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Round-off stability for multi-valued maps

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Abstract

An iterative procedure for a map T is said to be stable if the approximate sequence arising in numerical praxis converges to the point anticipated by the theoretical sequence. The study of stability of iterative procedures plays a vital role in computational analysis, game theory, computer programming, and fractal geometry. In generation of fractals, a sequence of approximations produces a stable set attractor only if the corresponding iterative procedure shows a stable behavior. The purpose of this article is to discuss stability of the Picard iterative procedure for a map T satisfying Zamfirescu multi-valued contraction on a metric space.

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Keywords: fixed point, stability of iterative procedures, Picard iterative procedure, Zamfirescu contraction, fractals

1 Introduction

Let (X, d) be a metric space and $T: X \to X$. The solution of a fixed point equation $Tx = x$ for any $x \in X$, is usually approximated by a sequence $\{x_n\}$ in X generated by an iterative procedure $f(T, x_n)$ that converges to a fixed point of T. However, in actual computations, we obtain an approximate sequence $\{y_n\}$ instead of the actual sequence $\{x_n\}$. Indeed, the approximate sequence $\{y_n\}$ is calculated in the following manner. First, we choose an initial approximation $x_0 \in X$. Then we compute $x_1 = f(T, x_0)$. But, due to rounding off or discretization of the function, we get an approximate value y_1, say, which is close enough to x_1, i.e., $y_1 \approx x_1$. Consequently, when computing x_2, we actually compute $y_2 \approx x_2$. In this way, we obtain an approximate sequence $\{y_n\}$ instead of the actual sequence $\{x_n\}$. The iterative procedure $f(T, x_n)$ is considered to be numerically stable if and only if the approximate sequence $\{y_n\}$ still converges to the desired solution of the equation $Tx = x$. Urabe [1] initiated the study of this problem. The study of stability of iterative procedures plays a significant role in numerical mathematics due to chaotic behavior of functions and discretization of computations in computer programming. For a detailed discussion on the role of stability of iterative procedures, one may refer to Czerwik et al. [2,3], Harder and Hicks [4-6], Lim [7], Matkowski and Singh [8], Ortega and Rheinboldt [9], Osilike [10,11], Ostrowski [12], Rhoades [13,14], Rus et al. [15] and Singh et al. [16].

However, Ostrowski [12] was the first to obtain the following classical stability result on metric spaces.

Theorem 1.1. *Let (X, d) be a complete metric space and $T: X \to X$ a Banach contraction with contraction constant q, i.e., $d(Tx, Ty) \le qd(x, y)$ for all $x, y \in X$, where* 0

$\leq q < 1$. *Let* p *be the fixed point of* T. *Let* $x_0 \in X$ *and* $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ *Suppose that* $\{y_n\}$ *is a sequence in* X *and* $\varepsilon_n = d(y_{n+1}, Ty_n)$. *Then*

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} \varepsilon_j.$$

Moreover, $\lim_n y_n = p$ *if and only if* $\lim_n \varepsilon_n = 0$.

This result has found a respectable place in numerical analysis and computer programming and further extended by Harder and Hicks [5,6], Jachymski [17], Osilike [10,11,18], Osilike and Udomene [19], Rhoades [13,14], Czerwik et al. [2] and Zhou [20].

The classical result on stability due to Ostrowski has been extended to multi-valued maps by Singh and Chadha [21] and further extended by Singh and Bhatnagar [22] and Singh et al. [23].

Furhter, stability of iterative procedures has a remarkable importance in fractal graphics while generating fractals. Its usefulness lies in the fact that in fractal graphics, fractal objects are generated by an infinite recursive process of successive approximations. An itertive procedure produces a sequence of results and tends towards one final object called a set attractor (fractal), which is independent of the initial choice. This stable character of set attractor is due to the stability of iterative procedure, else the system of underlying successive approximations would show chaotic behavior and never settle into a stationary state. However, fractals themselves have a variety of applications in digital imaging, mobile computing, architecture and construction, various branches of engineering and applied sciences. For recent potential applications of fractal geometry in related fields, one may refer to Batty and Longley [24], Buser et al. [25], Lee and Hsieh [26], Mistakeidis and Panagouli [27], Shaikh et al. [28] and Zmeskal et al. [29]. For connections of the round-off stability with the concept of limit shadowing for a fixed point problem involving multi-valued maps, one may refer to Petrusel and Rus [30].

The purpose of this article is to discuss the stability of Picard iterative procedure, i.e., $x_{n+1} \in f(T, x_n) = Tx_n$ for a map T satisfying Zamfirescu multi-valued contraction (*cf.* Definition 2.2).

2 Preliminaries

This section is primarily devoted to notations and definitions to be used in the sequel.

2.1 Multivalued contractions

Let (X, d) be a metric space and

$CB(X) = \{A: A$ is a nonempty closed bounded subset of $X\}$,

$CL(X) = \{A: A$ is a nonempty closed subset of $X\}$.

For $A, B \in CL(X)$ and $\varepsilon > 0$,

$$N(\varepsilon, A) = \{x \in X : d(x, a) < \varepsilon \quad \textit{for some} \quad a \in A\},$$
$$E_{A,B} = \{\varepsilon > 0 : A \subseteq N(\varepsilon, B),\ B \subseteq N(\varepsilon, A)\},$$

$$H(A, B) = \begin{cases} \mathit{inf}\ E_{A,B} & \text{if } E_{A,B} \neq \phi, \\ +\infty & \text{if } E_{A,B} = \phi. \end{cases}$$

H is called the generalized Hausdorff metric (resp. Hausdorff metric) for $CL(X)$ (resp. $CB(X)$) induced by d. For any nonempty subsets A, B of X, $d(A, B)$ will denote the gap between the subsets A and B, while we write $d(a, B)$ for $d(A, B)$ when $A = \{a\}$.

An orbit $O(x_0)$ of a multi-valued map T at a point x_0 is a sequence $\{x_n: x_n \in Tx_{n-1}, n = 1, 2,...\}$. For a single-valued map T, this orbit is $\{x_n: x_n = Tx_{n-1}, n = 1, 2,...\}$. A space X is said to be T-orbitally complete [31,32] if every Cauchy sequence which is contained in $O(x_0)$ for some initial point $x_0 \in X$ converges in X.

The study of fixed point theorems for multi-valued contractions was initiated by Markin [33] and Nadler [34]. The notion of multi-valued contractions have been generalized by many authors. For a good discussion on fixed point theorems for multi-valued contractions, one may refer to Ćirić [31,32], Czerwik [35,36], Neammanee and Kalwkhao [37] and Rus and Petrusel [15,30]. However for the sake of comparison, we consider the following three conditions.

Let (X, d) be a complete metric space and let $T: X \to CL(X)$. Then

Definition 2.1. (Nadler [34,38])

A map $T: X \to CL(X)$ is called a Nadler multi-valued contraction if

$$H(Tx,\ Ty) \leq qd(x,\ y)$$

for all $x, y \in X$, where $0 \leq q < 1$.

Definition 2.2. (Zamfirescu [39])

A map $T: X \to CL(X)$ is called a Zamfirescu multi-valued contraction if there exist real numbers α, β, and γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$ and $0 \leq \gamma < \frac{1}{2}$ such that for each $x, y \in X$ at least one of the following is true:

(i) $H(Tx, Ty) \leq \alpha d(x, y)$,

(ii) $H(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$,

(iii) $H(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

Definition 2.3. (Ćirić [31])

A map $T: X \to CL(X)$ is called a Ćirić generalized multi-valued contraction if there exists a nonnegative number q such that

$$H(Tx,\ Ty) \leq q \max\left\{d(x,\ y),\ d(x,\ Tx),\ d(y,\ Ty),\ \frac{1}{2}(d(x,\ Ty) + d(y,\ Tx))\right\} \qquad (2.1)$$

for all $x, y \in X$.

We remark that a Nadler multi-valued contraction (*cf.* Definition 2.1) is continuous, while T in Definition 2.2 need not be continuous. If we take $T: X \to X$, then (i) Definition 2.1 is the classical Banach contraction, (ii) Definition 2.2 is due to Zamfirescu [39] and (iii) Definition 2.3 is due to Ćirić [40]. In a comprehensive comparison of contractive maps, Rhoades [41] has listed 25 conditions for a single-valued map in a metric space. We remark that, for $T: X \to X$, the conditions given in Definition 2.1, 2.2, and 2.3 are respectively the conditions (1), (19), and (21'). For a comparison of contractive conditions for single valued maps more general than (21'), one may refer to Park [42] and see also Sessa and Cho [43]. Evidently, Nadler multi-valued contraction ⇒ Zamfirescu multi-valued contraction ⇒ Ćirić generalized multi-valued contraction.

We cite the following result due to Ćirić [31].

Theorem 2.1. *Let $T: X \to CL(X)$ be a Ćirić generalized multi-valued contraction such that X is T-orbitally complete. Then:*

(i) for each $x_0 \in X$, *there exists an orbit* $\{x_n\}$ *of* T *at* x_0 *and a point* $p \in X$ *such that* $\lim_n x_n = p$;

(ii) $p \in Tp$.

2.2 Stability of multivalued operators

Let X be a metric space and $T: X \to CL(X)$. For a point $x_0 \in X$, let

$$x_{n+1} \in f(T, x_n) \tag{2.2}$$

denote some iteration procedure. Let the sequence $\{x_n\}$ be convergent to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and set

$$\varepsilon_n = H(y_{n+1}, f(T, y_n)), \ n = 0, 1, 2, \ldots.$$

If $\lim_n \varepsilon_n = 0$ implies that $\lim_n y_n = p$ then the iteration process defined in (2.2) is said to be T-stable or stable with respect to T (*cf.* [21]).

Ostrowski's stablity theorem [12] says that Picard iterative procedure for (single-valued) Banach contraction is stable. Following is the extension of this theorem to multivalued contractions given by Singh and Chadha [21].

Theorem 2.2. *Let X be a complete metric space and* $T: X \to CL(X)$ *such that the condition given in Definition 2.1 holds for all* $x, y \in X$. *Let* x_0 *be an arbitrary point in X and* $\{x_n\}_{n=1}^{\infty}$ *an orbit for T at* x_0 *such that* $\{x_n\}_{n=1}^{\infty}$ *is convergent to a fixed point p of T. Let* $\{y_n\}$ *be a sequence in X, and set*

$$\varepsilon_n = H(y_{n+1}, Ty_n), \ n = 0, 1, 2, \ldots.$$

Then

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{j=0}^{n} q^{n-j} \varepsilon_j.$$

Further, if Tp is singleton then

$$\lim_n y_n = p \text{ if and only if } \lim_n \varepsilon_n = 0.$$

We shall need the following result.

Lemma 2.1. (Harder and Hicks [6])

If c is a real number such that $0 < |c| < 1$ *and* $\{b_k\}_{k=0}^{\infty}$ *is a sequence of real numbers such that* $\lim_{k\to\infty} b_k = 0$, *then* $\lim_{n\to\infty} (\sum_{k=0}^{n} c^{n-k} b_k) = 0$.

3 Main results

Theorem 3.1. *Let X be a complete metric space and* $T: X \to CL(X)$ *a Zamfirescu multi-valued contraction (cf. Definition 2.2). Let* x_0 *be an arbitrary point in X and* $\{x_n\}_{n=1}^{\infty}$ *an orbit for T at* x_0 *such that* $\{x_n\}_{n=1}^{\infty}$ *is convergent to a fixed point p of T. Let* $\{y_n\}_{n=1}^{\infty}$ *be a sequence in X and set* $\varepsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$ *Then*

(I) $d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k} \varepsilon_k$,

where $\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ *and* $n = 0, 1, \ldots$

Further, if Tp is singleton then

(II) $\lim_{n\to\infty} y_n = p$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof: Let $x, y \in X$. Since T is a Zamfirescu multi-valued contraction, T satisfies one of (i), (ii), and (iii). If (ii) holds, then

$$\begin{aligned} H(Tx, Ty) &\leq \beta[d(x, Tx) + d(y, Ty)] \\ &\leq \beta[H(x, Tx) + H(y, Ty)] \\ &\leq \beta[H(x, Tx) + H(y, x) + H(x, Tx) + H(Tx, Ty)] \\ &= \beta[2H(x, Tx) + d(y, x) + H(Tx, Ty)]. \end{aligned}$$

So,

$$H(Tx, Ty) \leq \frac{2\beta}{1-\beta} H(x, Tx) + \frac{\beta}{1-\beta} d(x, y).$$

If (iii) holds, then

$$\begin{aligned} H(Tx, Ty) &\leq \gamma[d(x, Ty) + d(y, Tx)] \\ &\leq \gamma[H(x, Ty) + H(y, Tx)] \\ &\leq \gamma[H(x, Tx) + H(Tx, Ty) + H(y, x) + H(x, Tx)] \\ &= \gamma[2H(x, Tx) + d(x, y) + H(Tx, Ty)], \end{aligned}$$

that is

$$H(Tx, Ty) \leq \frac{2\gamma}{1-\gamma} H(x, Tx) + \frac{\gamma}{1-\gamma} d(x, y).$$

Thus at least one of the following is true for any $x, y \in X$:

(i) $H(Tx, Ty) \leq \alpha d(x, y)$,

(ii') $H(Tx, Ty) \leq \frac{2\beta}{1-\beta} H(x, Tx) + \frac{\beta}{1-\beta} d(x, y)$.

(iii') $H(Tx, Ty) \leq \frac{2\gamma}{1-\gamma} H(x, Tx) + \frac{\gamma}{1-\gamma} d(x, y)$.

Let $\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$. Then

(i*) $H(Tx, Ty) \leq 2\delta H(x, Tx) + \delta d(x, y)$

for any $x, y \in X$.

Let n be a nonnegative integer. Since

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + d(x_{n+1}, y_{n+1}), \tag{3.1}$$

we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq H(Tx_n, y_{n+1}) \\ &\leq H(Tx_n, Ty_n) + H(Ty_n, y_{n+1}) \\ &\leq 2\delta H(x_n, Tx_n) + \delta d(x_n, y_n) + \varepsilon_n. \end{aligned} \tag{3.2}$$

Consequently

$$d(x_n, y_n) \leq 2\delta H(x_{n-1}, Tx_{n-1}) + \delta d(x_{n-1}, y_{n-1}) + \varepsilon_{n-1}. \tag{3.3}$$

Therefore using (3.2) and (3.3) in (3.1), we obtain

$$\begin{aligned} d(p, y_{n+1}) &\le d(p, x_{n+1}) + 2\delta H(x_n, Tx_n) + \varepsilon_n \\ &\quad + \delta[2\delta H(x_{n-1}, Tx_{n-1}) + \delta d(x_{n-1}, y_{n-1}) + \varepsilon_{n-1}] \\ &= d(p, x_{n+1}) + 2[\delta H(x_n, Tx_n) + \delta^2 H(x_{n-1}, Tx_{n-1})] \\ &\quad + \delta^2 d(x_{n-1}, y_{n-1}) + (\varepsilon_n + \delta\varepsilon_{n-1}). \end{aligned}$$

Repeat this process (n - 1) times to obtain

$$d(p, y_{n+1}) \le d(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k}\varepsilon_k.$$

This proves (I).

By (i*), we have

$$\begin{aligned} \varepsilon_n &= H(y_{n+1}, Ty_n) \\ &\le d(y_{n+1}, p) + H(p, Tp) + H(Tp, Ty_n) \\ &\le d(y_{n+1}, p) + H(p, Tp) + 2\delta H(p, Tp) + \delta d(p, y_n). \end{aligned}$$

This yields $\varepsilon_n \to 0$ as $n \to \infty$, since $Tp = \{p\}$ by hypothesis.

Conversly, suppose that $\varepsilon_n \to 0$ as $n \to \infty$.

First, we assert that $\lim_{k\to\infty} H(x_k, Tx_k) = 0$, if $Tp = \{p\}$. For

$$H(x_k, Tx_k) \le H(x_k, \{p\}) \le d(x_k, \{p\}) + H(Tp, Tx_k). \tag{3.4}$$

Since T is a Zamfirescu multi-valued contraction, it follows from (i), (ii), and (iii), (cf. Definition 2.2), that $\{Tx_k\}$ is a Cauchy sequence. Consequently, $Tx_k \to Tp$ as $k \to \infty$. So making $k \to \infty$, (3.4) yields the assertion.

Note that $0 \le \delta < 1$.

If $\delta = 0$, then (I) yields $\lim_{n\to\infty} y_n = p$. So assume that $0 < \delta < 1$.

Then $\delta^{n+1} d(x_0, y_0) \to 0$ as $n \to \infty$.

Since $\lim_{k\to\infty} H(x_k, Tx_k) = 0$, $\lim_{k\to\infty} \varepsilon_k = 0$. Therefore, by Lemma (2.1),

$$\sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) \to 0 \text{ and } \sum_{k=0}^{n} \delta^{n-k}\varepsilon_k \to 0 \text{ as } n \to \infty.$$

Hence from (I), $\lim_{n\to\infty} y_n = p$.□

We remark that the second term on the right-hand side of the conclusion (I) vanishes when $\beta = \gamma = 0$. So we have the following.

Corollary 3.1. *Theorem 2.2.*

Proof: It comes from Theorem 3.1 when $\alpha = q$ and $\beta = \gamma = 0$.

Corollary 3.2. (Harder and Hicks [6])

Let (X, d) be a complete metric space and let $T: X \to X$ be a Zamfirescu contraction. Let p be the fixed point of T. Let $x_0 \in X$, and put $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$, so that $\lim_{n\to\infty} x_n = p$. Let $\{y_n\}_{n=0}^{\infty}$ be a sequence in X and set $\varepsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \ldots$. Then

(Ia) $d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} d(x_k, x_{k+1}) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k} \varepsilon_k,$ *where* $\delta = max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$ *and* $n = 0, 1, \ldots$.

Further, if Tp is singleton then

(IIa) $\lim_{n\to\infty} y_n = p$ *if and only if* $\lim_{n\to\infty} \varepsilon_n = 0$.

Proof: It is exactly derivable from Theorem 3.1 when $\varepsilon_n = H(y_{n+1}, Ty_n) = d(y_{n+1}, Ty_n)$ when T is single valued. Further, $H(x_n, Tx_n) = d(x_n, x_{n+1})$, if the map T is single-valued.

We remark that $p \in X$ in (II) of Theorem 3.1, is not required to be the unique fixed point of T. The related condition emphasizes that Tp contains just one point.

The following, due to an idea of Singh and Whitfield [[44], p. 226] and Singh and Chadha [[21], p. 190], is another extension of Corollary 3.1.

Theorem 3.2. *Let all the hypotheses of Theorem 3.1 hold, wherein the definition of* ε_n *is replaced as follows:*

$$\varepsilon_n = d(y_{n+1}, p_n), p_n \in Ty_n, n = 0, 1, 2\ldots.$$

Then

$$\text{(III)} \qquad d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k}(\varepsilon_k + H_k),$$

where $H_k = H(x_{k+1}, Tx_k)$.

Further, if Tp is singleton then

(IVa) $\lim_{n\to\infty} y_n = p$ *then* $\lim_{n\to\infty} \varepsilon_n = 0$,

(IVb) If T is continuous and $\lim_{n\to\infty} \varepsilon_n = 0$ *then* $\lim_{n\to\infty} y_n = p$.

Proof: Since T is Zamfirescu multi-valued contraction in Theorem 3.2, we recall by (i*) the following property: $H(Tx_n, Ty_n) \leq 2\delta H(x_n, Tx_n) + \delta d(x_n, y_n)$ for any $x_n, y_n \in X$. Therefore for any nonnegative integer n,

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq d(x_{n+1}, p_n) + d(p_n, y_{n+1}) \\ &\leq H(x_{n+1}, Ty_n) + \varepsilon_n \\ &\leq H(x_{n+1}, Tx_n) + H(Tx_n, Ty_n) + \varepsilon_n \\ &\leq H_n + 2\delta H(x_n, Tx_n) + \delta d(x_n, y_n) + \varepsilon_n \\ &\leq H_n + 2\delta H(x_n, Tx_n) + \delta[H_{n-1} + 2\delta H(x_{n-1}, Tx_{n-1}) \\ &\quad + \delta d(x_{n-1}, y_{n-1}) + \varepsilon_{n-1}] + \varepsilon_n \\ &\leq \delta^2 d(x_{n-1}, y_{n-1}) + 2\delta[H(x_n, Tx_n) + \delta H(x_{n-1}, Tx_{n-1})] \\ &\quad + \delta(H_{n-1} + \varepsilon_{n-1}) + (H_n + \varepsilon_n). \end{aligned}$$

Inductively,

$$d(x_{n+1}, y_{n+1}) \leq \sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k}(H_k + \varepsilon_k),$$

and the relation (III) follows as in the proof of (I).

To prove (IVa), first assume that $y_n \to p$ as $n \to \infty$.

Then $\varepsilon_n = d(y_{n+1}, p_n) \leq H(y_{n+1}, Ty_n)$.

This, as in proof of Theorem 3.1, gives $\lim_n \varepsilon_n = 0$.

Now assume that T is continuous and $\lim_n \varepsilon_n = 0$. From (III),

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} H(x_k, Tx_k) + \delta^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} \delta^{n-k} t_k,$$

where $t_k = (\varepsilon_k + H_k)$. In view of the (corresponding part of the) proof of Theorem 3.1, it is sufficient to show that the sequence $\{t_k\}$ is convergent to 0. Since, by one of the assumptions, the sequence $\{\varepsilon_k\}$ is convergent to 0, it is enough to show that $\{H_n\}$ is also convergent to 0. Since T is continuous,

$$\lim_n H_n = \lim_n H(x_{n+1}, Tx_n) = H(p, Tp) = 0.$$

This completes the proof.□

Corollary 3.3. (Singh and Chadha [21, Theorem 3])

Let all the hypotheses of Theorem 2.2 hold, wherein the definition of ε_n *is replaced by the following*

$$\varepsilon_n = d(y_{n+1}, p_n), p_n \in Ty_n, n = 0, 1, 2 \ldots.$$

Then

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + q^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} q^{n-k}(H_k + \varepsilon_k),$$

where $H_k = H(x_{k+1}, Tx_k)$*. Further, if Tp is singleton then*

$$\lim_n y_n = p \text{ if and only if } \lim_n \varepsilon_n = 0.$$

Proof: Recall that a Nadler multi-valued contraction is continuous. So it comes from the fact that Definition 2.1 implies Definition 2.2.

It seems interesting to answer the following

Question: Can one replace Zamfirescu multi-valued contraction in Theorems 3.1 and 3.2 by the Ćirić generalized multi-valued contraction?

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Authors' contributions
All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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