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Fixed points of (ψ,ϕ) contractions on rectangular metric spaces

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Abstract
Existence and uniqueness of fixed points of a general class of (ψ,ϕ) contractive mappings on complete rectangular metric spaces are discussed. One of the theorems is a generalization of a fixed point theorem recently introduced by Lakzian and Samet. Fixed points of (ψ,ϕ) contractions under conditions involving rational expressions are also investigated. Several particular cases and applications as well as an illustrative example are given.
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1 Introduction and preliminaries

Fixed point theory has been one of the most rapidly developing fields in analysis during the last few decades. Wide application potential of this theory has accelerated the research activities which resulted in an enormous increase in publications [1–5]. In a large class of studies the classical concept of a metric space has been generalized in different directions by partly changing the conditions of the metric. Among these generalizations, one can mention the partial metric spaces introduced by Matthews [6, 7] (see also [1, 2, 8–10]), and rectangular metric spaces defined by Branciari [11].

Branciari defined a rectangular metric space (RMS) by replacing the sum at the right-hand side of the triangle inequality by a three-term expression. He also proved an analog of the Banach Contraction Principle. The intriguing nature of these spaces has attracted attention, and fixed points theorems for various contractions on rectangular metric spaces have been established (see, *e.g.*, [12–17]).

In 1969 Boyd and Wong [18] defined a class of contractive mappings called ϕ contractions. In 1997, Alber and Guerre-Delabriere [19] generalized this concept by introducing weak ϕ contraction. A self-mapping T on a metric space (X,d) is said to be weak ϕ contractive if there exists a map $\phi:[0,+\infty)\longrightarrow[0,+\infty)$ with $\phi(0)=0$ and $\phi(t)>0$ for all $t>0$ such that

$$d(Tx,Ty)\leq d(x,y)-\phi\bigl(d(x,y)\bigr), \tag{1.1}$$

for all $x,y\in X$. Contractions of this type have been studied by many authors (see, *e.g.*, [20, 21]). The larger class of (ψ,ϕ) weakly contractive mappings has also been a subject of interest (see, *e.g.*, [10, 22–25]).

In a recent paper, Lakzian and Samet [26] stated and proved a fixed point theorem for the (ψ,ϕ) weakly contractive mappings on complete rectangular metric spaces. They also provided interesting examples as particular cases of such mappings. In this paper, we generalize the result of Lakzian and Samet [26] and, in addition, investigate maps satisfying rational type contractive conditions. We also give applications and an example.

We state some basic definitions and notations to be used throughout this paper. Rectangular metric spaces are defined as follows.

Definition 1 ([11]) Let X be a nonempty set and let $d : X \times X \longrightarrow [0,\infty]$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of which is different from x and y.

$$\begin{aligned} &(\mathrm{RM1}) \quad d(x,y) = 0 \quad \text{if and only if } x = y,\\ &(\mathrm{RM2}) \quad d(x,y) = d(y,x),\\ &(\mathrm{RM3}) \quad d(x,y) \le d(x,u) + d(u,v) + d(v,y). \end{aligned} \tag{1.2}$$

Then the map d is called a rectangular metric and the pair (X,d) is called a rectangular metric space.

The concepts of convergence, Cauchy sequence and completeness in a RMS are defined below.

Definition 2

(1) A sequence $\{x_n\}$ in a RMS (X,d) is RMS convergent to a limit x if and only if $d(x_n,x) \to 0$ as $n \to \infty$.
(2) A sequence $\{x_n\}$ in a RMS (X,d) is RMS Cauchy if and only if for every $\varepsilon > 0$ there exists a positive integer $N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $n > m > N(\varepsilon)$.
(3) A RMS (X,d) is called complete if every RMS Cauchy sequence in X is RMS convergent.

We also use the following modified notations of Lakzian and Samet [26].

Let Ψ denote the set of all continuous functions $\psi : [0,+\infty) \to [0,+\infty)$ for which $\psi(t) = 0$ if and only if $t = 0$. Nondecreasing functions which belong to the class Ψ are also known as altering distance functions (see [27]).

In their paper, Lakzian and Samet [26] stated the following fixed point theorem.

Theorem 3 *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\psi\bigl(d(Tx,Ty)\bigr) \le \psi\bigl(d(x,y)\bigr) - \phi\bigl(d(x,y)\bigr)$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where ψ is nondecreasing. Then T has a unique fixed point in X.

In this paper, we give a generalization of Theorem 3 for a larger class of (ψ,ϕ) weakly contractive mappings and improve the results obtained by Lakzian and Samet. Moreover, Theorem 3 can be considered as a particular case of our generalized theorem.

2 Main results and applications

We present our main results in this section. First, we state the following fixed point theorem.

Theorem 4 *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\psi\bigl(d(Tx, Ty)\bigr) \le \psi\bigl(M(x,y)\bigr) - \phi\bigl(M(x,y)\bigr) + Lm(x,y) \tag{2.1}$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where $L > 0$, the function ψ is nondecreasing and

$$\begin{aligned} M(x,y) &= \max\bigl\{d(x,y), d(x,Tx), d(y,Ty)\bigr\}, \\ m(x,y) &= \min\bigl\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\}. \end{aligned} \tag{2.2}$$

Then T has a unique fixed point in X.

Proof First, we prove the existence part. Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\} \subset X$ as

$$x_n = Tx_{n-1}, \quad n = 1, 2, 3, \ldots.$$

Assume that $x_n \neq x_{n+1} = Tx_n$ for all $n \ge 1$. Substitute $x = x_{n-1}$ and $y = x_n$ in (2.1) and note that

$$m(x_{n-1}, x_n) = \min\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\bigr\} = 0.$$

Then we obtain,

$$\begin{aligned} \psi\bigl(d(Tx_{n-1}, Tx_n)\bigr) &= \psi\bigl(d(x_n, x_{n+1})\bigr) \\ &\le \psi\bigl(M(x_{n-1}, x_n)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr) + Lm(x_{n-1}, x_n) \\ &= \psi\bigl(M(x_{n-1}, x_n)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\bigl\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\} \\ &= \max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\}. \end{aligned}$$

If $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$, then we have

$$\psi\bigl(d(x_n, x_{n+1})\bigr) \le \psi\bigl(d(x_n, x_{n+1})\bigr) - \phi\bigl(d(x_n, x_{n+1})\bigr),$$

which implies $\phi(d(x_n, x_{n+1})) = 0$, and hence $d(x_n, x_{n+1}) = 0$. Then $x_n = x_{n+1} = Tx_n$, which contradicts the initial assumption. Therefore, we must have $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, that is,

$$\psi\bigl(d(x_n, x_{n+1})\bigr) \le \psi\bigl(d(x_{n-1}, x_n)\bigr) - \phi\bigl(d(x_{n-1}, x_n)\bigr) \le \psi\bigl(d(x_{n-1}, x_n)\bigr). \tag{2.4}$$

Since ψ is nondecreasing, then $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \geq 1$, that is, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and positive. Hence, it converges to a positive number, say $s > 0$. Taking limit as $n \to \infty$ in (2.4), we obtain

$$\psi(s) \leq \psi(s) - \phi(s),$$

which leads to $\phi(s) = 0$ and hence to $s = 0$. Thus,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Next, we will show that T has a periodic point, that is, there exist a positive integer p and a point $z \in X$ such that $z = T^p z$. Assume the contrary, that is, T has no periodic point. Then, all elements of the sequence $\{x_n\}$ are distinct, *i.e.*, $x_n \neq x_m$ for all $n \neq m$. Suppose also that $\{x_n\}$ is not a RMS Cauchy sequence. Therefore, there exists $\varepsilon > 0$ for which one can find subsequences $\{x_{n(i)}\}$ and $\{x_{m(i)}\}$ of $\{x_n\}$ with $n(i) > m(i) > i$ such that

$$d(x_{m(i)}, x_{n(i)}) \geq \varepsilon, \tag{2.6}$$

where $n(i)$ is the smallest integer satisfying (2.6), that is,

$$d(x_{m(i)}, x_{n(i)-1}) < \varepsilon. \tag{2.7}$$

We apply the rectangular inequality (RM3) and use (2.6) and (2.7) to obtain

$$\begin{aligned} \varepsilon &\leq d(x_{m(i)}, x_{n(i)}) \\ &\leq d(x_{m(i)}, x_{n(i)-2}) + d(x_{n(i)-2}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{n(i)}) \\ &\leq \varepsilon + d(x_{n(i)-2}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{n(i)}). \end{aligned} \tag{2.8}$$

Taking limit as $i \to \infty$ in (2.8) and using (2.5), we get

$$\lim_{i\to\infty} d(x_{n(i)}, x_{m(i)}) = \varepsilon. \tag{2.9}$$

Employing the rectangular inequality (RM3) once again, we write the following inequalities:

$$\begin{aligned} d(x_{n(i)}, x_{m(i)}) &\leq d(x_{n(i)}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{m(i)-1}) + d(x_{m(i)-1}, x_{m(i)}), \\ d(x_{n(i)-1}, x_{m(i)-1}) &\leq d(x_{n(i)-1}, x_{n(i)}) + d(x_{n(i)}, x_{m(i)}) + d(x_{m(i)}, x_{m(i)-1}), \end{aligned} \tag{2.10}$$

from which we obtain

$$\varepsilon \leq \lim_{i\to\infty} d(x_{n(i)-1}, x_{m(i)-1}) \leq \varepsilon, \tag{2.11}$$

using (2.5) and (2.9); and therefore,

$$\lim_{i\to\infty} d(x_{n(i)-1}, x_{m(i)-1}) = \varepsilon. \tag{2.12}$$

Now we substitute $x = x_{n(i)-1}$ and $y = x_{m(i)-1}$ in (2.1), which yields

$$\begin{aligned}\psi\big(d(Tx_{n(i)-1}, Tx_{m(i)-1})\big) &= \psi\big(d(x_{n(i)}, x_{m(i)})\big)\\ &\leq \psi\big(M(x_{n(i)-1}, x_{m(i)-1})\big) - \phi\big(M(x_{n(i)-1}, x_{m(i)-1})\big)\\ &\quad + Lm(x_{n(i)-1}, x_{m(i)-1}),\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}&M(x_{n(i)-1}, x_{m(i)-1}) = \max\{d(x_{n(i)-1}, x_{m(i)-1}), d(x_{n(i)-1}, x_{n(i)}), d(x_{m(i)-1}, x_{m(i)})\},\\ &m(x_{n(i)-1}, x_{m(i)-1}) = \min\{d(x_{n(i)-1}, x_{n(i)}), d(x_{m(i)-1}, x_{m(i)}), d(x_{n(i)-1}, x_{m(i)}), d(x_{m(i)-1}, x_{n(i)})\}.\end{aligned}$$

Clearly, as $i \to \infty$ we have $M(x_{n(i)-1}, x_{m(i)-1}) \to \max\{\varepsilon, 0, 0\} = \varepsilon$ and $m(x_{n(i)-1}, x_{m(i)-1}) \to 0$ due to (2.5) and (2.12). Then letting $i \to \infty$ in (2.13), we get

$$0 \leq \psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon) + 0, \tag{2.14}$$

which implies $\phi(\varepsilon) = 0$, and hence $\varepsilon = 0$. This contradicts the assumption that $\{x_n\}$ is not RMS Cauchy, thus, $\{x_n\}$ must be RMS Cauchy. Since (X, d) is complete, then $\{x_n\}$ converges to a limit, say $u \in X$. Let $x = x_n$ and $y = u$ in (2.1). This gives

$$\psi\big(d(Tx_n, Tu)\big) \leq \psi\big(M(x_n, u)\big) - \phi\big(M(x_n, u)\big) + Lm(x_n, u), \tag{2.15}$$

with

$$\begin{aligned}&M(x_n, u) = \max\{d(x_n, u), d(x_n, Tx_n), d(u, Tu)\},\\ &m(x_n, u) = \min\{d(x_n, Tx_n), d(u, Tu), d(x_n, Tu), d(u, Tx_n)\}.\end{aligned}$$

Note that $m(x_n, u) \to 0$ as $n \to \infty$. If $M(x_n, u) = d(x_n, u)$ or $M(x_n, u) = d(x_n, x_{n+1})$, then we have $M(x_n, u) \to 0$ as $n \to \infty$, due to (2.5) and the fact that the sequence $\{x_n\}$ converges to u. Regarding the continuity of ψ and ϕ, we have

$$0 \leq \psi\Big(\lim_{n\to\infty} d(Tx_n, Tu)\Big) \leq \lim_{n\to\infty}\big[\psi\big(M(x_n, u)\big) - \phi\big(M(x_n, u)\big) + Lm(x_n, u)\big] = 0.$$

Hence, $\lim_{n\to\infty} d(Tx_n, Tu) = 0$, that is, in either case, we end up with $x_{n+1} = Tx_n \to Tu$. Since X is Hausdorff, we deduce that $u = Tu$. If, on the other hand, $M(x_n, u) = d(u, Tu)$ passing to limit as $n \to \infty$ in (2.15), we get $\phi(M(u, Tu)) = 0$, hence $d(u, Tu) = 0$, that is $u = Tu$. This result contradicts the assumption that T has no periodic points. Therefore, T has a periodic point, that is, $z = T^p z$ for some $z \in X$ and a positive integer p.

If $p = 1$, then $z = Tz$, so z is a fixed point of T. Let $p > 1$. We claim that the fixed point of T is $T^{p-1}z$. Suppose the contrary, that is, $T^{p-1}z \neq T(T^{p-1}z)$. Then $d(T^{p-1}z, T^p x) > 0$ and so is $\phi(d(T^{p-1}z, T^p x))$. Letting $x = T^{p-1}z$ and $y = T^p z$ in (2.1), we have

$$\begin{aligned}\psi\big(d(z, Tz)\big) &= \psi\big(d\big(T^p z, T^{p+1}z\big)\big)\\ &\leq \psi\big(M\big(T^{p-1}z, T^p z\big)\big) - \phi\big(M\big(T^{p-1}z, T^p z\big)\big) + Lm\big(T^{p-1}z, T^p z\big),\end{aligned} \tag{2.16}$$

where

$$M(T^{p-1}z, T^p z) = \max\{d(T^{p-1}z, T^p z), d(T^{p-1}z, T^p z), d(T^p z, T^{p+1}z)\},$$
$$m(T^{p-1}z, T^p z) = \min\{d(T^{p-1}z, T^p z), d(T^p z, T^{p+1}z), d(T^{p-1}z, T^{p+1}z), d(T^p z, T^p z)\} = 0.$$

For $M(T^{p-1}z, T^p z) = d(T^p z, T^{p+1}z)$, (2.16) becomes

$$\psi(d(z, Tz)) = \psi(d(T^p z, T^{p+1}z)) \leq \psi(d(T^p z, T^{p+1}z)) - \phi(d(T^p z, T^{p+1}z)).$$

Thus, we get $\phi(d(T^p z, T^{p+1}z)) = 0$ and hence $d(T^p z, T^{p+1}z) = d(z, Tz) = 0$, which is not possible since $p > 1$. If, on the other hand, $M(T^{p-1}z, T^p z) = d(T^{p-1}z, T^p z)$, then (2.16) turns into

$$\begin{aligned} \psi(d(z, Tz)) = \psi(d(T^p z, T^{p+1}z)) &\leq \psi(d(T^{p-1}z, T^p z)) - \phi(d(T^{p-1}z, T^p z)) \\ &< \psi(d(T^{p-1}z, T^p z)), \end{aligned} \tag{2.17}$$

and taking into account the fact that ψ is nondecreasing, we deduce

$$d(z, Tz) < d(T^{p-1}z, T^p z).$$

Now we write $x = T^{p-2}z$ and $y = T^{p-1}z$ in (2.1) and get

$$\begin{aligned} \psi(d(T^{p-1}z, T^p z)) \leq{}& \psi(M(T^{p-2}z, T^{p-1}z)) - \phi(M(T^{p-2}z, T^{p-1}z)) \\ &+ Lm(T^{p-2}z, T^{p-1}z), \end{aligned} \tag{2.18}$$

where

$$M(T^{p-2}z, T^{p-1}z) = \max\{d(T^{p-2}z, T^{p-1}z), d(T^{p-2}z, T^{p-1}z), d(T^{p-1}z, T^p z)\},$$
$$\begin{aligned} m(T^{p-2}z, T^{p-1}z) = \min\{&d(T^{p-2}z, T^{p-1}z), d(T^{p-1}z, T^p z), \\ &d(T^{p-2}z, T^p z), d(T^{p-1}z, T^{p-1}z)\} = 0. \end{aligned}$$

For $M(T^{p-2}z, T^{p-1}z) = d(T^{p-1}z, T^p z)$, we obtain

$$\psi(d(T^{p-1}z, T^p z)) \leq \psi(d(T^{p-1}z, T^p z)) - \phi(d(T^{p-1}z, T^p z)),$$

which is possible only if $\phi(d(T^{p-1}z, T^p z)) = 0$ and hence, $d(T^{p-1}z, T^p z) = 0$. However, we assumed that $d(T^{p-1}z, T^p z) > 0$. Thus, we must have $M(T^{p-2}z, T^{p-1}z) = d(T^{p-2}z, T^{p-1}z)$, so that

$$\begin{aligned} \psi(d(T^{p-1}z, T^p z)) &\leq \psi(d(T^{p-2}z, T^{p-1}z)) - \phi(d(T^{p-2}z, T^{p-1}z)) \\ &\leq \psi(d(T^{p-2}z, T^{p-1}z)), \end{aligned} \tag{2.19}$$

which implies $d(T^{p-2}z, T^{p-1}z) \geq d(T^{p-1}z, T^p z)$ since ψ is nondecreasing. This leads to

$$0 < d(z, Tz) = d(T^p z, T^{p+1}z) < d(T^{p-1}z, T^p z) \leq d(T^{p-2}z, T^{p-1}z).$$

We continue in this way and end up with the inequality

$$0 < d(z, Tz) = d\big(T^p z, T^{p+1} z\big) < d\big(T^{p-1} z, T^p z\big) \leq d\big(T^{p-2} z, T^{p-1} z\big) \leq \cdots \leq d(z, Tz),$$

which yields $d(z, Tz) < d(z, Tz)$. Therefore, the assumption $d(T^{p-1}z, T^p x) > 0$ is wrong, that is, $d(T^{p-1}z, T^p x) = 0$ and $T^{p-1}z$ is the fixed point of T.

Finally, to prove the uniqueness, we assume that T has two distinct fixed points, say z and w. Then letting $x = z$ and $y = w$ in (2.1), we have

$$\psi\big(d(z, w)\big) = \psi\big(d(Tz, Tw)\big) \leq \psi\big(M(z, w)\big) - \phi\big(M(z, w)\big) + Lm(z, w), \tag{2.20}$$

where

$$\begin{aligned} M(z, w) &= \max\big\{d(z, w), d(z, Tz), d(w, Tw)\big\} = d(z, w), \\ m(z, w) &= \min\big\{d(z, Tz), d(w, Tw), d(w, Tz), d(z, Tw)\big\} = 0. \end{aligned}$$

Thus, we have

$$\psi\big(d(z, w)\big) \leq \psi\big(d(z, w)\big) - \phi\big(d(z, w)\big), \tag{2.21}$$

implying $\phi(d(z, w)) = 0$, and hence $d(z, w) = 0$, which completes the proof of the uniqueness. □

It is worth mentioning that the Theorem 3.1 given in [26] is a particular case of Theorem 4. We next give some consequences of Theorem 4.

Corollary 5 *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\psi\big(d(Tx, Ty)\big) \leq \psi\big(M(x, y)\big) - \phi\big(M(x, y)\big) \tag{2.22}$$

for all $x, y \in X$ and $\psi, \phi \in \Psi$, where the function ψ is nondecreasing and

$$M(x, y) = \max\big\{d(x, y), d(x, Tx), d(y, Ty)\big\}. \tag{2.23}$$

Then T has a unique fixed point in X.

Proof Observe that

$$\begin{aligned} \psi\big(d(Tx, Ty)\big) &\leq \psi\big(M(x, y)\big) - \phi\big(M(x, y)\big) \\ &\leq \psi\big(M(x, y)\big) - \phi\big(M(x, y)\big) \\ &\quad + L\min\big\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\} \end{aligned}$$

for some $L > 0$. Then by Theorem 4, T has a unique fixed point in X. □

Corollary 6 *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$d(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty)\} \tag{2.24}$$

for all $x, y \in X$ and some $0 \le k < 1$. Then T has a unique fixed point in X.

Proof Let $\psi(t) = t$ and $\phi(t) = (1 - k)t$. Then by Corollary 5, T has a unique fixed point. □

Corollary 7 *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\begin{aligned} d(Tx, Ty) \le {} & k\big[d(x, y) + d(x, Tx) + d(y, Ty)\big] \\ & + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \end{aligned} \tag{2.25}$$

for all $x, y \in X$ and some $0 \le k < \frac{1}{3}$ and $L > 0$. Then T has a unique fixed point in X.

Proof Obviously,

$$\begin{aligned} & k\big[d(x, y) + d(x, Tx) + d(y, Ty)\big] + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \\ & \quad \le 3k \max\{d(x, y), d(x, Tx), d(y, Ty)\} + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Let $\psi(t) = t$ and $\phi(t) = (1 - 3k)t$. Then by Theorem 4, T has a unique fixed point. □

Our next corollary is concerned with weak ϕ contractions.

Corollary 8 *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$d(Tx, Ty) \le M(x, y) - \phi\big(M(x, y)\big) + Lm(x, y) \tag{2.26}$$

for all $x, y \in X$, where

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty)\}, \\ m(x, y) &= \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \end{aligned}$$

Then T has a unique fixed point in X.

Proof Let $\psi(t) = t$. Then by Theorem 4, T has a unique fixed point. □

As the second result, we state the following existence and uniqueness theorem under conditions involving rational expressions.

Theorem 9 *Let (X, d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\psi\big(d(Tx, Ty)\big) \le \psi\big(M(x, y)\big) - \phi\big(M(x, y)\big) \tag{2.27}$$

for all $x, y \in X$ *and* $\psi, \phi \in \Psi$, *where* ψ *is nondecreasing and*

$$M(x,y) = \max\left\{d(x,y), d(y,Ty)\frac{1 + d(x,Tx)}{1 + d(x,y)}\right\}. \tag{2.28}$$

Then T *has a unique fixed point in* X.

Proof Let $x_0 \in X$ be an arbitrary point. Define the sequence $\{x_n\} \subset X$ as

$$x_n = Tx_{n-1}, \quad n = 1, 2, 3, \ldots.$$

Assume that $x_n \neq x_{n+1} = Tx_n$ for all $n \geq 1$. Substituting $x = x_{n-1}$ and $y = x_n$ in (2.27), we obtain

$$\psi\bigl(d(Tx_{n-1}, Tx_n)\bigr) = \psi\bigl(d(x_n, x_{n+1})\bigr) \leq \psi\bigl(M(x_{n-1}, x_n)\bigr) - \phi\bigl(M(x_{n-1}, x_n)\bigr), \tag{2.29}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), d(x_n, Tx_n)\frac{1 + d(x_{n-1}, Tx_{n-1})}{1 + d(x_{n-1}, x_n)}\right\} \\ &= \max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\}. \end{aligned}$$

For the rest of the proof, one can follow the same steps as in the proof of Theorem 4. □

Setting $\psi(t) = t$ and $\phi(t) = (1 - k)t$, we obtain the following particular result.

Corollary 10 *Let* (X, d) *be a Hausdorff and complete RMS and let* $T : X \to X$ *be a self-map satisfying*

$$d(Tx, Ty) \leq k \max\left\{d(x,y), d(y,Ty)\frac{1 + d(x,Tx)}{1 + d(x,y)}\right\} \tag{2.30}$$

for all $x, y \in X$ *and some* $k \in [0, 1)$. *Then* T *has a unique fixed point in* X.

We also generalize the applications of Theorem 3.1 in [26] given by Lakzian and Samet. Let Λ be the set of functions $f : [0, +\infty) \to [0, +\infty)$ such that

(1) f is Lebesgue integrable on each compact subset of $[0, +\infty)$;
(2) $\int_0^\varepsilon f(t)\,dt > 0$ for every $\varepsilon > 0$.

For this class of functions, we can state the following results.

Theorem 11 *Let* (X, d) *be a Hausdorff and complete RMS and let* $T : X \to X$ *be a self-map satisfying*

$$\int_0^{d(Tx,Ty)} f(t)\,dt \leq \int_0^{M(x,y)} f(t)\,dt - \int_0^{M(x,y)} g(t)\,dt + Lm(x,y) \tag{2.31}$$

for all $x, y \in X$ *and* $f, g \in \Lambda$, *where* $L > 0$ *and*

$$M(x,y) = \max\bigl\{d(x,y), d(x,Tx), d(y,Ty)\bigr\},$$

$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Then T has a unique fixed point in X.

Proof Let $\psi(t) = \int_0^t f(u)\,du$ and $\phi(t) = \int_0^t g(u)\,du$. Then ψ and ϕ are functions in Ψ, and moreover, the function ψ is nondecreasing. By Theorem 4, T has a unique fixed point. □

Corollary 12 *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\int_0^{d(Tx,Ty)} f(u)\,du \le k\int_0^{M(x,y)} f(u)\,du + Lm(x,y) \tag{2.32}$$

for all $x,y \in X$ and $f \in \Lambda$ and some $0 \le k < 1$, $L > 0$, where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$
$$m(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Then T has a unique fixed point in X.

Proof Let $g(t) = (1-k)f(t)$. Then by Theorem 11, T has a unique fixed point. □

Theorem 13 *Let (X,d) be a Hausdorff and complete RMS and let $T : X \to X$ be a self-map satisfying*

$$\int_0^{d(Tx,Ty)} f(t)\,dt \le \int_0^{M(x,y)} f(t)\,dt - \int_0^{M(x,y)} g(t)\,dt \tag{2.33}$$

for all $x,y \in X$ and $f,g \in \Lambda$, where

$$M(x,y) = \max\left\{d(x,y), d(y,Ty)\frac{1+d(x,Tx)}{1+d(x,y)}\right\}.$$

Then T has a unique fixed point in X.

Proof Let $\psi(t) = \int_0^t f(u)\,du$ and $\phi(t) = \int_0^t g(u)\,du$. Then ψ and ϕ are functions in Ψ, and moreover, the function ψ is nondecreasing. By Theorem 9, T has a unique fixed point. □

Finally, we give an illustrative example of (ψ,ϕ) contraction defined on a generalized metric space.

Example 14 Let $X = A \cup B$, where $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$ and $B = [1,2]$. Define the generalized metric d on X as follows:

$$d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0.3, \qquad d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = 0.2,$$
$$d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = 0.6, \qquad d\left(\frac{1}{2},\frac{1}{2}\right) = d\left(\frac{1}{3},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{5}\right) = 0$$

and

$$d(x,y) = |x-y| \quad \text{if } x,y \in B \text{ or } x \in A, y \in B \text{ or } x \in B, y \in A.$$

It is clear that d does not satisfy the triangle inequality on A. Indeed,

$$0.6 = d\left(\frac{1}{2},\frac{1}{4}\right) \geq d\left(\frac{1}{2},\frac{1}{3}\right) + d\left(\frac{1}{3},\frac{1}{4}\right) = 0.5.$$

Notice that (RM3) holds, so d is a rectangular metric.

Let $T: X \to X$ be defined as

$$Tx = \begin{cases} \frac{1}{5} & \text{if } x \in [1,2], \\ \frac{1}{4} & \text{if } x \in \{\frac{1}{2},\frac{1}{3},\frac{1}{4}\}, \\ \frac{1}{3} & \text{if } x = \frac{1}{5}. \end{cases}$$

Define $\psi(t) = t$ and $\phi(t) = \frac{t}{5}$. Then T satisfies the conditions of Theorem 4 and has a unique fixed point on X, *i.e.*, $x = \frac{1}{4}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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