p + p\langle x_0 - x, J_p(x)\rangle + \|x\|^p \\
&= \|z\|^p - \|x_0\|^p - p\langle z - x_0, J_p(x)\rangle \\
&\geq p\langle z - x_0, J_p(x_0)\rangle - p\langle z - x_0, J_p(x)\rangle \\
&= p\langle z - x_0, J_p(x_0) - J_p(x)\rangle \\
&\geq 0,
\end{aligned}$$

which proves (2.6). This completes the proof. □

Proposition 2.6 *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E, let $x \in E$ and $R_C x \in C$ with*

$$\|y - x\| = \|y - R_C x\| + \|R_C x - x\|, \quad \forall y \in L[x, R_C x] \cap C.$$

Then we have

$$\Psi(y, R_C x) + \Psi(R_C x, x) \leq \Psi(y, x), \quad \forall y \in L[x, R_C x] \cap C.$$

Proof It follows from Proposition 2.5 that

$$\begin{aligned}
&\Psi(y,x) - \Psi(y, R_C x) - \Psi(R_C x, x) \\
&\quad = \|y\|^p - p\langle y - x, J_p(x)\rangle - \|x\|^p + \|y - x\| - \|y\|^p \\
&\qquad + p\langle y - R_C x, J_p(R_C x)\rangle + \|R_C x\|^p - \|y - R_C x\| - \|R_C x\|^p \\
&\qquad + p\langle R_C x - x, J_p(xP\rangle + \|x\|^p - \|R_C x - x\| \\
&\quad = -p\langle y - x, J_p(x)\rangle + p\langle y - R_C x, J_p(R_C x)\rangle + p\langle R_C x - x, J_p(x)\rangle \\
&\quad = p\langle y - R_C x, J_p(R_C x) - J_p(x)\rangle \\
&\quad \geq 0, \quad \forall y \in L[x, R_C x] \cap C.
\end{aligned}$$

This completes the proof. □

3 Main results

Throughout this section, unless otherwise stated, we assume that $T : E \to 2^{E^*}$ is a occasionally pseudomonotone maximal monotone operator. In this section, we study the following algorithm $\{x_n\}$ in a smooth Banach space E, which is an extension of (1.2):

$$\begin{cases} x_0 \in E, \\ 0 = v_n + \frac{1}{r_n}(J_p(y_n) - J_p(x_n)), \quad v_n \in Ty_n, \\ H_n = \{z \in E : \langle z - y_n, v_n\rangle \leq 0\}, \\ W_n = \{z \in E : \langle z - x_n, J_p(x_n) - J_p(x_n)\rangle \leq 0\}, \\ x_{n+1} = R_{H_n \cap W_n} x_n, \quad \forall n \geq 0, \end{cases} \tag{3.1}$$

where $\{r_n\}$ is a sequence of positive real numbers.

First, we investigate the condition under which the algorithm (3.1) is well defined. Rockafellar [23] proved the following theorem.

Theorem 3.1 *Let E be a reflexive, strictly convex and smooth Banach space, and let $T : E \to 2^{E^*}$ be a monotone operator. Then T is maximal if and only if $R(J_p + rT) = E^*$ for all $r > 0$.*

By the appropriate modification of arguments in Theorem 3.1, we can prove the following.

Theorem 3.2 *Let E be a reflexive, strictly convex and smooth Banach space, and let $T : E \to 2^{E^*}$ be an occasionally pseudomonotone operator. Then T is maximal if and only if $R(J_p + rT) = E^*$ for all $r > 0$.*

Using Theorem 3.2, we can show the following result.

Proposition 3.3 *Let E be a reflexive, strictly convex and smooth Banach space. If $T^{-1}0 \neq \emptyset$, then the sequence generated $\{x_n\}$ by* (3.1) *is well defined.*

Proof From the definition of the sequence $\{x_n\}$, it is obvious that both H_n and W_n are closed convex sets. Let $w \in T^{-1}0$. From Theorem 3.2, there exists $(y_0, v_0) \in E \times E^*$ such that

$$0 = v_0 + \frac{1}{r_0}\big(J_p(y_0) - J_p(x_0)\big), \quad v_0 \in Ty_0.$$

Since T is occasionally pseudomonotone and $\langle y_0 - w, 0\rangle = 0 \geq 0$, from $Tw \ni 0$, it follows that

$$\langle y_0 - w, v_0\rangle \geq 0$$

for some $v_0 \in Ty_0$. It follows that $w \in H_0$. On the other hand, it is clear that $w \in W_0 = E$. Then $w \in H_0 \cap W_0$ and so $x_1 = R_{H_0 \cap W_0} x_0$ is well defined. Suppose that $w \in H_{n-1} \cap W_{n-1}$ is well defined for some $n \geq 1$. Again, by Theorem 3.2, we obtain $(y_n, v_n) \in E \times E^*$ such that

$$0 = v_n + \frac{1}{r_n}\big(J_p(y_n) - J_p(x_n)\big), \quad v_n \in Ty_n.$$

Then since T is occasionally pseudomonotone and $\langle y_n - w, 0\rangle = 0 \geq 0$, from $Tw \ni 0$, it follows that

$$\langle y_n - w, v_n\rangle \geq 0$$

for some $v_n \in Ty_n$, and so $w \in H_n$. It follows from Proposition 2.5 that

$$\begin{aligned}\big\langle w - x_n, J_p(x_0) - J_p(x_n)\big\rangle &= \big\langle w - R_{H_{n-1}\cap W_{n-1}} x_0, J_p(x_0) - J_p(R_{H_{n-1}} \cap W_{n-1} x_0)\big\rangle \\ &\leq 0,\end{aligned}$$

which implies $w \in W_n$. Therefore, $w \in H_n \cap W_n$ and hence $x_{n-1} = R_{H_n \cap W_n} x_0$ is well defined. Then by induction, the sequence $\{x_n\}$ generated by (3.1) is well defined for each $n \geq 0$. This completes the proof. □

Remark 3.1 From the above proof, we obtain

$$T^{-1}0 \subset H_n \cap W_n, \quad \forall n \geq 0.$$

Now, we are ready to prove our main theorem.

Theorem 3.4 *Let E be a reflexive, strictly convex and uniformly smooth Banach space. If $T^{-1}0 \neq \emptyset$, ϕ satisfies the condition* (2.2) *and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$, then the sequence $\{x_n\}$ generated by* (3.1) *converges strongly to $R_{T^{-1}0} x_0$.*

Proof It follows from the definition of W_{n+1} and Proposition 2.5 that $x_{n+1} = R_{W_{n+1}}x_0$. Further, from $x_0 \in L(x_n, R_{W_{n+1}}x_0) \cap W_{n-1}$ and Proposition 2.6, we have

$$\Psi(x_n, R_{W_{n+1}}x_0) + \Psi(R_{W_{n+1}}x_0, x_0) \leq \Psi(x_n, x_0)$$

and hence

$$\Psi(x_n, x_{n+1}) + \Psi(x_{n+1}, x_0) \leq \Psi(x_n, x_0). \tag{3.2}$$

Since the sequence $\{\Psi(x_n, x_0)\}$ is monotone decreasing and bounded below by 0, it follows that $\liminf_{n\to\infty} \Psi(x_n, x_0)$ exists and, in particular, $\{\Psi(x_n, x_0)\}$ is bounded. Then by (2.3), $\{x_n\}$ is also bounded. This implies that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$ for some $w \in E$.

Now, we show that $w \in T^{-1}0$. It follows from (3.2) that $\Psi(x_n, x_{n+1}) \to 0$. On the other hand, we have

$$\begin{aligned}
&\Psi(R_{H_n}x_n, x_n) - \Psi(y_n, x_n) \\
&\quad = \|R_{H_n}x_n\|^p - p\langle R_{H_n}x_n - x_n, J_p(x_n)\rangle - \|x_n\|^p \\
&\qquad + \|R_{H_n}x_n - x_n\| - \|y_n\|^p + p\langle y_n - x_n, J_p(x_n)\rangle + \|x_n\|^p - \|y_n - x_n\| \\
&\quad = \|R_{H_n}x_n\|^p - \|y_n\|^p + p\langle y_n - R_{H_n}x_n, J_p(x_n)\rangle + \|R_{H_n}x_n - y_n\| \\
&\quad \geq p\langle R_{H_n}x_n - y_n, J_p(y_n)\rangle + p\langle y_n - R_{H_n}x_n, J_p(x_n)\rangle + \|R_{H_n}x_n - y_n\| \\
&\quad = p\langle y_n - R_{H_n}x_n - y_n, J_p(x_n) - J_p(y_n)\rangle + \|R_{H_n}x_n - y_n\|.
\end{aligned}$$

Since $R_{H_n}x_n \in H_n$ and $0 = v_n + \frac{1}{r_n}(J_p(y_n) - J_p(x_n))$, it follows that

$$\langle y_n - R_{H_n}x_n - y_n J_p(x_n) - J_p(y_n)\rangle \geq 0,$$

and so $\Psi(R_{H_n}x_n, x_n) \geq \Psi(y_n, x_n)$. Further, since $x_{n+1} \in H_n$, we have

$$\Psi(x_{n+1}, x_n) \geq \Psi(R_{H_n}x_n, x_n),$$

which yields that

$$\Psi(x_{n+1}, x_n) \geq \Psi(R_{H_n}x_n, x_n) \geq \Psi(y_n, x_n).$$

Then it follows from $\Psi(x_n, x_{n+1}) \to 0$ that $\Psi(y_n, x_n) \to 0$. Consequently, by Proposition 2.3, we have $y_n - x_n \to 0$, which implies $y_{n_i} \rightharpoonup w$. Moreover, since J is uniformly norm-to-norm continuous on bounded subsets and $\liminf_{n\to\infty} r_n > 0$, we obtain

$$v_n = -\frac{1}{r_n}\big(J_p(y_n) - J_p(x_n)\big) \to 0.$$

It follows from $v_n \in Ty_n$ with $v_n \to 0$ and $y_{n_i} \rightharpoonup w$ that

$$\lim_{i,n\to\infty} \langle z - y_{n_i}, v_n\rangle = \langle z - w, 0\rangle = 0, \quad \forall z \in D(T).$$

Then, since T is occasionally pseudomonotone, it follows that $\langle z-w, z'\rangle = 0$ for some $z' \in Tz$. Therefore, from the maximality of T, we obtain $w \in T^{-1}0$. Let $w^* \in R_{T^{-1}0}x_0$. Now, from $x_{n+1} = R_{H_n \cap W_n}x_0$ and $w^* \in T^{-1}0 \subset L(x_n, R_{W_{n+1}}x_0) \cap H_n \cap W_n$, we have

$$\Psi(x_{n+1}, x_0) \le \Psi\big(w^*, x_0\big).$$

Then we have

$$\begin{aligned}\Psi\big(x_n, w^*\big) &= \Psi(x_n, x_0) + \Psi\big(x_0, w^*\big) - p\big\langle x_n, x_0, J_p\big(w^*\big) - J_p(x_0)\big\rangle \\ &\quad + \big\|x_n - w^*\big\| - \|x_n - x_0\| - \big\|x_0 - w^*\big\| \\ &\le \Psi\big(w^*, x_0\big) + \Psi\big(x_0, w^*\big) - p\big\langle x_n - x_0, J_p\big(w^*\big) - J_p(x_0)\big\rangle \\ &\quad + \big\|x_n - w^*\big\| - \|x_n - x_0\| - \big\|x_0 - w^*\big\|,\end{aligned}$$

which yields

$$\begin{aligned}\limsup_{i\to\infty} \Psi\big(x_{ni}, w^*\big) &\le \Psi\big(w^*, x_0\big) + \Psi\big(x_0, w^*\big) - p\big\langle w - x_0, J_p\big(w^*\big) - J_p(x_0)\big\rangle \\ &\quad + \big\|w - w^*\big\| - \|w - x_0\| - \big\|x_0 - w^*\big\|.\end{aligned}$$

Thus, from Proposition 2.5, we have

$$\begin{aligned}&\Psi\big(w^*, x_0\big) + \Psi\big(x_0, w^*\big) - p\big\langle w - x_0, J_p\big(w^*\big) - J_p(x_0)\big\rangle \\ &\quad + \big\|w - w^*\big\| - \|w - x_0\| - \big\|x_0 - w^*\big\| \\ &= p\big\langle w - w^*, J_p(x_0) - J_p\big(w^*\big)\big\rangle \\ &\le 0.\end{aligned}$$

Then we obtain $\limsup_{i\to\infty} \Psi(x_{n_i}, w^*) \le 0$, and hence $\Psi(x_{n_i}, w^*) \to 0$. It follows from Proposition 2.3 that $x_{n_i} \to w^*$. This means that the whole sequence $\{x_n\}$ generated by (3.1) converges weakly to w^* and each weakly convergent subsequence of $\{x_n\}$ converges strongly to w^*. Therefore, $\{x_n\}$ converges strongly to $w^* \in R_{T^{-1}0}x_0$. This completes the proof. □

4 An application

Let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Then the subdifferential ∂f of f is defined by

$$\partial f(x) = \big\{v \in E^* : f(y) - f(x) \ge \langle y - x, v\rangle, \forall y \in E\big\}.$$

Using Theorem 3.4, we consider the problem of finding a minimizer of the function f.

Theorem 4.1 *Let E be reflexive, strictly convex and uniformly smooth Banach space, and let $f : E \to (-\infty, \infty]$ be a proper convex lower semicontinuous function. Assume that $\{r_n\} \subset$*

$(0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$ *and* $\{x_n\}$ *is the sequence generated by*

$$\begin{cases} x_0 \in E, \\ y_n = \operatorname{argmin}_{z\in E}\{f(z) + \frac{1}{pr_n}\|z\|^p - \frac{1}{r_n}\langle z, J_p(x_n)\rangle\}, \\ 0 = v_n + \frac{1}{r_n}(J_p(y_n) - J_p(x_n)), \quad v_n \in Ty_n, \\ H_n = \{z \in E : \langle z - y_n, v_n\rangle \le 0\}, \\ W_n = \{z \in E : \langle z - x_n, J_p(x_n)\rangle \le 0\}, \\ x_{n+1} = R_{H_n\cap W_n} x_0, \quad \forall n \ge 0. \end{cases} \tag{4.1}$$

If $(\partial f)^{-1} \neq \emptyset$, *then the sequence* $\{x_n\}$ *generated by* (4.1) *converges strongly to the minimizer of* f.

Proof Since $f : E \to (-\infty,\infty]$ is a proper convex lower semicontinuous function, by Rockafelllar [23], the subdifferential ∂f of f is a maximal monotone operator and so it is also an occasionally pseudomonotone maximal operator. We also know that

$$y_n = \operatorname*{argmin}_{z\in E}\left\{f(z) + \frac{1}{pr_n}\|z\|^p - \frac{1}{r_n}\langle z, J_p(x_n)\rangle\right\}$$

is equivalent to the following:

$$\frac{1}{r_n}\big(J_p(z) - J_p(x_n)\big) \in \partial f(y_n), \quad \forall z \in E.$$

This implies that

$$0 \in \partial f(y_n) + \frac{1}{r_n}\big(J_p(y_n) - J_p(x_n)\big).$$

Thus, we have $v_n \in \partial f(y_n)$ such that $0 = v_n + \frac{1}{r_n}(J_p(y_n) - J_p(x_n))$. Therefore, using Theorem 3.4, we get the conclusion. This completes the proof. □

5 Concluding remarks

We presented a modified proximal-type algorithm with the varied degree of rate of the convergence depending upon the choice of p $(1 < p < \infty)$ for an occasionally pseudomonotone operator, which is a generalization of a monotone operator, to extend Kamimura and Takahashi's result to more general Banach spaces which are not necessarily uniformly convex like locally uniformly Banach spaces. As an application, we consider the problem of finding a minimizer of a convex function in a more general setting of Banach spaces than what Kamimura and Takahashi have considered.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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