Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations

Fangfang Yan, Yongfu Su* and Qiansheng Feng

*Correspondence:
suyongfu@gmail.com
Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, P.R. China

Abstract

The aim of this paper is to extend the results of Harjani and Sadarangani and some other authors and to prove a new fixed point theorem of a contraction mapping in a complete metric space endowed with a partial order by using altering distance functions. Our theorem can be used to investigate a large class of nonlinear problems. As an application, we discuss the existence of a solution for a periodic boundary value problem.

Keywords: contraction mapping principle; partially ordered metric spaces; fixed point; altering distance function; differential equation

1 Introduction

The Banach contraction principle is a classical and powerful tool in nonlinear analysis. Weak contractions are generalizations of Banach's contraction mapping studied by several authors. In [1–8], the authors prove some types of weak contractions in complete metric spaces respectively. In particular, the existence of a fixed point for weak contraction and generalized contractions was extended to partially ordered metric spaces in [2, 9–18]. Among them, the altering distance function is basic concept. Such functions were introduced by Khan *et al.* in [1], where they present some fixed point theorems with the help of such functions. Firstly, we recall the definition of an altering distance function.

Definition 1.1 An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies

(a) ψ is continuous and nondecreasing.
(b) $\psi = 0$ if and only if $t = 0$.

Recently, Harjani and Sadarangani proved some fixed point theorems for weak contraction and generalized contractions in partially ordered metric spaces by using the altering distance function in [11, 19] respectively. Their results improve the theorems of [2, 3].

Theorem 1.1 [11] *Let $(X,\le)$ be a partially ordered set, and suppose that there exists a metric $d \in X$ such that (X,d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$d\big(f(x),f(y)\big) \le d(x,y) - \psi\big(d(x,y)\big) \quad \textit{for } x \ge y,$$

where $\psi : [0,\infty) \longrightarrow [0,\infty)$ is continuous and nondecreasing function such that ψ is positive in $(0,\infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Theorem 1.2 [19] *Let $(X,\leq)$ be a partially ordered set, and suppose that there exists a metric $d \in X$ such that (X,d) is a complete metric space. Let $f : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi d\big(f(x),f(y)\big) \leq \psi\big(d(x,y)\big) - \phi\big(d(x,y)\big) \quad \text{for } x \geq y,$$

where ψ and ϕ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Subsequently, Amini-Harandi and Emami proved another fixed point theorem for contraction type maps in partially ordered metric spaces in [10]. The following class of functions is used in [10].

Let $\Re$ denote the class of functions $\beta : [0,\infty) \longrightarrow [0,1)$ which satisfies the condition $\beta(t_n) \longrightarrow 1 \Rightarrow t_n \longrightarrow 0$.

Theorem 1.3 [10] *Let $(X,\leq)$ be a partially ordered set, and suppose that there exists a metric d such that (X,d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose that there exists $\beta \in \Re$ such that*

$$d\big(f(x),f(y)\big) \leq \beta\big(d(x,y)\big)d(x,y) \quad \text{for each } x,y \in X \text{ with } x \geq y.$$

Assume that either f is continuous or M is such that if an increasing sequence $x_n \to x \in X$, then $x_n \leq x$, $\forall n$. Besides, if for each $x,y \in X$, there exists $z \in m$ which is comparable to x and y, then f has a unique fixed point.

The purpose of this paper is to extend the results of [10, 11, 19] and to obtain a new contraction mapping principle in partially ordered metric spaces. The result is more generalized than the results of [10, 11, 19] and other works. The main theorems can be used to investigate a large class of nonlinear problems. In this paper, we also present some applications to first- and second-order ordinary differential equations.

2 Main results

We first recall the following notion of a monotone nondecreasing function in a partially ordered set.

Definition 2.1 If $(X,\leq)$ is a partially ordered set and $T : X \to X$, we say that T is monotone nondecreasing if $x,y \in X$, $x \leq y \Rightarrow T(x) \leq T(y)$.

This definition coincides with the notion of a nondecreasing function in the case where $X = R$ and $\leq$ represents the usual total order in R.

We shall need the following lemma in our proving.

Lemma 2.1 *If ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.*

Proof Since $\phi(t) < \psi(t)$ and ϕ, ψ are continuous, we have

$$0 \leq \phi(0) = \lim_{t\to 0} \phi(t) \leq \lim_{t\to 0} \psi(t) = \psi(0) = 0.$$

This finishes the proof. □

In what follows, we prove the following theorem which is the generalized type of Theorem 1.1-1.3.

Theorem 2.1 *Let X be a partially ordered set and suppose that there exists a metric d in x such that (X,d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big), \quad \forall x \geq y,$$

where ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Since T is a nondecreasing function, we obtain, by induction, that

$$x_o \leq Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \cdots \tag{1}$$

Put $x_{n+1} = Tx_n$. Then for each integer $n \geq 1$, as the elements x_{n+1} and x_n are comparable, from (1) we get

$$\psi\big(d(x_{n+1},x_n)\big) = \psi\big(d(Tx_n,Tx_{n-1})\big) \leq \phi\big(d(x_n,x_{n-1})\big). \tag{2}$$

Using the condition of Theorem 2.1, we have

$$d(x_{n+1},x_n) < d(x_n,x_{n-1}). \tag{3}$$

Hence the sequence $d(x_{n+1},x_n)$ is decreasing, and consequently, there exists $r \geq 0$ such that

$$d(x_{n+1},x_n) \to r,$$

as $n \to \infty$. Letting $n \to \infty$ in (2), we get

$$\psi(r) \leq \phi(r).$$

By using the condition of Theorem 2.1, we have $r = 0$, and hence

$$d(x_{n+1},x_n) \to 0, \tag{4}$$

as $n \to \infty$. In what follows, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{5}$$

for all $k \geq 1$. Further, corresponding to m_k, we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ satisfying (5). Then

$$d(x_{n_{k-1}}, x_{m_{k-1}}) < \varepsilon. \tag{6}$$

From (5) and (6), we have

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{m_k}) < d(x_{n_k}, x_{n_{k-1}}) + \varepsilon.$$

Letting $k \to \infty$ and using (4), we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{7}$$

By using the triangular inequality, we have

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k}),$$
$$d(x_{n_{k-1}}, x_{m_{k-1}}) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_{k-1}}).$$

Letting $k \to \infty$ in the above two inequalities and (4) and (7), we have

$$\lim_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) = \varepsilon. \tag{8}$$

As $n_k > m_k$ and $x_{n_{k-1}}$ and $x_{m_{k-1}}$ are comparable, using (1), we have

$$\psi\bigl(d(x_{n_k}, x_{m_k})\bigr) \leq \phi\bigl(d(x_{n_{k-1}}, x_{m_{k-1}})\bigr).$$

Letting $k \to \infty$ and taking into account (7) and (8), we have

$$\psi(\varepsilon) \leq \phi(\varepsilon).$$

From the condition of Theorem 2.1, we get $\varepsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and, since X is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Moreover, the continuity of T implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz$$

and this proves that z is a fixed point. This completes the proof. □

In what follows, we prove that Theorem 2.1 is still valid for T not necessarily being continuous, assuming the following hypothesis in X:

$$\text{If } (x_n) \text{ is a nondecreasing sequence in } X \text{ such that } x_n \to x, \text{ then } x_n \leq x \text{ for all } n \in N. \tag{9}$$

Theorem 2.2 *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies (9). Let $T : X \to X$ be a nondecreasing mapping such that*

$$\psi\big(d(Tx, Ty)\big) \leq \phi\big(d(x, y)\big), \quad \forall x \geq y,$$

where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with the conditions $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Following the proof of Theorem 2.1, we only have to check that $T(z) = z$. As (x_n) is a nondecreasing sequence in X and $\lim_{n\to\infty} x_n = z$, the condition (9) gives us that $x_n \leq z$ for every $n \in N$, and consequently,

$$\psi\big(d\big(x_{n+1}, T(z)\big)\big) = \psi\big(d\big(T(x_n), T(z)\big)\big) \leq \phi\big(d(x_n, z)\big).$$

Letting $n \to \infty$ and taking into account that ψ is an altering distance function, we have

$$\psi\big(d\big(z, T(z)\big)\big) \leq \phi(0).$$

Using Lemma 2.1, we have $\phi(0) = 0$, which implies $\Psi(d(z, T(z))) = 0$. Thus $d(z, T(z)) = 0$ or equivalently, $T(z) = z$. □

Now, we present an example where it can be appreciated that the hypotheses in Theorems 2.1 and Theorems 2.2 do not guarantee the uniqueness of the fixed point. The example appears in [17].

Let $X = \{(1, 0), (0, 1)\} \subset R^2$ and consider the usual order $(x, y) \leq (z, t) \Leftrightarrow x \leq z$, $y \leq t$. Thus, (x, y) is a partially ordered set whose different elements are not comparable. Besides, (X, d_2) is a complete metric space and d_2 is the Euclidean distance. The identity map $T(x, y) = (x, y)$ is trivially continuous and nondecreasing, and the condition (9) of Theorem 2.2 is satisfied since the elements in X are only comparable to themselves. Moreover, $(1, 0) \leq T(1, 0) = (1, 0)$ and T has two fixed points in X.

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorems 2.1 and 2.2. This condition is as follows:

$$\text{for } x, y \in X, \text{ there exists a lower bound or an upper bound.} \tag{10}$$

In [17], it is proved that the condition (10) is equivalent to

$$\text{for } x, y \in X, \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{11}$$

Theorem 2.3 *Adding the condition (11) to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), we obtain the uniqueness of the fixed point of T.*

Proof Suppose that there exist $z, y \in X$ which are fixed points. We distinguish the following two cases:

Case 1. If y is comparable to z, then $T^n(y) = y$ is comparable to $T^n(z) = z$ for $n = 0, 1, 2, \ldots$ and

$$\begin{aligned}\psi\bigl(d(z,y)\bigr) &= \psi\bigl(d\bigl(T^n(z), T^n(y)\bigr)\bigr)\\ &\le \phi\bigl(d\bigl(T^{n-1}(z), T^{n-1}(y)\bigr)\bigr)\\ &\le \phi\bigl(d(z,y)\bigr).\end{aligned}$$

By the condition $\psi(t) > \phi(t)$ for $t > 0$, we obtain $d(z, y) = 0$ and this implies $z = y$.

Case 2. If y is not comparable to z, then there exists $x \in X$ comparable to y and z. Monotonicity of T implies that $T^n(x)$ is comparable to $T^n(y)$ and to $T^n(z) = z$, for $n = 0, 1, 2, \ldots$. Moreover,

$$\begin{aligned}\psi\bigl(d\bigl(z, T^n(x)\bigr)\bigr) &= \psi\bigl(d\bigl(T^n(z), T^n(x)\bigr)\bigr)\\ &\le \phi\bigl(d\bigl(T^{n-1}(z), T^{n-1}(x)\bigr)\bigr)\\ &= \phi\bigl(d\bigl(z, T^{n-1}(x)\bigr)\bigr).\end{aligned} \tag{12}$$

Hence, ψ is an altering distance function and the condition of $\psi(t) > \phi(t)$ for $t > 0$. This gives us that $\{d(z, f^n(x))\}$ is a nonnegative decreasing sequence, and consequently, there exists γ such that

$$\lim_{n\to\infty} d\bigl(z, T^n(x)\bigr) = \gamma.$$

Letting $n \to \infty$ in (12) and taking into account that ψ and Φ are continuous functions, we obtain

$$\psi(\gamma) \le \phi(\gamma).$$

This and the condition of Theorem 2.1 implies $\phi(\gamma) = 0$, and consequently, $\gamma = 0$.

Analogously, it can be proved that

$$\lim_{n\to\infty} d\bigl(y, T^n(x)\bigr) = 0.$$

Finally, as

$$\lim_{n\to\infty} d\bigl(z, T^n(x)\bigr) = \lim_{n\to\infty} d\bigl(y, T^n(x)\bigr) = 0,$$

the uniqueness of the limit gives us $y = z$. This finishes the proof. □

Remark 2.1 Under the assumption of Theorem 2.3, it can be proved that for every $x \in X$, $\lim_{n\to\infty} T^n(x) = z$, where z is the fixed point (*i.e.*, the operator f is Picard).

Remark 2.2 Theorem 1.1 is a particular case of Theorem 2.1 for ψ, the identity function, and $\phi(x) = x - \psi(x)$.

Theorem 1.2 is a particular case of Theorem 2.1 for $\phi(x) = \psi(x) - \phi_{1.2}(x)$, $\phi_{1.2}$ is an altering function in Theorem 1.2. Theorem 1.3 is a particular case of Theorem 2.1 for ψ, the identity function, and $\phi(x) = \psi(x)x$.

3 Application to ordinary differential equations

In this section, we present two examples where our Theorems 2.2 and 2.3 can be applied. The first example is inspired by [17]. We study the existence of a solution for the following first-order periodic problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T] \\ u(0) = u(T), \end{cases} \tag{13}$$

where $T > 0$ and $f : I \times R \longrightarrow R$ is a continuous function. Previously, we considered the space $C(I)$ ($I = [0, T]$) of continuous functions defined on I. Obviously, this space with the metric given by

$$d(x, y) = \sup\{|x(t) - y(t)| : t \in I\}, \quad \text{for } x, y \in C(I)$$

is a complete metric space. $C(I)$ can also be equipped with a partial order given by

$$x, y \in C(I), \quad x \leq y \quad \Leftrightarrow \quad x(T) \leq y(t) \quad \text{for } t \in I.$$

Clearly, $(C(I), \leq)$ satisfies the condition (10) since for $x, y \in C(I)$, the functions $\max\{x, y\}$ and $\min\{x, y\}$ are the least upper and the greatest lower bounds of x and y, respectively. Moreover, in [17] it is proved that $(C(I), \leq)$ with the above mentioned metric satisfies the condition (9).

Now, we give the following definition.

Definition 3.1 A lower solution for (13) is a function $\alpha \in C^{(1)}(I)$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), & \text{for } t \in I, \\ \alpha(0) \leq \alpha(T). \end{cases}$$

Theorem 3.1 *Consider the problem (13) with $f : I \times R \longrightarrow R$ continuous, and suppose that there exist $\lambda, \alpha > 0$ with*

$$\alpha \leq \left(\frac{2\lambda(e^{\lambda T} - 1)}{T(e^{\lambda T} + 1)}\right)^{\frac{1}{2}}$$

such that for $x, y \in R$ with $x \geq y$,

$$0 \leq f(t, x) + \lambda x - [f(t, y) + \lambda y] \leq \alpha\sqrt{\ln[(x - y)^2 + 1]}.$$

Then the existence of a lower solution for (13) provides the existence of a unique solution of (13).

Proof The problem (13) can be written as

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), & \text{for } t \in I = [0, T], \\ u(0) = u(T). \end{cases}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\,ds,$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s < t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t < s \le T. \end{cases}$$

Define $F : C(I) \to C(I)$ by

$$(Fu)(t) = \int_0^T G(t,s)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\,ds.$$

Note that if $u \in C(I)$ is a fixed point of F, then $u \in C^1(I)$ is a solution of (13). In what follows, we check that the hypotheses in Theorems 2.2 and 2.3 are satisfied. The mapping F is nondecreasing for $u \ge v$; using our assumption, we can obtain

$$f(t,u) + \lambda u \ge f(t,v) + \lambda v,$$

which implies, since $G(t,s) > 0$, that for $t \in I$,

$$\begin{aligned} (Fu)(t) &= \int_0^T G(t,s)\big[f\big(s, u(s)\big) + \lambda u(s)\big]\,ds \\ &\ge \int_0^T G(t,s)\big[f\big(s, v(s)\big) + \lambda v(s)\big]\,ds = (Fv)(t). \end{aligned}$$

Besides, for $u \ge v$, we have

$$\begin{aligned} d(Fu, Fv) &= \sup_{t\in I}\big|(Fu)(t) - (Fv)(t)\big| \\ &= \sup_{t\in I}\big((Fu)(t) - (Fv)(t)\big) \\ &= \sup_{t\in I}\int_0^T G(t,s)\big[f\big(s,u(s)\big) + \lambda u(s) - f\big(s,v(s)\big) - \lambda v(s)\big]\,ds \\ &\le \sup_{t\in I}\int_0^T G(t,s)\alpha\sqrt{\ln\big[\big(u(s)-v(s)\big)^2 + 1\big]}\,ds. \end{aligned} \tag{14}$$

Using the Cauchy-Schwarz inequality in the last integral, we get

$$\begin{aligned} &\int_0^T G(t,s)\alpha\sqrt{\ln\big[\big(u(s)-v(s)\big)^2 + 1\big]}\,ds \\ &\quad\le \left(\int_0^T G(t,s)^2\,ds\right)^{\frac{1}{2}} \left(\int_0^T \alpha^2 \ln\big[\big(u(s)-v(s)\big)^2 + 1\big]\,ds\right)^{\frac{1}{2}}. \end{aligned} \tag{15}$$

The first integral gives us

$$\begin{aligned}\int_0^T G(t,s)^2\,ds &= \int_0^t G(t,s)^2\,ds + \int_t^T G(t,s)^2\,ds \\ &= \int_0^t \frac{e^{2\lambda(T+s-t)}}{(e^{\lambda T}-1)^2}\,ds + \int_t^T \frac{e^{2\lambda(s-t)}}{(e^{\lambda T}-1)^2}\,ds \\ &= \frac{1}{2\lambda(e^{\lambda T}-1)^2}(e^{2\lambda T-1)} \\ &= \frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}. \end{aligned} \tag{16}$$

The second integral in (15) gives us the following estimate:

$$\begin{aligned}\int_0^T \alpha^2 \ln\big[\big(u(s)-v(s)\big)^2+1\big]\,ds &\le \alpha^2 \ln\big[\|u-v\|^2+1\big]\cdot T \\ &= \alpha^2 \ln\big[d(u,v)^2+1\big]\cdot T. \end{aligned} \tag{17}$$

Taking into account (14)-(17), we have

$$\begin{aligned} d(Fu,Fv) &\le \sup_{t\in I}\left(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\right)^{\frac{1}{2}} \cdot \big(\alpha^2 \ln\big[d(u,v)^2+1\big]\cdot T\big)^{\frac{1}{2}} \\ &= \left(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\right)^{\frac{1}{2}} \cdot \alpha \cdot \sqrt{T} \cdot \big(\ln\big[d(u,v)^2+1\big]\big)^{\frac{1}{2}}, \end{aligned}$$

and from the last inequality, we obtain

$$d(Fu,Fv)^2 \le \frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)} \cdot \alpha^2 \cdot T \cdot \ln\big[d(u,v)^2+1\big]$$

or equivalently,

$$2\lambda\big(e^{\lambda T}-1\big)d(Fu,Fv)^2 \le \big(e^{\lambda T}+1\big)\cdot \alpha^2 \cdot T \cdot \ln\big[d(u,v)^2+1\big].$$

By our assumption, as

$$\alpha \le \left(\frac{2\lambda(e^{\lambda T}-1)}{T(e^{\lambda T}+1)}\right)^{\frac{1}{2}},$$

the last inequality gives us

$$2\lambda\big(e^{\lambda T}-1\big)d(Fu,Fv)^2 \le 2\lambda\big(e^{\lambda T}-1\big)\cdot \ln\big[d(u,v)^2+1\big],$$

and hence,

$$d(Fu,Fv)^2 \le \ln\big[d(u,v)^2+1\big]. \tag{18}$$

Put $\psi(x) = x^2$ and $\phi = \ln(x^2 + 1)$. Obviously, ψ is an altering distance function, $\psi(x)$ and $\phi(x)$ satisfy the condition of $\psi(x) > \phi(x)$ for $x > 0$. From (18), we obtain for $u \geq v$,

$$\psi\bigl(d(Fu, Fv)\bigr) \leq \phi\bigl(d(u, v)\bigr).$$

Finally, let $\alpha(t)$ be a lower solution for (13). We claim that $\alpha \leq F(\alpha)$. In fact

$$\alpha'(t) + \lambda\alpha(t) \leq f\bigl(t, \alpha(t)\bigr) + \lambda\alpha(t), \quad \text{for } t \in I.$$

Multiplying by $e^{\lambda t}$

$$\bigl(\alpha(t)e^{\lambda t}\bigr)' \leq \bigl[f\bigl(t, \alpha(t)\bigr) + \lambda\alpha(t)\bigr]e^{\lambda t}, \quad \text{for } t \in I,$$

we get

$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t \bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]e^{\lambda s}\,ds, \quad \text{for } t \in I. \tag{19}$$

As $\alpha(0) \leq \alpha(T)$, the last inequality gives us

$$\alpha(0)e^{\lambda t} \leq \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T \bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]e^{\lambda s}\,ds,$$

and so

$$\alpha(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]\,ds.$$

This and (19) give us

$$\alpha(t)e^{\lambda t} \leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1}\bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]\,ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]\,ds$$

and consequently,

$$\begin{aligned}
\alpha(t) &\leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}\,ds + \int_0^t \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}\bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]\,ds \\
&= \int_0^T G(t, s)\bigl[f\bigl(s, \alpha(s)\bigr) + \lambda\alpha(s)\bigr]\,ds \\
&= (F\alpha)(t), \quad \text{for } t \in I.
\end{aligned}$$

Finally, Theorems 2.2 and 2.3 give that F has an unique fixed point. □

The second example where our results can be applied is the following two-point boundary value problem of the second-order differential equation

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x), & x \in [0, \infty], t \in [0, 1], \\ x(0) = x(1) = 0. \end{cases} \tag{20}$$

It is well known that $x \in C^2[0,1]$, a solution of (20), is equivalent to $x \in C[0,1]$, a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\, ds, \quad \text{for } t \in [0,1],$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{21}$$

Theorem 3.2 *Consider the problem (20) with $f : I \times R \to [0,\infty)$ continuous and nondecreasing with respect to the second variable, and suppose that there exists $0 \le \alpha \le 8$ such that for $x, y \in R$ with $y \ge x$,*

$$f(t,y) - f(t,x) \le \alpha \sqrt{\ln\bigl[(y-x)^2 + 1\bigr]}. \tag{22}$$

Then our problem (20) has a unique nonnegative solution.

Proof Consider the cone

$$P = \bigl\{x \in C[0,1] : x(t) \ge 0\bigr\}.$$

Obviously, (P,d) with $d(x,y) = \sup\{|x(t) - y(t)| : t \in [0,1]\}$ is a complete metric space. Consider the operator given by

$$(Tx)(t) = \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\, ds, \quad \text{for } x \in P,$$

where $G(t,s)$ is the Green function appearing in (21).

As f is nondecreasing with respect to the second variable, then for $x, y \in P$ with $y \ge x$ and $t \in [0,1]$, we have

$$(Ty)(t) = \int_0^1 G(t,s) f\bigl(s, y(s)\bigr)\, ds \ge \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\, ds \ge (Tx)(t),$$

and this proves that T is a nondecreasing operator.

Besides, for $y \ge x$ and taking into account (22), we can obtain

$$\begin{aligned} d(Ty, Tx) &= \sup_{t\in[0,1]} \bigl|(Ty)(t) - (Tx)(t)\bigr| \\ &= \sup_{t\in[0,1]} \bigl((Ty)(t) - (Tx)(t)\bigr) \\ &= \sup_{t\in[0,1]} \int_0^1 G(t,s)\bigl(f\bigl(s, y(s)\bigr) - f\bigl(s, x(s)\bigr)\bigr)\, ds \\ &\le \sup_{t\in[0,1]} \int_0^1 G(t,s) \alpha \sqrt{\ln\bigl[(y-x)^2 + 1\bigr]} \end{aligned}$$

$$\begin{aligned} &\le \sup_{t\in[0,1]} \int_0^1 G(t,s)\alpha\sqrt{\ln\bigl[\|y-x\|^2+1\bigr]}\,ds \\ &= \alpha\sqrt{\ln\bigl[\|y-x\|^2+1\bigr]}\sup_{t\in[0,1]}\int_0^1 G(t,s)\,ds. \end{aligned} \tag{23}$$

It is easy to verify that

$$\int_0^1 G(t,s)\,ds = \frac{-t^2}{2} + \frac{t}{2}$$

and that

$$\sup_{t\in[0,1]}\int_0^1 G(t,s)\,ds = \frac{1}{8}.$$

These facts, the inequality (23), and the hypothesis $0 < \alpha \le 8$ give us

$$\begin{aligned} d(Ty,Tx) &\le \frac{\alpha}{8}\sqrt{\ln\bigl[\|y-x\|^2+1\bigr]} \\ &\le \sqrt{\ln\bigl[\|y-x\|^2+1\bigr]} = \sqrt{\ln\bigl[d(x,y)^2+1\bigr]}. \end{aligned}$$

Hence

$$d(Ty,Tx)^2 \le \ln\bigl[d(x,y)^2+1\bigr].$$

Put $\psi(x) = x^2$, $\phi(x) = \ln(x^2+1)$; obviously, ψ is an altering distance function, ψ and ϕ satisfy the condition $\psi(x) > \phi(x)$, for $x > 0$. From the last inequality, we have

$$\psi\bigl(d(Tx,Ty)\bigr) \le \phi\bigl(d(x,y)\bigr).$$

Finally, as f and G are nonnegative functions,

$$(T0)(t) = \int_0^1 G(t,s)f(s,0)\,ds \ge 0.$$

Theorems 2.2 and 2.3 tell us that F has a unique nonnegative solution. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally to the writing of the present article. All authors read and approved the final manuscript.

Acknowledgements
This project is supported by the National Natural Science Foundation of China under the grant (11071279).

Received: 28 April 2012 Accepted: 30 August 2012 Published: 18 September 2012

References

1. Khan, MS, Swaleh, M, Sessa, S: Fixed point theorems by altering distances between the points. Bull. Aust. Math. Soc. **30**(1), 1-9 (1984)
2. Dhutta, P, Choudhury, B: A generalization of contractions in partially ordered metric spaces. Appl. Anal. **87**, 109-116 (2008)
3. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683-2693 (2001)
4. Aydi, H, Karapnar, E, Bessem, S: Fixed point theorems in ordered abstract spaces. Fixed Point Theory Appl. **2012**, 76 (2012)
5. Nieto, JJ, Pouso, RL, Rodríguez-López, R: A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. Proc. Am. Math. Soc. **135**, 2505-2517 (2007)
6. Gordji, ME, Baghani, H, Kim, GH: Coupled fixed point theorems for contraction mapping induced by cone ball-metric in partially ordered spaces. Discrete Dyn. Nat. Soc. **2012**, art. no. 981517 (2012)
7. Sintunavarat, W, Cho, YJ, Kumam, P: Existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions. Fixed Point Theory Appl. **2012**, 128 (2012)
8. Obersnel, F, Omari, P, Rivetti, S: A generalization of Geraghty's theorem in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal., Real World Appl. **13**, 2830-2852 (2012)
9. Sastry, K, Babu, G: Some fixed point theorems by altering distance between the points. Indian J. Pure Appl. Math. **30**, 641-647 (1999)
10. Amini-Harandi, A, Emami, H: A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. Nonlinear Anal. **72**, 2238-2242 (2010)
11. Harjani, J, Sadarangni, K: Fixed point theorems for weakly contraction mappings in partially ordered sets. Nonlinear Anal. **71**, 3403-3410 (2009)
12. Burgic, D, Kalabusic, S, Kulenovic, M: Global attractivity results for mixed monotone mappings in partially ordered complete metric spaces. Fixed Point Theory Appl. **2009**, Article ID 762478 (2009)
13. Ciric, L, Cakid, N, Rajovic, M, Uma, J: Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2008**, Article ID 131294 (2008)
14. Gnana Bhaskar, T, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
15. Lakshmikantham, V, Ciric, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
16. Nieto, JJ, Rodriguez-López, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)
17. Nieto, JJ, Rodriguez-López, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. **23**, 2205-2212 (2007)
18. O'Regan, D, Petrusel, A: Fixed point theorems for generalized contractions in ordered metric spaces. J. Math. Anal. Appl. **341**, 1241-1252 (2008)
19. Harjani, J, Sadarangni, K: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. **72**, 1188-1197 (2010)

doi:10.1186/1687-1812-2012-152
Cite this article as: Yan et al.: **A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations.** *Fixed Point Theory and Applications* 2012 **2012**:152.