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An implicit iterative algorithm with errors for two families of generalized asymptotically nonexpansive mappings

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Abstract

In this paper, an implicit iterative algorithm with errors is considered for two families of generalized asymptotically nonexpansive mappings. Strong and weak convergence theorems of common fixed points are established based on the implicit iterative algorithm.

Mathematics Subject Classification (2000) 47H09 · 47H10 · 47J25

Keywords: Asymptotically nonexpansive mapping, common fixed point, implicit iterative algorithm, generalized asymptotically nonexpansive mapping

1 Introduction

In nonlinear analysis theory, due to applications to complex real-world problems, a growing number of mathematical models are built up by introducing constraints which can be expressed as subproblems of a more general problem. These constraints can be given by fixed-point problems, see, for example, [1-3]. Study of fixed points of nonlinear mappings and its approximation algorithms constitutes a topic of intensive research efforts. Many well-known problems arising in various branches of science can be studied by using algorithms which are iterative in their nature. The well-known convex feasibility problem which captures applications in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning is to find a point in the intersection of common fixed point sets of a family of nonexpansive mappings, see, for example, [3-5].

For iterative algorithms, the most oldest and simple one is Picard iterative algorithm. It is known that T enjoys a unique fixed point, and the sequence generated in Picard iterative algorithm can converge to the unique fixed point. However, for more general nonexpansive mappings, Picard iterative algorithm fails to convergence to fixed points of nonexpansive even that it enjoys a fixed point.

Recently, Mann-type iterative algorithm and Ishikawa-type iterative algorithm (implicit and explicit) have been considered for the approximation of common fixed points of nonlinear mappings by many authors, see, for example, [6-24]. A classical convergence theorem of nonexpansive mappings has been established by Xu and Ori [23]. In 2006, Chang et al. [6] considered an implicit iterative algorithm with errors for asymptotically nonexpansive mappings in a Banach space. Strong and weak convergence theorems are

established. Recently, Cianciaruso et al. [9] considered an Ishikawa-type iterative algorithm for the class of asymptotically nonexpansive mappings. Strong and weak convergence theorems are also established. In this paper, based on the class of generalized asymptotically nonexpansive mappings, an Ishikawa-type implicit iterative algorithm with errors for two families of mappings is considered. Strong and weak convergence theorems of common fixed points are established. The results presented in this paper mainly improve the corresponding results announced in Chang et al. [6], Chidume and Shahzad [7], Cianciaruso et al. [9], Guo and Cho [10], Khan et al. [12], Plubtieng et al. [14], Qin et al. [15], Shzhzad and Zegeye [18], Thakur [21], Thianwan and Suantai [22], Xu and Ori [23], Zhou and Chang [24].

2 Preliminaries

Let C be a nonempty closed convex subset of a Banach space E . Let $T : C \rightarrow C$ be a mapping. Throughout this paper, we use $F(T)$ to denote the fixed point set of T .

Recall the following definitions.

T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

T is said to be *asymptotically nonexpansive* if there exists a positive sequence $\{h_n\} \subset [1, \infty)$ with $h_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\|, \quad \forall x, y \in C, n \geq 1.$$

It is easy to see that every nonexpansive mapping is asymptotically nonexpansive with the asymptotical sequence $\{1\}$. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [25] in 1972. It is known that if C is a nonempty bounded closed convex subset of a uniformly convex Banach space E , then every asymptotically nonexpansive mapping on C has a fixed point. Further, the set $F(T)$ of fixed points of T is closed and convex. Since 1972, a host of authors have studied weak and strong convergence problems of implicit iterative processes for such a class of mappings.

T is said to be *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following inequality holds:

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0. \tag{2.1}$$

Putting $\zeta_n = \max\{0, \sup_{x, y \in C} (\|T^n x - T^n y\| - \|x - y\|)\}$, we see that $\zeta_n \rightarrow 0$ as $n \rightarrow \infty$. Then, (2.1) is reduced to the following:

$$\|T^n x - T^n y\| \leq \|x - y\| + \zeta_n, \quad \forall x, y \in C, n \geq 1.$$

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck et al. [26] (see also Kirk [27]) as a generalization of the class of asymptotically nonexpansive mappings. It is known that if C is a nonempty closed convex and bounded subset of a real Hilbert space, then every asymptotically nonexpansive self mapping in the intermediate sense has a fixed point; see [28] more details.

T is said to be *generalized asymptotically nonexpansive* if it is continuous and there exists a positive sequence $\{h_n\} \subset [1, \infty)$ with $h_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\limsup_{n \rightarrow \infty} \sup_{x, y \in C} (\|T^n x - T^n y\| - h_n \|x - y\|) \leq 0. \tag{2.2}$$

Putting $\zeta_n = \max\{0, \sup_{x, y \in C} (\|T^n x - T^n y\| - h_n \|x - y\|)\}$, we see that $\zeta_n \rightarrow 0$ as $n \rightarrow \infty$. Then, (2.2) is reduced to the following:

$$\|T^n x - T^n y\| \leq h_n \|x - y\| + \xi_n, \quad \forall x, y \in C, n \geq 1.$$

We remark that if $h_n \equiv 1$, then the class of generalized asymptotically nonexpansive mappings is reduced to the class of asymptotically nonexpansive mappings in the intermediate.

In 2006, Chang et al. [6] considered the following implicit iterative algorithms for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$ with $\{\alpha_n\}$ a real sequence in $(0, 1)$, $\{u_n\}$ a bounded sequence in C and an initial point $x_0 \in C$:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 + u_1, \\ x_2 &= \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2 + u_2, \\ &\dots \\ x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N + u_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^n x_{N+1} + u_{N+1}, \\ &\dots \\ x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N} + u_{2N}, \\ x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1} + u_{2N+1}, \\ &\dots \end{aligned}$$

The above table can be rewritten in the following compact form:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_{i(n)}^{j(n)} x_n + u_n, \quad \forall n \geq 1,$$

where for each $n \geq 1$ fixed, $j(n) - 1$ denotes the quotient of the division of n by N and $i(n)$ the rest, i.e., $n = (j(n) - 1)N + i(n)$.

Based on the implicit iterative algorithm, they obtained, under the assumption that $C + C \subset C$, weak and strong convergence theorems of common fixed points for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$; see [6] for more details.

Recently, Cianciaruso et al. [9] considered a Ishikawa-like iterative algorithm for the class of asymptotically nonexpansive mappings in a Banach space. To be more precise, they introduced and studied the following implicit iterative algorithm with errors.

$$\begin{cases} \gamma_n = (1 - \beta_n - \delta_n)x_n + \beta_n T_{i(n)}^{j(n)} x_n + \delta_n v_n, \\ x_n = (1 - \alpha_n - \gamma_n)x_{n-1} + \alpha_n T_{i(n)}^{j(n)} \gamma_n + \gamma_n u_n, \end{cases} \quad \forall n \geq 1, \tag{2.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are real number sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequence in C . Weak and strong convergence theorems are established in a uniformly convex Banach space; see [29] for more details.

In this paper, motivated and inspired by the results announced in Chang et al. [6], Chidume and Shahzad [7], Cianciaruso et al. [9], Guo and Cho [10], Plubtieng et al. [14], Qin et al. [15], Shzhzad and Zegeye [18], Thakur [21], Thianwan and Suantai [22], Xu and Ori [23], Zhou and Chang [24], we consider the following Ishikawa-like implicit iteration algorithm with errors for two finite families of generalized asymptotically nonexpansive mappings $\{T_1, T_2, \dots, T_N\}$ and $\{S_1, S_2, \dots, S_N\}$.

$$\begin{aligned}
 x_0 &\in C, \\
 x_1 &= \alpha_1 x_0 + \beta_1 T_1(\alpha'_1 x_1 + \beta'_1 S_1 x_1 + \gamma'_1 v_1) + \gamma_1 u_1, \\
 x_2 &= \alpha_2 x_1 + \beta_2 T_2(\alpha'_2 x_2 + \beta'_2 S_2 x_2 + \gamma'_2 v_2) + \gamma_2 u_2, \\
 &\dots \\
 x_N &= \alpha_N x_{N-1} + \beta_N T_N(\alpha'_N x_N + \beta'_N S_N x_N + \gamma'_N v_N) + \gamma_N u_N, \\
 x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_{N+1}(\alpha'_{N+1} x_{N+1} + \beta'_{N+1} S_{N+1} x_{N+1} \\
 &\quad + \gamma'_{N+1} v_{N+1}) + \gamma_{N+1} u_{N+1}, \\
 &\dots \\
 x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_{2N}(\alpha'_{2N} x_{2N} + \beta'_{2N} S_{2N} x_{2N} + \gamma'_{2N} v_{2N}) \\
 &\quad + \gamma_{2N} u_{2N}, \\
 x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_{2N+1}(\alpha'_{2N+1} x_{2N+1} + \beta'_{2N+1} S_{2N+1} x_{2N+1} \\
 &\quad + \gamma'_{2N+1} v_{2N+1}) + \gamma_{2N+1} u_{2N+1}, \\
 &\dots,
 \end{aligned}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, and $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for each $n \geq 1$. We have rewritten the above table in the following compact form:

$$x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)}(\alpha'_n x_n + \beta'_n S_{i(n)}^{j(n)} x_n + \gamma'_n v_n) + \gamma_n u_n, \quad n \geq 1,$$

where for each $n \geq 1$ fixed, $j(n) - 1$ denotes the quotient of the division of n by N and $i(n)$ the rest, i.e., $n = (j(n) - 1)N + i(n)$.

Putting $\gamma_n = \alpha'_n x_n + \beta'_n S_n x_n + \gamma'_n v_n$, we have the following composite iterative algorithm:

$$\begin{cases} \gamma_n = \alpha'_n x_n + \beta'_n S_{i(n)}^{j(n)} x_n + \gamma'_n v_n, \\ x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)} \gamma_n + \gamma_n u_n, \end{cases} \quad n \geq 1. \tag{2.4}$$

We remark that the implicit iterative algorithm (2.4) is general which includes (2.3) as a special case.

Now, we show that (2.4) can be employed to approximate fixed points of generalized asymptotically nonexpansive mappings which is assumed to be Lipschitz continuous. Let T_i be a L_i^i -Lipschitz generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$ such that $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ and S_i be a L_i^i -Lipschitz generalized asymptotically nonexpansive mapping with sequences $\{k_n^i\} \subset [1, \infty)$ such that $k_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Define a mapping $W_n : C \rightarrow C$ by

$$W_n(x) = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)}(\alpha'_n x + \beta'_n S_{i(n)}^{j(n)} x + \gamma'_n v_n) + \gamma_n u_n, \quad \forall n \geq 1.$$

It follows that

$$\begin{aligned}
 &\| W_n(x) - W_n(y) \| \\
 &\leq \beta_n \| T_{i(n)}^{j(n)}(\alpha'_n x + \beta'_n S_{i(n)}^{j(n)} x + \gamma'_n v_n) - T_{i(n)}^{j(n)}(\alpha'_n y + \beta'_n S_{i(n)}^{j(n)} y + \gamma'_n v_n) \| \\
 &\leq \beta_n L(\alpha'_n \| x - y \| + \beta'_n \| S_{i(n)}^{j(n)} x - S_{i(n)}^{j(n)} y \|) \\
 &\leq \beta_n L(\alpha'_n + \beta'_n L) \| x - y \|, \quad \forall x, y \in C,
 \end{aligned}$$

where

$$L = \max \{L_t^1, \dots, L_t^N, L_s^1, \dots, L_s^N\}. \tag{2.5}$$

If $\beta_n L(\alpha'_n + \beta'_n L) < 1$ for all $n \geq 1$, then W_n is a contraction. By Banach contraction mapping principal, we see that there exists a unique fixed point $x_n \in C$ such that

$$x_n = W_n(x_n) = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)}(\alpha'_n x + \beta'_n S_{i(n)}^{j(n)} x + \gamma'_n v_n) + \gamma_n u_n, \quad \forall n \geq 1.$$

That is, the implicit iterative algorithm (2.4) is well defined.

The purpose of this paper is to establish strong and weak convergence theorem of fixed points of generalized asymptotically nonexpansive mappings based on (2.4).

Next, we recall some well-known concepts.

Let E be a real Banach space and $U_E = \{x \in E : \|x\| = 1\}$. E is said to be *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \varepsilon \quad \text{implies} \quad \left\| \frac{x+y}{2} \right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex.

Recall that E is said to satisfy *Opial's condition* [30] if for each sequence $\{x_n\}$ in E , the condition that the sequence $x_n \rightarrow x$ weakly implies that

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\|$$

for all $y \in E$ and $y \neq x$. It is well known [30] that each l^p ($1 \leq p < \infty$) and Hilbert spaces satisfy Opial's condition. It is also known [29] that any separable Banach space can be equivalently renormed to that it satisfies Opial's condition.

Recall that a mapping $T : C \rightarrow C$ is said to be *demiclosed* at the origin if for each sequence $\{x_n\}$ in C , the condition $x_n \rightarrow x_0$ weakly and $Tx_n \rightarrow 0$ strongly implies $Tx_0 = 0$. T is said to be *semicompact* if any bounded sequence $\{x_n\}$ in C satisfying $\lim_{n \rightarrow \infty} \|x_n - Tx_n\| = 0$ has a convergent subsequence.

In order to prove our main results, we also need the following lemmas.

Lemma 2.1. [20] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer. If $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} b_n < \infty$, then $\lim_{n \rightarrow \infty} a_n$ exists.

Lemma 2.2. [17] *Let E be a uniformly convex Banach space and $0 < \lambda \leq t_n \leq \eta < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \quad \limsup_{n \rightarrow \infty} \|y_n\| \leq r$$

and

$$\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

The following lemma can be obtained from Qin et al. [31] or Sahu et al. [32] immediately.

Lemma 2.3. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a Lipschitz generalized asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at origin.*

3 Main results

Now, we are ready to give our main results in this paper.

Theorem 3.1. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ and $S_i : C \rightarrow C$ be a uniformly L_s^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{k_n^i\} \subset [1, \infty)$, where $k_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in C and $e_n = \max\{h_n, k_n\}$, where $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$ and $k_n = \sup\{k_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (2.4). Put*

$$\mu_n^i = \max\{0, \sup_{x, y \in C} (\|T_i^n x - T_i^n y\| - h_n^i \|x - y\|)\}$$

and

$$v_n^i = \max\{0, \sup_{x, y \in C} (\|S_i^n x - S_i^n y\| - k_n^i \|x - y\|)\}.$$

Let $\xi_n = \max\{\mu_n, v_n\}$, where $v_n = \max\{v_n^i : 1 \leq i \leq N\}$ and $\mu_n = \max\{\mu_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (e_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$;
- (c) $\beta_n L(\alpha'_n + \beta'_n L) < 1$, where L is defined in (2.5);
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that $\lambda \leq \alpha_n, \alpha'_n \leq \eta$.

Then

$$\lim_{n \rightarrow \infty} \|x_n - T_r x_n\| = \lim_{n \rightarrow \infty} \|x_n - S_r x_n\| = 0, \quad \forall r \in \{1, 2, \dots, N\}.$$

Proof. Fixing $f \in \mathcal{F}$, we see that

$$\begin{aligned} \|\gamma_n - f\| &= \|\alpha'_n x_n + \beta'_n S_{i(n)}^{j(n)} x_n + \gamma'_n v_n - f\| \\ &\leq \alpha'_n \|x_n - f\| + \beta'_n \|S_{i(n)}^{j(n)} x_n - f\| + \gamma'_n \|v_n - f\| \\ &\leq \alpha'_n \|x_n - f\| + \beta'_n e_{j(n)} \|x_n - f\| + \beta'_n \xi_{j(n)} + \gamma'_n \|v_n - f\| \\ &\leq e_{j(n)} \|x_n - f\| + \beta'_n \xi_{j(n)} + \gamma'_n \|v_n - f\| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\|x_n - f\| \\ &= \|\alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)} \gamma_n + \gamma_n u_n - f\| \\ &\leq \alpha_n \|x_{n-1} - f\| + \beta_n \|T_{i(n)}^{j(n)} \gamma_n - f\| + \gamma_n \|u_n - f\| \\ &\leq \alpha_n \|x_{n-1} - f\| + \beta_n e_{j(n)} \|\gamma_n - f\| + \beta_n \xi_{j(n)} + \gamma_n \|u_n - f\|. \\ &\leq \alpha_n \|x_{n-1} - f\| + (1 - \alpha_n) e_{j(n)} \|\gamma_n - f\| + \beta_n \xi_{j(n)} + \gamma_n \|u_n - f\|. \end{aligned} \tag{3.2}$$

Substituting (3.1) into (3.2), we see that

$$\begin{aligned} & \|x_n - f\| \\ & \leq \alpha_n \|x_{n-1} - f\| + (1 - \alpha_n)e_{j(n)}(e_{j(n)} \|x_n - f\| + \beta'_n \xi_{j(n)} + \gamma'_n \|v_n - f\|) \\ & \quad + \beta_n \xi_{j(n)} + \gamma_n \|u_n - f\|. \\ & \leq \alpha_n \|x_{n-1} - f\| + (1 - \alpha_n)e_{j(n)}^2 \|x_n - f\| + (1 + e_{j(n)})\xi_{j(n)} \\ & \quad + e_{j(n)}\gamma'_n \|v_n - f\| + \gamma_n \|u_n - f\|. \end{aligned}$$

Notice that $\sum_{n=1}^{\infty} (e_n - 1) < \infty$. We see from the restrictions (b) and (d) that there exists a positive integer n_0 such that

$$(1 - \alpha_n)e_{j(n)}^2 \leq R < 1, \quad \forall n \geq n_0,$$

where $R = (1 - \lambda)(1 + \frac{\lambda}{2-2\lambda})$. It follows that

$$\begin{aligned} \|x_n - f\| & \leq \left(1 + \frac{(1 - \alpha_n)(e_{j(n)}^2 - 1)}{1 - (1 - \alpha_n)e_{j(n)}^2}\right) \|x_{n-1} - f\| \\ & \quad + \frac{(1 + e_{j(n)})\xi_{j(n)} + e_{j(n)}\gamma'_n \|v_n - f\| + \gamma_n \|u_n - f\|}{1 - (1 - \alpha_n)e_{j(n)}^2} \\ & \leq \left(1 + \frac{(1 + M_1)(e_{j(n)} - 1)}{1 - R}\right) \|x_{n-1} - f\| \\ & \quad + \frac{(1 + M_1)\xi_{j(n)} + M_1M_2\gamma'_n + M_3\gamma_n}{1 - R}, \end{aligned} \tag{3.3}$$

where $M_1 = \sup_{n \geq 1} \{e_n\}$, $M_2 = \sup_{n \geq 1} \{\|v_n - f\|\}$, and $M_3 = \sup_{n \geq 1} \{\|u_n - f\|\}$. In view of Lemma 2.1, we see that $\lim_{n \rightarrow \infty} \|x_n - f\|$ exists for each $f \in \mathcal{F}$. This implies that the sequence $\{x_n\}$ is bounded. Next, we assume that $\lim_{n \rightarrow \infty} \|x_n - f\| = d > 0$. From (3.1), we see that

$$\begin{aligned} & \|T_{i(n)}^{j(n)}\gamma_n - f + \gamma_n(u_n - T_{i(n)}^{j(n)}\gamma_n)\| \\ & \leq \|T_{i(n)}^{j(n)}\gamma_n - f\| + \gamma_n \|u_n - T_{i(n)}^{j(n)}\gamma_n\| \\ & \leq e_{j(n)} \| \gamma_n - f \| + \xi_{j(n)} + \gamma_n \|u_n - T_{i(n)}^{j(n)}\gamma_n\| \\ & \leq e_{j(n)}^2 \|x_n - f\| + e_{j(n)}\xi_{j(n)} + e_{j(n)}\gamma'_n \|v_n - f\| + \xi_{j(n)} \\ & \quad + \gamma_n \|u_n - T_{i(n)}^{j(n)}\gamma_n\|. \end{aligned}$$

This implies from the restrictions (a) and (b) that

$$\limsup_{n \rightarrow \infty} \|T_{i(n)}^{j(n)}\gamma_n - f + \gamma_n(u_n - T_{i(n)}^{j(n)}\gamma_n)\| \leq d.$$

Notice that

$$\|x_{n-1} - f + \gamma_n(u_n - T_{i(n)}^{j(n)}\gamma_n)\| \leq \|x_{n-1} - f\| + \gamma_n \|u_n - T_{i(n)}^{j(n)}\gamma_n\|.$$

This shows from the restriction (a) that

$$\limsup_{n \rightarrow \infty} \|x_{n-1} - f + \gamma_n(u_n - T_{i(n)}^{j(n)}\gamma_n)\| \leq d.$$

On the other hand, we have

$$\begin{aligned} d &= \lim_{n \rightarrow \infty} \|x_n - f\| \\ &= \lim_{n \rightarrow \infty} \|\alpha_n(x_{n-1} - f + \gamma_n(u_n - T_{i(n)}^{j(n)}\gamma_n)) \\ &\quad + (1 - \alpha_n)(T_{i(n)}^{j(n)}\gamma_n - f + \gamma_n(u_n - T_{i(n)}^{j(n)}\gamma_n))\|. \end{aligned}$$

It follows from Lemma 2.2 that

$$\lim_{n \rightarrow \infty} \|T_{i(n)}^{j(n)}\gamma_n - x_{n-1}\| = 0. \tag{3.4}$$

Notice that

$$\|x_n - x_{n-1}\| \leq \beta_n \|T_{i(n)}^{j(n)}\gamma_n - x_{n-1}\| + \gamma_n \|u_n - x_{n-1}\|.$$

It follows from (3.4) and the restriction (a) that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n-1}\| = 0. \tag{3.5}$$

This implies that

$$\lim_{n \rightarrow \infty} \|x_n - x_{n+l}\| = 0, \quad \forall l = 1, 2, \dots, N. \tag{3.6}$$

Notice that

$$\|x_n - f + \gamma'_n(v_n - S_{i(n)}^{j(n)}x_n)\| \leq \|x_n - f\| + \gamma'_n \|v_n - S_{i(n)}^{j(n)}x_n\|$$

and

$$\begin{aligned} &\|S_{i(n)}^{j(n)}x_n - f + \gamma'_n(v_n - S_{i(n)}^{j(n)}x_n)\| \\ &\leq \|S_{i(n)}^{j(n)}x_n - f\| + \gamma'_n \|v_n - S_{i(n)}^{j(n)}x_n\| \\ &\leq e_{j(n)} \|x_n - f\| + \xi_{j(n)} + \gamma'_n \|v_n - S_{i(n)}^{j(n)}x_n\|. \end{aligned}$$

which in turn imply that

$$\limsup_{n \rightarrow \infty} \|x_n - f + \gamma'_n(v_n - S_{i(n)}^{j(n)}x_n)\| \leq d$$

and

$$\limsup_{n \rightarrow \infty} \|S_{i(n)}^{j(n)}x_n - f + \gamma'_n(v_n - S_{i(n)}^{j(n)}x_n)\| \leq d.$$

On the other hand, we have

$$\begin{aligned} \|x_n - f\| &= \|\alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)}\gamma_n + \gamma_n u_n - f\| \\ &\leq \alpha_n \|x_{n-1} - f\| + \beta_n \|T_{i(n)}^{j(n)}\gamma_n - f\| + \gamma_n \|u_n - f\| \\ &\leq \alpha_n \|x_{n-1} - T_{i(n)}^{k(n)}\gamma_n\| + \|T_{i(n)}^{j(n)}\gamma_n - f\| + \gamma_n \|u_n - f\| \\ &\leq \alpha_n \|x_{n-1} - T_{i(n)}^{k(n)}\gamma_n\| + e_{j(n)} \|x_n - f\| + \xi_{j(n)} + \gamma_n \|u_n - f\|, \end{aligned}$$

from which it follows that $\liminf_{n \rightarrow \infty} \|y_n - f\| \geq d$. In view of (3.1), we see that $\limsup_{n \rightarrow \infty} \|y_n - f\| \leq d$. This proves that

$$\lim_{n \rightarrow \infty} \|y_n - f\| = d.$$

Notice that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|y_n - f\| &= \lim_{n \rightarrow \infty} \|\alpha'_n(x_n - f + \gamma'_n(v_n - S_{i(n)}^{j(n)}x_n)) \\ &\quad + (1 - \alpha'_n)(S_{i(n)}^{j(n)}x_n - f + \gamma'_n(v_n - S_{i(n)}^{j(n)}x_n))\|. \end{aligned}$$

This implies from Lemma 2.2 that

$$\lim_{n \rightarrow \infty} \|S_{i(n)}^{j(n)}x_n - x_n\| = 0. \tag{3.7}$$

On the other hand, we have

$$\begin{aligned} &\|T_{i(n)}^{j(n)}x_n - x_n\| \\ &\leq \|T_{i(n)}^{j(n)}x_n - T_{i(n)}^{j(n)}y_n\| + \|T_{i(n)}^{j(n)}y_n - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &\leq \|T_{i(n)}^{j(n)}x_n - T_{i(n)}^{j(n)}y_n\| + \|T_{i(n)}^{j(n)}y_n - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &\leq L \|x_n - y_n\| + \|T_{i(n)}^{j(n)}y_n - x_{n-1}\| + \|x_{n-1} - x_n\| \\ &\leq L\beta'_n \|S_{i(n)}^{j(n)}x_n - x_n\| + L\gamma'_n \|v_n - x_n\| + \|T_{i(n)}^{j(n)}y_n - x_{n-1}\| \\ &\quad + \|x_{n-1} - x_n\|. \end{aligned}$$

This combines with (3.4), (3.5), and (3.7) gives that

$$\lim_{n \rightarrow \infty} \|T_{i(n)}^{j(n)}x_n - x_n\| = 0. \tag{3.8}$$

Since $n = (j(n) - 1)N + i(n)$, where $i(n) \in \{1, 2, \dots, N\}$, we see that

$$\begin{aligned} \|x_n - S_{i(n)}x_n\| &\leq \|x_n - S_{i(n)}^{j(n)}x_n\| + \|S_{i(n)}^{j(n)}x_n - S_{i(n)}x_n\| \\ &\leq \|x_n - S_{i(n)}^{j(n)}x_n\| + L \|S_{i(n)}^{j(n)-1}x_n - x_n\| \\ &\leq \|x_n - S_{i(n)}^{j(n)}x_n\| + L(\|S_{i(n)}^{j(n)-1}x_n - S_{i(n-N)}^{j(n)-1}x_{n-N}\| \\ &\quad + \|S_{i(n-N)}^{j(n)-1}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|). \end{aligned} \tag{3.9}$$

Notice that

$$j(n - N) = j(n) - 1 \quad \text{and} \quad i(n - N) = i(n).$$

This in turn implies that

$$\begin{aligned} \|S_{i(n)}^{j(n)-1}x_n - S_{i(n-N)}^{j(n)-1}x_{n-N}\| &= \|S_{i(n)}^{j(n)-1}x_n - S_{i(n)}^{j(n)-1}x_{n-N}\| \\ &\leq L \|x_n - x_{n-N}\| \end{aligned} \tag{3.10}$$

and

$$\|S_{i(n-N)}^{j(n)-1}x_{n-N} - x_{n-N}\| = \|S_{i(n-N)}^{j(n-N)}x_{n-N} - x_{n-N}\|. \tag{3.11}$$

Substituting (3.10) and (3.11) into (3.9) yields that

$$\begin{aligned} \|x_n - S_{i(n)}x_n\| &\leq \|x_n - S_{i(n)}^{j(n)}x_n\| + L(L\|x_n - x_{n-N}\| \\ &\quad + \|S_{i(n-N)}^{j(n-N)}x_{n-N} - x_{n-N}\|). \end{aligned}$$

It follows from (3.6) and (3.7) that

$$\lim_{n \rightarrow \infty} \|x_n - S_{i(n)}x_n\| = 0. \tag{3.12}$$

In particular, we see that

$$\begin{cases} \lim_{j \rightarrow \infty} \|x_{jN+1} - S_1x_{jN+1}\| = 0, \\ \lim_{j \rightarrow \infty} \|x_{jN+2} - S_2x_{jN+2}\| = 0, \\ \vdots \\ \lim_{j \rightarrow \infty} \|x_{jN+N} - S_Nx_{jN+N}\| = 0. \end{cases}$$

For any $r, s = 1, 2, \dots, N$, we obtain that

$$\begin{aligned} &\|x_{jN+s} - S_r x_{jN+s}\| \\ &\leq \|x_{jN+s} - x_{jN+r}\| + \|x_{jN+r} - S_r x_{jN+r}\| + \|S_r x_{jN+r} - S_r x_{jN+s}\| \\ &\leq (1 + L)\|x_{jN+s} - x_{jN+r}\| + \|x_{jN+r} - S_r x_{jN+r}\|. \end{aligned}$$

Letting $j \rightarrow \infty$, we arrive at

$$\lim_{j \rightarrow \infty} \|x_{jN+s} - S_r x_{jN+s}\| = 0,$$

which is equivalent to

$$\lim_{n \rightarrow \infty} \|x_n - S_r x_n\| = 0. \tag{3.13}$$

Notice that

$$\begin{aligned} \|x_n - T_{i(n)}x_n\| &\leq \|x_n - T_{i(n)}^{j(n)}x_n\| + \|T_{i(n)}^{j(n)}x_n - T_{i(n)}x_n\| \\ &\leq \|x_n - T_{i(n)}^{j(n)}x_n\| + L\|T_{i(n)}^{j(n)-1}x_n - x_n\| \\ &\leq \|x_n - T_{i(n)}^{j(n)}x_n\| + L(\|T_{i(n)}^{j(n)-1}x_n - T_{i(n-N)}^{j(n)-1}x_{n-N}\| \\ &\quad + \|T_{i(n-N)}^{j(n)-1}x_{n-N} - x_{n-N}\| + \|x_{n-N} - x_n\|). \end{aligned} \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} \|T_{i(n)}^{j(n)-1}x_n - T_{i(n-N)}^{j(n)-1}x_{n-N}\| &= \|T_{i(n)}^{j(n)-1}x_n - T_{i(n)}^{j(n)-1}x_{n-N}\| \\ &\leq L\|x_n - x_{n-N}\| \end{aligned} \tag{3.15}$$

and

$$\|T_{i(n-N)}^{j(n)-1}x_{n-N} - x_{n-N}\| = \|T_{i(n-N)}^{j(n-N)}x_{n-N} - x_{n-N}\|. \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) yields that

$$\begin{aligned} \|x_n - T_{i(n)}x_n\| &\leq \|x_n - T_{i(n)}^{k(n)}x_n\| + L(L\|x_n - x_{n-N}\| \\ &\quad + \|T_{i(n-N)}^{k(n-N)}x_{n-N} - x_{n-N}\|). \end{aligned}$$

It follows from (3.6) and (3.8) that

$$\lim_{n \rightarrow \infty} \|x_n - T_{i(n)}x_n\| = 0. \tag{3.17}$$

In particular, we see that

$$\begin{cases} \lim_{k \rightarrow \infty} \|x_{jN+1} - T_1 x_{jN+1}\| = 0, \\ \lim_{k \rightarrow \infty} \|x_{jN+2} - T_2 x_{jN+2}\| = 0, \\ \vdots \\ \lim_{k \rightarrow \infty} \|x_{jN+N} - T_N x_{jN+N}\| = 0. \end{cases}$$

For any $r, s = 1, 2, \dots, N$, we obtain that

$$\begin{aligned} & \|x_{jN+s} - T_r x_{jN+s}\| \\ & \leq \|x_{jN+s} - x_{jN+r}\| + \|x_{jN+r} - T_r x_{jN+r}\| + \|T_r x_{jN+r} - T_r x_{jN+s}\| \\ & \leq (1 + L) \|x_{jN+s} - x_{jN+r}\| + \|x_{jN+r} - T_r x_{jN+r}\|. \end{aligned}$$

Letting $j \rightarrow \infty$, we arrive

$$\lim_{j \rightarrow \infty} \|x_{jN+s} - T_r x_{jN+s}\| = 0,$$

which is equivalent to

$$\lim_{n \rightarrow \infty} \|x_n - T_r x_n\| = 0. \tag{3.18}$$

This completes the proof. \square

Now, we are in a position to give weak convergence theorems.

Theorem 3.2. *Let E be a real Hilbert space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ and $S_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{k_n^i\} \subset [1, \infty)$, where $k_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in C and $e_n = \max\{h_n, k_n\}$, where $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$ and $k_n = \sup\{k_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\beta'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (2.4). Put $v_n^i = \max\{0, \sup_{x, y \in C} (\|S_i^n x - S_i^n y\| - k_n^i \|x - y\|)\}$ and $v_n^i = \max\{0, \sup_{x, y \in C} (\|S_i^n x - S_i^n y\| - k_n^i \|x - y\|)\}$. Let $\xi_n = \max\{\mu_n, v_n\}$, where $v_n = \max\{v_n^i : 1 \leq i \leq N\}$ and $v_n = \max\{v_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:*

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (e_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$;
- (c) $\beta_n L(\alpha'_n + \beta'_n L) < 1$, where L is defined in (2.5);
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that $\lambda \leq \alpha_n, \alpha'_n \leq \eta$.

Then the sequence $\{x_n\}$ converges weakly to some point in \mathcal{F} .

Proof. Since E is a Hilbert space and $\{x_n\}$ is bounded, we can obtain that there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ converges weakly to $x^* \in C$. It follows from (3.13) and (3.18) that

$$\lim_{p \rightarrow \infty} \|x_{n_p} - T_r x_{n_p}\| = \lim_{p \rightarrow \infty} \|x_{n_p} - S_r x_{n_p}\| = 0.$$

Since $I - S_r$ and $I - T_r$ are demiclosed at origin by Lemma 2.3, we see that $x^* \in \mathcal{F}$. Next, we show that the whole sequence $\{x_n\}$ converges weakly to x^* . Suppose the contrary. Then there exists some subsequence $\{x_{n_q}\}$ of $\{x_n\}$ such that $\{x_{n_q}\}$ converges weakly to $x^{**} \in C$. In the same way, we can show that $x^{**} \in \mathcal{F}$. Notice that we have proved that $\lim_{n \rightarrow \infty} \|x_n - f\|$ exists for each $f \in \mathcal{F}$. By virtue of Opial's condition of E , we have

$$\begin{aligned} \liminf_{p \rightarrow \infty} \|x_{n_p} - x^*\| &< \liminf_{p \rightarrow \infty} \|x_{n_p} - x^{**}\| \\ &= \liminf_{q \rightarrow \infty} \|x_{n_q} - x^{**}\| \\ &< \liminf_{q \rightarrow \infty} \|x_{n_q} - x^*\|. \end{aligned}$$

This is a contradiction. Hence, $x^* = x^{**}$. This completes the proof. \square

Remark 3.3. Theorem 3.2 which includes the corresponding results announced in Chang et al. [6], Chidume and Shahzad [7], Guo and Cho [10], Plubtieng et al. [14], Qin et al. [15], Thakur [21], Thianwan and Suantai [22], Xu and Ori [23], and Zhou and Chang [24] as special cases mainly improves the results of Cianciaruso et al. [9] in the following aspects.

- (1) Extend the mappings from one family of mappings to two families of mappings;
- (2) Extend the mappings from the class of asymptotically nonexpansive mappings to the class of generalized asymptotically nonexpansive mappings.

If $S_r = I$ for each $r \in \{1, 2, \dots, N\}$ and $\gamma'_n = 0$, then Theorem 3.2 is reduced to the following.

Corollary 3.4. *Let E be a real Hilbert space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C and $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in the following process:*

$$x_0 \in C, \quad x_n = \alpha_n x_{n-1} + \beta_n T_{i(n)}^{j(n)} x_n + \gamma_n u_n, \quad n \geq 1. \tag{3.19}$$

Put $\mu_n^i = \max\{0, \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - h_n^i \|x - y\|)\}$. Let $\mu_n = \max\{\mu_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$;
- (c) $\beta_n L < 1$, where $L = \max\{L_i^i : 1 \leq i \leq N\}$;
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that $\lambda \geq \alpha_n, \alpha'_n \leq \eta$.

Then the sequence $\{x_n\}$ converges weakly to some point in \mathcal{F} .

Next, we are in a position to state strong convergence theorems in a Banach space.

Theorem 3.5. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized*

asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ and $S_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{k_n^i\} \subset [1, \infty)$, where $k_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in C and $e_n = \max\{h_n, k_n\}$, where $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$ and $k_n = \sup\{k_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (2.4). Put

$$\mu_n^i = \max\{0, \sup_{x, \gamma \in C} (\|T_i^n x - T_i^n \gamma\| - h_n^i \|x - \gamma\|)\}$$

and

$$v_n^i = \max\{0, \sup_{x, \gamma \in C} (\|S_i^n x - S_i^n \gamma\| - k_n^i \|x - \gamma\|)\}.$$

Let $\xi_n = \max\{\mu_n, v_n\}$, where $v_n = \max\{v_n^i : 1 \leq i \leq N\}$ and $\mu_n = \max\{\mu_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (e_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$;
- (c) $\beta_n L(\alpha'_n + \beta'_n L) < 1$, where L is defined in (2.5);
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that $\lambda \geq \alpha_n, \alpha'_n \leq \eta$.

If one of mappings in $\{T_1, T_2, \dots, T_N\}$ or one of mappings in $\{S_1, S_2, \dots, S_N\}$ are semicompact, then the sequence $\{x_n\}$ converges strongly to some point in \mathcal{F} .

Proof. Without loss of generality, we may assume that S_1 are semicompact. It follows from (3.13) that

$$\lim_{n \rightarrow \infty} \|x_n - S_1 x_n\| = 0.$$

By the semicompactness of S_1 , we see that there exists a subsequence $\{x_{n_p}\}$ of $\{x_n\}$ such that $x_{n_p} \rightarrow w \in C$ strongly. From (3.13) and (3.18), we have

$$\|w - S_r w\| \leq \|w - x_{n_p}\| + \|x_{n_p} - S_r x_{n_p}\| + \|S_r x_{n_p} - S_r w\|$$

and

$$\|w - T_r w\| \leq \|w - x_{n_p}\| + \|x_{n_p} - T_r x_{n_p}\| + \|T_r x_{n_p} - T_r w\|$$

Since S_r and T_r are Lipschitz continuous, we obtain that $w \in \mathcal{F}$. From Theorem 3.1, we know that $\lim_{n \rightarrow \infty} \|x_n - f\|$ exists for each $f \in \mathcal{F}$. That is, $\lim_{n \rightarrow \infty} \|x_n - w\|$ exists. From $x_{n_p} \rightarrow w$, we have

$$\lim_{n \rightarrow \infty} \|x_n - w\| = 0.$$

This completes the proof of Theorem 3.5. \square

If $S_r = I$ for each $r \in \{1, 2, \dots, N\}$ and $\gamma'_n = 0$, then Theorem 3.5 is reduced to the following.

Corollary 3.6. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^1 -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C and $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in*

(3.19). Put $\mu_n^i = \max\{0, \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - h_n^i \|x - y\|)\}$. Let $\mu_n = \max\{\mu_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$;
- (c) $\beta_n L < 1$, where $L = \max\{L_i^i : 1 \leq i \leq N\}$;
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that $\lambda \geq \alpha_n, \alpha'_n \leq \eta$.

If one of mappings in $\{T_1, T_2, \dots, T_N\}$ is semicompact, then the sequence $\{x_n\}$ converges strongly to some point in \mathcal{F} .

Theorem 3.7. Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ and $S_i : C \rightarrow C$ be a uniformly L_s^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{k_n^i\} \subset [1, \infty)$, where $k_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \cap \bigcap_{i=1}^N F(S_i) \neq \emptyset$. Let $\{u_n\}, \{v_n\}$ be bounded sequences in C and $e_n = \max\{h_n, k_n\}$, where $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$ and $k_n = \sup\{k_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha'_n\}, \{\beta'_n\}$ and $\{\gamma'_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (2.4). Put $\mu_n^i = \max\{0, \sup_{x,y \in C} (\|T_i^n x - T_i^n y\| - h_n^i \|x - y\|)\}$ and $\nu_n^i = \max\{0, \sup_{x,y \in C} (\|S_i^n x - S_i^n y\| - k_n^i \|x - y\|)\}$. Let $\xi_n = \max\{\mu_n, \nu_n\}$, where $\nu_n = \max\{\nu_n^i : 1 \leq i \leq N\}$ and $\nu_n = \max\{\nu_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \gamma'_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (e_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$;
- (c) $\beta_n L(\alpha'_n + \beta'_n L) < 1$, where L is defined in (2.5);
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that $\lambda \geq \alpha_n, \alpha'_n \leq \eta$.

If there exists a nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and $g(m) > 0$ for all $m \in (0, \infty)$ such that

$$\max_{1 \leq r \leq N} \{\|x - S_r x\|\} + \max_{1 \leq r \leq N} \{\|x - T_r x\|\} \geq g(\text{dist}(x, \mathcal{F})), \quad \forall x \in C,$$

then the sequence $\{x_n\}$ converges strongly to some point in \mathcal{F} .

Proof. In view of (3.13) and (3.18) that $g(\text{dist}(x_n, \mathcal{F})) \rightarrow 0$, which implies $\text{dist}(x_n, \mathcal{F}) \rightarrow 0$. Next, we show that the sequence $\{x_n\}$ is Cauchy. In view of (3.3), we obtain by putting

$$a_n = \frac{(1 + M_1)(e_{j(n)} - 1)}{1 - R} \quad \text{and} \quad b_n = \frac{(1 + M_1)\xi_{j(n)} + M_1 M_2 \gamma'_n + M_3 \gamma_n}{1 - R}$$

that

$$\|x_n - f\| \leq (1 + a_n) \|x_{n-1} - f\| + b_n.$$

It follows, for any positive integers m, n , where $m > n > n_0$, that

$$\|x_m - p\| \leq B \|x_n - p\| + B \sum_{i=n+1}^{\infty} b_i + b_m,$$

where $B = \exp\{\sum_{n=1}^{\infty} a_n\}$. It follows that

$$\begin{aligned} \|x_n - x_m\| &\leq \|x_n - f\| + \|x_m - f\| \\ &\leq (1 + B) \|x_n - f\| + B \sum_{i=n+1}^{\infty} b_i + b_m. \end{aligned}$$

Taking the infimum over all $f \in \mathcal{F}$, we arrive at

$$\|x_n - x_m\| \leq (1 + B) \text{dist}(x_n, \mathcal{F}) + B \sum_{i=n+1}^{\infty} b_i + b_m.$$

In view of $\sum_{n=1}^{\infty} b_n < \infty$ and $\text{dist}(x_n, \mathcal{F}) \rightarrow 0$, we see that $\{x_n\}$ is a Cauchy sequence in C and so $\{x_n\}$ converges strongly to some $x^* \in C$. Since T_r and S_r are Lipschitz for each $r \in \{1, 2, \dots, N\}$, we see that \mathcal{F} is closed. This in turn implies that $x^* \in \mathcal{F}$. This completes the proof. \square

If $S_r = I$ for each $r \in \{1, 2, \dots, N\}$ and $\gamma'_n = 0$, then Theorem 3.7 is reduced to the following.

Corollary 3.8. *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E . Let $T_i : C \rightarrow C$ be a uniformly L_i^i -Lipschitz and generalized asymptotically nonexpansive mapping with a sequence $\{h_n^i\} \subset [1, \infty)$, where $h_n^i \rightarrow 1$ as $n \rightarrow \infty$ for each $1 \leq i \leq N$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in C and $h_n = \sup\{h_n^i : 1 \leq i \leq N\}$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for each $n \geq 1$. Let $\{x_n\}$ be a sequence generated in (3.19). Put $\mu_n^i = \max\{0, \sup_{x, \gamma \in C} (\|T_i^n x - T_i^n \gamma\| - h_n^i \|x - \gamma\|)\}$. Let $\mu_n = \max\{\mu_n^i : 1 \leq i \leq N\}$. Assume that the following restrictions are satisfied:*

- (a) $\sum_{n=1}^{\infty} \gamma_n < \infty$;
- (b) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \mu_n < \infty$;
- (c) $\beta_n L < 1$, where $L = \max\{L_i^i : 1 \leq i \leq N\}$;
- (d) there exist constants $\lambda, \eta \in (0, 1)$ such that, $\lambda \geq \alpha_n, \alpha'_n \leq \eta$.

If there exists a nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ and $g(m) > 0$ for all $m \in (0, \infty)$ such that

$$\max_{1 \leq r \leq N} \{\|x - T_r x\|\} \geq g(\text{dist}(x, \mathcal{F})), \quad \forall x \in C,$$

then the sequence $\{x_n\}$ converges strongly to some point in \mathcal{F} .

Acknowledgements

The authors are indebted to the referees for their helpful comments. The work was supported partially by Natural Science Foundation of Zhejiang Province (Y6110270).

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Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

Received: 13 April 2011 Accepted: 27 September 2011 Published: 27 September 2011

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doi:10.1186/1687-1812-2011-58

Cite this article as: Agarwal et al.: An implicit iterative algorithm with errors for two families of generalized asymptotically nonexpansive mappings. *Fixed Point Theory and Applications* 2011 **2011**:58.

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