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Fixed point theorems and Δ-convergence theorems for generalized hybrid mappings on CAT(0) spaces

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Abstract

In this paper, we introduce generalized hybrid mapping on CAT(0) spaces. The class of generalized hybrid mappings contains the class of nonexpansive mappings, nonspreading mappings, and hybrid mappings. We study the fixed point theorems of generalized hybrid mappings on CAT(0) spaces. We also consider some iteration processes for generalized hybrid mappings on CAT(0) spaces, and our results generalize some results of fixed point theorems on CAT(0) spaces and Hilbert spaces.

Keywords: nonexpansive mapping, fixed point, generalized hybrid mapping, CAT(0) spaces

1 Introduction

Fixed point theory in CAT(0) spaces was first studied by Kirk [1,2]. He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then, the fixed point theory for single-valued and multivalued mappings in CAT(0) spaces has been rapidly developed, and many papers have appeared (e.g., see [3-6] and related references.)

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0, \ell] \subseteq R$ to X such that $c(0) = x$, $c(\ell) = y$, and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, \ell]$. In particular, c is an isometry and $d(x, y) = \ell$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, and x_3 in X (the vertices of Δ and a geodesic segment between each pair of vertices (the edge of Δ). A comparison triangle for geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}_2$ such that $d_{\mathbb{E}_2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

CAT(0): Let Δ be a geodesic triangle in X, and let $\overline{\Delta}$ be a comparison triangle for Δ. Then, Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$, $d(x, y) \leq d_{\mathbb{E}_2}(\bar{x}, \bar{y})$. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include Pre-Hilbert spaces [7], R-trees [8], the complex Hilbert ball with a hyperbolic metric [9], and many others.

If x, y_1, y_2 are points in a CAT(0) space, and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2).$$

This is the (CN) inequality of Bruhat and Tits [10]. In fact, a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality ([[7], p. 163]).

In 2008, Dhompongsa and Panyanak [11] gave the following result, and the proof is similar to the proof of remark in [[12], p. 374].

Lemma 1.1. [11] Let X be a CAT(0) space. Then,

$$d((1-t)x \oplus ty, z) \leq (1-t)d(x, z) + td(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

By the above lemma, we know that CAT(0) space is a convex metric space. Indeed, a metric space X with a convex structure if there exists a mapping $W : X \times X \times [0, 1] \to X$ such that

$$d(W(x, y, t), z) \leq td(x, z) + (1-t)d(y, z)$$

for all $x, y, z \in X$ and $t \in [0, 1]$ and call this space X a convex metric space [13]. Furthermore, Takahashi [13] has proved that

$$d(x, y) = td(x, W(x, y, t)) + (1-t)d(y, W(x, y, t))$$

for all $x, y, z \in X$ and $t \in [0, 1]$ when X is a convex metric space with a convex structure. So, we also get the following result, and it is proved in [11].

Lemma 1.2. [11] Let X be a CAT(0) space, and $x, y \in X$. For each $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that $d(x, z) = td(x, y)$ and $d(y, z) = (1 - t)d(x, y)$.

For convenience, from now on we will use the notation $z = (1 - t)x \oplus ty$. Therefore, we have:

$$z = (1-t)x \oplus ty \Leftrightarrow z \in [x, y], d(x, z) = td(x, y), \text{and } d(y, z) = (1-t)d(x, y).$$

Let C be a nonempty closed convex subset of a CAT(0) space (X, d). A mapping $T : C \to C$ is called a nonexpansive mapping if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T if $Tx = x$. Let $F(T)$ denote the set of fixed points of T.

Now, we introduce the following nonlinear mappings on CAT(0) spaces.

Definition 1.1. Let C be a nonempty closed convex subset of a CAT(0) space X. We say $T : C \to X$ is a generalized hybrid mapping if there are functions a_1, a_2, a_3, k_1, $k_2 : C \to [0, 1)$ such that

(P1) $d^2(Tx, Ty)$

$$\leq a_1(x)d^2(x,y) + a_2(x)d^2(Tx,y) + a_3(x)d^2(Ty,x) + k_1(x)d^2(Tx,x) + k_2(x)d^2(Ty,y)$$

for all $x, y \in C$;

(P2) $a_1(x) + a_2(x) + a_3(x) \leq 1$ for all $x, y \in C$;

(P3) $2k_1(x) < 1 - a_2(x)$ and $k_2(x) < 1 - a_3(x)$ for all $x \in C$.

Remark 1.1. In Definition 1.1, if $a_1(x) = 1$ and $a_2(x) = a_3(x) = k_1(x) = k_2(x) = 0$ for all $x \in C$, then T is a nonexpansive mapping.

In 2008, Kohsaka and Takahashi [14] introduced nonspreading mappings on Banach spaces. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be a nonspreading mapping if $2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||Ty - x||^2$ for all $x, y \in C$ (for detail, refer to [15]).

In 2010, Takahashi [16] introduced hybrid mapping on Hilbert spaces. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be hybrid if $3||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2 + ||x - Ty||^2$ for all $x, y \in C$.

In 2011, Takahashi and Yao [17] also introduced two nonlinear mappings in Hilbert spaces. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be TJ-1 if $2||Tx - Ty||^2 \leq ||x - y||^2 + ||Tx - y||^2$ for all $x, y \in C$. A mapping $T : C \to C$ is said to be TJ-2 if $3||Tx - Ty||^2 \leq 2||Tx - y||^2 + ||Ty - x||^2$ for all $x, y \in C$.

Now, we give the definitions of nonspreading mapping, TJ-1, TJ-2, hybrid mappings on CAT(0) spaces. In fact, these are special cases of generalized hybrid mapping on CAT(0) spaces.

Definition 1.2. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Then, $T : C \to C$ is said to be a nonspreading mapping if $2d^2(Tx, Ty) \leq d^2(Tx, y) + d^2(Ty, x)$ for all $x, y \in C$.

Definition 1.3. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Then, $T : C \to C$ is said to be hybrid if $3d^2(Tx, Ty) \leq d^2(x, y)+d^2(Tx, y)+d^2(x, Ty)$ for all $x, y \in C$.

Definition 1.4. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Then, $T : C \to C$ is said to be TJ-1 if $2d^2(Tx, Ty) \leq d^2(x, y) + d^2(Tx, y)$ for all $x, y \in C$.

Definition 1.5. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Then, $T : C \to C$ is said to be TJ-2 if $3d^2(Tx, Ty) \leq 2d^2(Tx, y) + d^2(Ty, x)$ for all $x, y \in C$.

On the other hand, we observe that construction of approximating fixed points of nonlinear mappings is an important subject in the theory of nonlinear mappings and its applications in a number of applied areas. Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T, S : C \to C$ be two mappings.

In 1953, Mann [18] gave an iteration process:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, n \geq 0,$$

where the initial guess x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ is a sequence in the interval [0, 1].

In 1974, Ishikawa [19] gave an iteration process which is defined recursively by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n := (1-\beta_n)x_n + \beta_n T x_n \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

In 1986, Das and Debata [20] studied a two mappings's iteration on the pattern of the Ishikawa iteration:

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n := (1-\beta_n)x_n + \beta_n S x_n \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

In 2007, Agarwal et al. [21] introduced the following iterative process:

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := (1-\alpha_n)T x_n + \alpha_n T y_n, \\ y_n := (1-\beta_n)x_n + \beta_n T x_n, \end{cases} \tag{1.2}$$

where the initial guess x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

In 2011, Khan and Abbas [22] modified (1.1) and (1.2) for two nonexpansive mappings S and T in CAT(0) spaces as follows.

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := (1-\alpha_n)x_n \oplus \alpha_n T y_n, \\ y_n := (1-\beta_n)x_n \oplus \beta_n S x_n, \end{cases} \tag{1.3}$$

and

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := (1-\alpha_n)T x_n \oplus \alpha_n T y_n, \\ y_n := (1-\beta_n)x_n \oplus \beta_n T x_n, \end{cases} \tag{1.4}$$

where the initial guess x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

Let D be a nonempty closed convex subset of a complete CAT(0) space (X, d). For each $x \in X$, there exists a unique element $y \in D$ such that $d(x, y) = \min_{z \in D} d(x, z)$[7]. In the sequel, let $P_D : X \to D$ be defined by

$$P_D(x) = y \Leftrightarrow d(x, y) = \min_{z \in D} d(x, z).$$

And we call P_D the metric projection from the complete CAT(0) space X onto a nonempty closed convex subset D of X. Note that P_D is a nonexpansive mapping [7].

Now, let C be a nonempty closed convex subset of a complete CAT(0) space X, let $T, S : C \to X$ be two nonexpansive mappings, and we modified (1.3) and (1.4) as follows:

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)x_n \oplus \alpha_n T y_n), \\ y_n := P_C((1-\beta_n)x_n \oplus \beta_n S x_n), \end{cases} \tag{1.5}$$

and

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)Tx_n \oplus \alpha_n T y_n), \\ y_n := P_C((1-\beta_n)x_n \oplus \beta_n T x_n), \end{cases} \tag{1.6}$$

where the initial guess x_0 is taken in C arbitrarily, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval [0, 1].

In this paper, we study the fixed point theorems of generalized hybrid mappings on CAT(0) spaces. Next, we also consider iteration process (1.5), (1.6), or Mann's type for generalized hybrid mappings on CAT(0) spaces, and our results improve or generalize recent results on fixed point theorems on CAT(0) spaces or Hilbert spaces.

2 Preliminaries

In this paper, we need the following definitions, notations, lemmas, and related results.

Lemma 2.1. [11] Let X be a CAT(0) space. Then,

$$d^2((1-t)x \oplus ty, z) \leq (1-t)d^2(x,z) + td^2(y,z) - t(1-t)d^2(x,y)$$

for all $t \in [0, 1]$ and $x, y, z \in X$.

Definition 2.1. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X, and let C be a subset of X. Now, we use the following notations:

(i) $r(x, \{x_n\}) := \limsup_{n\to\infty} d(x, x_n)$.
(ii) $r(\{x_n\}) := \inf_{x\in X} r(x, \{x_n\})$.
(iii) $r_C(\{x_n\}) := \inf_{x\in C} r(x, \{x_n\})$.
(iv) $A(\{x_n\}) := \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.
(iv) $A_C(\{x_n\}) := \{x \in C : r(x, \{x_n\}) = r_C(\{x_n\})\}$.

Note that $x \in X$ is called an asymptotic center of $\{x_n\}$ if $x \in A(\{x_n\})$. It is known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point [23].

Definition 2.2. [6] Let (X, d) be a CAT(0) space. A sequence $\{x_n\}$ in X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. That is, $A(\{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-limit of $\{x_n\}$.

In 2008, Kirk and Panyanak [6] gave the following result for nonexpansive mappings on CAT(0) spaces.

Theorem 2.1. [6] Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping. Let $\{x_n\}$ be a bounded sequence in C with $\Delta\text{-}\lim_n x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $x \in C$ and $Tx = x$.

Lemma 2.2. [6] Let (X, d) be a CAT(0) space. Then, every bounded sequence in X has a Δ-convergent subsequence.

Lemma 2.3. [24] Let C be a nonempty closed convex subset of a CAT(0) space X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is in C.

Lemma 2.4. [11] Let C be a nonempty closed convex subset of a CAT(0) space (X, d).

Let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$, and let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$. Suppose that $\lim_{n\to\infty} d(x_n, u)$ exists. Then, $x = u$.

Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d), and let C be a nonempty closed convex subset of X which contains $\{x_n\}$. We denote the notation

$$x_n \rightharpoonup w \text{ iff } \Phi(w) = \inf_{x\in C} \Phi(x),$$

where $\Phi(x) := \limsup_{n\to\infty} d(x_n, x)$. Then, we observe that

$$A(\{x_n\}) = \{x \in X : \Phi(x) = \inf_{u\in X} \Phi(u)\}, \text{ and}$$
$$A_C(\{x_n\}) = \{x \in C : \Phi(x) = \inf_{u\in C} \Phi(u)\}.$$

Remark 2.1. Let $\{x_n\}$ be a bounded sequence in a CAT(0) space (X, d), and let C be a nonempty closed convex subset of X which contains $\{x_n\}$. If $x_n \rightharpoonup w$, then $w \in C$.

Proof. There exist $\bar{x} \in X$ and $\bar{y} \in C$ such that $A(\{x_n\}) = \{\bar{x}\}$ and $A_C(\{x_n\}) = \{\bar{y}\}$. By Lemma 2.3, $\bar{x} = \bar{y}$. Hence,

$$\Phi(\bar{y}) = \Phi(\bar{x}) \leq \Phi(w) = \inf_{x\in C} \Phi(x) = \Phi(\bar{y}).$$

Hence, $w \in A(\{x_n\})$ and $w = \bar{x} \in C$. □

Lemma 2.5. [25] Let C be a nonempty closed convex subset of a CAT(0) space (X, d), and let $\{x_n\}$ be a bounded sequence in C. If $\Delta\text{-}\lim_n x_n = x$, then $x_n \rightharpoonup x$.

Proposition 2.1. Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d), and let $T : C \to X$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Then, $F(T)$ is a closed convex subset of C.

Proof. If $\{x_n\}$ is a sequence in $F(T)$ and $\lim_{n\to\infty} x_n = x$. Then, we have:

$$d^2(Tx, x_n) \leq d^2(x, x_n) + \frac{k_1(x)}{1 - a_2(x)} d^2(Tx, x).$$

This implies that

$$(1 - \frac{k_1(x)}{1 - a_2(x)}) d^2(Tx, x) \leq 0.$$

Then, $Tx = x$ and $F(T)$ is a closed set.

Next, we want to show that $F(T)$ is a convex set. If $x, y \in F(T) \subseteq C$ and $z \in [x, y]$, then there exists $t \in [0, 1]$ such that $z = tx \oplus (1 - t)y$. Since C is convex, $z \in C$.

Furthermore,

$$\begin{aligned}
&d^2(Tz,z)\\
&\le td^2(Tz,x)+(1-t)d^2(Tz,y)-t(1-t)d^2(x,y)\\
&\le td^2(z,x)+\tfrac{tk_1(z)}{1-a_2(z)}d^2(Tz,z)+(1-t)d^2(z,y)+\tfrac{(1-t)k_1(z)}{1-a_2(z)}d^2(Tz,z)-t(1-t)d^2(x,y)\\
&\le t(1-t)^2d^2(x,y)+\tfrac{k_1(z)}{1-a_2(z)}d^2(Tz,z)+t^2(1-t)d^2(x,y)-t(1-t)d^2(x,y)\\
&\le \tfrac{k_1(z)}{1-a_2(z)}d^2(Tz,z).
\end{aligned}$$

Hence, $Tz = z$ and $F(T)$ is a convex set. □

Remark 2.2. Let C be a nonempty closed convex subset of a complete CAT(0) space (X, d), and let $T : C \to X$ be any one of nonspreading mapping, TJ-1 mapping, TJ-2 mapping, and hybrid mapping. If $F(T) \neq \emptyset$, then $F(T)$ is a closed convex subset of C.

3 Fixed point theorems on complete CAT(0) spaces

The following theorem establishes a demiclosed principle for a generalized hybrid mapping on CAT(0) spaces.

Theorem 3.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to X$ be a generalized hybrid mapping. Let $\{x_n\}$ be a bounded sequence in C with $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $x \in C$ and $Tx = x$.

Proof. Since $x_n \rightharpoonup x$, we know that $x \in C$ and $\Phi(x) = \inf_{u\in C}\Phi(u)$, where $\Phi(u) := \limsup_{n\to\infty} d(x_n, u)$. Furthermore, we know that $\Phi(x) = \inf\{\Phi(u) : u \in X\}$. Since T is a generalized hybrid,

$$\begin{aligned}
&d^2(Tx_n,Tx)\\
&\le a_1(x)d^2(x,x_n)+a_2(x)d^2(Tx,x_n)+a_3(x)d^2(Tx_n,x)+k_1(x)d^2(Tx,x)+\\
&\quad k_2(x)d^2(Tx_n,x_n)\\
&\le a_1(x)d^2(x,x_n)+a_2(x)(d(Tx,Tx_n)+d(Tx_n,x_n))^2+a_3(x)(d(Tx_n,x_n)+d(x_n,x))^2\\
&\quad +k_1(x)d^2(Tx,x)+k_2(x)d^2(Tx_n,x_n).
\end{aligned}$$

Then, we have:

$$\limsup_{n\to\infty} d^2(Tx_n,Tx)\le \limsup_{n\to\infty} d^2(x,x_n)+\frac{k_1(x)}{(1-a_2(x))}d^2(x,Tx).$$

This implies that

$$\begin{aligned}
&\limsup_{n\to\infty} d^2(x_n,Tx)\\
&\le \limsup_{n\to\infty}\,(d(x_n,Tx_n)+d(Tx_n,Tx))^2\\
&\le \limsup_{n\to\infty} d^2(Tx_n,Tx)\\
&\le \limsup_{n\to\infty} d^2(x,x_n)+\frac{k_1(x)}{1-a_2(x)}d^2(x,Tx).
\end{aligned}$$

Besides, by (CN) inequality, we have:

$$d^2(x_n,\frac{1}{2}x\oplus\frac{1}{2}Tx)\le\frac{1}{2}d^2(x_n,x)+\frac{1}{2}d^2(x_n,Tx)-\frac{1}{4}d^2(x,Tx).$$

So,

$$
\begin{aligned}
&\limsup_{n\to\infty} d^2(x_n, \frac{1}{2}x \oplus \frac{1}{2}Tx) \\
&\leq \frac{1}{2}\limsup_{n\to\infty} d^2(x_n, x) + \frac{1}{2}\limsup_{n\to\infty} d^2(x_n, Tx) - \frac{1}{4}d^2(x, Tx) \\
&\leq \limsup_{n\to\infty} d^2(x_n, x) + \frac{k_1(x)}{2(1-a_2(x))}d^2(x, Tx) - \frac{1}{4}d^2(x, Tx).
\end{aligned}
$$

So,

$$(\frac{1}{4} - \frac{k_1(x)}{2(1-a_2(x))})d^2(x, Tx) \leq \limsup_{n\to\infty} d^2(x_n, x) - \limsup_{n\to\infty} d^2(x_n, \frac{1}{2}x \oplus \frac{1}{2}Tx).$$

That is,

$$(\frac{1}{4} - \frac{k_1(x)}{2(1-a_2(x))})d^2(x, Tx) \leq (\Phi(x))^2 - (\Phi(\frac{1}{2}x \oplus \frac{1}{2}Tx))^2 \leq 0.$$

Therefore, $Tx = x$. □

By Theorem 3.1 and Lemma 2.5, it is easy to get the conclusion.

Corollary 3.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to X$ be a generalized hybrid mapping. Let $\{x_n\}$ be a bounded sequence in C with $\Delta\text{-}\lim_n x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tx = x$.

Theorem 3.1 generalizes Theorem 2.1 since the class of generalized hybrid mappings contains the class of nonexpansive mappings on CAT(0) spaces. Furthermore, we also get the following result.

Corollary 3.2. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to X$ be any one of nonspreading mapping, TJ-1 mapping, TJ-2 mapping, and hybrid napping. Let $\{x_n\}$ be a bounded sequence in C with $x_n \rightharpoonup x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tx = x$.

Corollary 3.3. [14-17] Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to H$ be a any one of nonspreading mapping, hybrid mapping, TJ-1 mapping, and TJ-2 mapping. Let $\{x_n\}$ be a sequence in C with $\{x_n\}$ converges weakly to $x \in C$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $x \in C$ and $Tx = x$.

Proof. For each $x, y \in H$, let $d(x, y) := ||x - y||$. Clearly, a real Hilbert space H is a CAT(0) space, and C is a nonempty closed convex subset of a CAT(0) space H, and T is generalized hybrid. Since $\{x_n\}$ converges weakly to x, $\{x_n\}$ is a bounded sequence.

Since H is a real Hilbert space,

$$\limsup_{n\to\infty} ||x_n - x|| \leq \limsup_{n\to\infty} ||x_n - y||, \quad \text{for each } y \in C.$$

This implies that $x_n \rightharpoonup x$. By Theorem 3.1, $Tx = x$ and the proof is completed. □

Theorem 3.2. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a generalized hybrid mapping with $k_1(x) = k_2(x) = 0$ for all $x \in C$. Then, the following conditions are equivalent:

(i) $\{T^n x\}$ is bounded for some $x \in C$;
(ii) $F(T) \neq \emptyset$.

Proof. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. For each $n \in \mathbb{N}$, let $x_n := T^n x$. Since $\{x_n\}$ is bounded, there exists $\bar{x} \in X$ such that $A(\{x_n\}) = \{\bar{x}\}$. By Lemma 2.3, $\bar{x} \in C$. Furthermore, we have:

$$d^2(x_n, T\bar{x}) \leq a_1(\bar{x})d^2(\bar{x}, x_{n-1}) + a_2(\bar{x})d^2(T\bar{x}, x_{n-1}) + a_3(\bar{x})d^2(x_n, \bar{x}).$$

This implies that

$$\begin{aligned}
&\limsup_{n\to\infty} d^2(x_n, T\bar{x}) \\
&\leq a_1(\bar{x}) \limsup_{n\to\infty} d^2(\bar{x}, x_{n-1}) + a_2(\bar{x}) \limsup_{n\to\infty} d^2(T\bar{x}, x_{n-1}) + a_3(\bar{x}) \limsup_{n\to\infty} d^2(x_n, \bar{x}) \\
&\leq (a_1(\bar{x}) + a_3(\bar{x})) \limsup_{n\to\infty} d^2(x_n, \bar{x}) + a_2(\bar{x}) \limsup_{n\to\infty} d^2(x_n, T\bar{x}).
\end{aligned}$$

Then

$$(\Phi(T\bar{x}))^2 = \limsup_{n\to\infty} d^2(x_n, T\bar{x}) \leq \limsup_{n\to\infty} d^2(x_n, \bar{x}) = (\Phi(\bar{x}))^2.$$

Since $A(\{x_n\}) = \{\bar{x}\}$, $T\bar{x} = \bar{x}$. This shows that $F(T) \neq \emptyset$. It is easy to see that (ii) implies (i). □

By Theorem 3.2, it is easy to get the following results.

Corollary 3.4. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be any one of nonspreading mapping, TJ-1 mapping, TJ-2 mapping, hybrid mapping, and nonexpansive mapping. Then, $\{T^n x\}$ is bounded for some $x \in C$ if and only if $F(T) \neq \emptyset$.

Corollary 3.5. [1,2] Let C be a nonempty bounded closed convex subset of a complete CAT(0) space X, and let $T : C \to C$ be a nonexpansive mapping. Then, $F(T) \neq \emptyset$.

Corollary 3.6. [14-17,26] Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \to C$ be any one of nonspreading mapping, TJ-1 mapping, TJ- 2 mapping, hybrid mapping, and nonexpansive mapping. Then, $\{T^n x\}$ is bounded for some $x \in C$ if and only if $F(T) \neq \emptyset$.

4 Δ-convergent theorems

In the sequel, we need the following lemmas. By Lemmas 2.2-2.4 and Theorem 3.1, and following the similar argument as in the proof of Lemma 2.10 in [11], we have the following result.

Lemma 4.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X, and let $T : C \to X$ be a generalized hybrid mapping. If $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$, where $\omega_w(x_n) := \cup A(\{u_n\})$ and $\{u_n\}$ is any subsequence of $\{x_n\}$. Furthermore, $\omega_w(x_n)$ consists of exactly one point.

Remark 4.1. The conclusion of Lemma 4.1 is still true if $T : C \to X$ is any one of nonexpansive mapping, nonspreading mapping, TJ-1 mapping, TJ-2 mapping, and hybrid mapping.

Theorem 4.1. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : C \to X$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in [0, 1]. Let $\{x_n\}$ be defined by

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)x_n \oplus \alpha_n Tx_n). \end{cases}$$

Assume $\liminf_{n\to\infty} \alpha_n[(1-\alpha_n) - \frac{k_2(w)}{1-a_3(w)}] > 0$ for all $w \in F(T)$. Then, $\{x_n\}$ Δ-converges to a point of $F(T)$.

Proof. Clearly, $\{x_n\} \subseteq C$. Take any $w \in F(T)$ and let w be fixed. Then,

$$d^2(Tx, w) \le d^2(w, x) + \frac{k_2(w)}{1-a_3(w)} d^2(Tx, x)$$

for all $x \in C$. Hence, by Lemma 2.1,

$$\begin{aligned} & d^2(x_{n+1}, w) \\ = & d^2(P_C((1-\alpha_n)x_n \oplus \alpha_n Tx_n), w) \\ \le & d^2((1-\alpha_n)x_n \oplus \alpha_n Tx_n, w) \\ \le & (1-\alpha_n)d^2(x_n, w) + \alpha_n d^2(Tx_n, w) - \alpha_n(1-\alpha_n)d^2(x_n, Tx_n) \\ \le & d^2(x_n, w) + \alpha_n[\frac{k_2(w)}{1-a_3(w)} - (1-\alpha_n)]d^2(Tx_n, x_n). \end{aligned}$$

By assumption, there exists $\delta > 0$ and $M \in \mathbb{N}$ such that

$$\alpha_n[(1-\alpha_n) - \frac{k_2(w)}{1-a_3(w)}] \ge \delta > 0$$

for all $n \ge M$. Without loss of generality, we may assume that

$$\alpha_n[(1-\alpha_n) - \frac{k_2(w)}{1-a_3(w)}] > 0$$

for all $n \in \mathbb{N}$. Hence, $\{d(x_n, w)\}$ is decreasing, $\lim_{n\to\infty} d(x_n, w)$ exists, and $\{x_n\}$ is bounded.

Then

$$\lim_{n\to\infty} \alpha_n[(1-\alpha_n) - \frac{k_2(w)}{1-a_3(w)}]d^2(x_n, Tx_n) = 0.$$

This implies that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By Lemma 4.1, there exists $\bar{x} \in C$ such that $\omega_w(\{x_n\}) = \{\bar{x}\} \subseteq F(T)$. So, $\Delta - \lim_n x_n = \bar{x}$ and the proof is completed. □

Theorem 4.2. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : C \to X$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0, 1]. Let $\{x_n\}$ be defined as

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n), \\ y_n := P_C((1-\beta_n)x_n \oplus \beta_n Tx_n). \end{cases}$$

Assume that:

(i) $k_2(w) = 0$ for all $w \in F(T)$;
(ii) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$.

Then $\{x_n\}$ Δ-converges to a point of $F(T)$.

Proof. Take any $w \in F(T)$ and let w be fixed. Then, by (i), $d(Tx, w) \leq d(x, w)$ for all $x \in C$. By Lemma 2.1, we have:

$$\begin{aligned}
&d^2(y_n, w)\\
&= d^2(P_C((1-\beta_n)x_n \oplus \beta_n Tx_n), w)\\
&\leq d^2((1-\beta_n)x_n \oplus \beta_n Tx_n, w)\\
&\leq (1-\beta_n)d^2(x_n, w) + \beta_n d^2(Tx_n, w) - \beta_n(1-\beta_n)d(x_n, Tx_n)\\
&\leq d^2(x_n, w) - \beta_n(1-\beta_n)d(x_n, Tx_n)\\
&\leq d^2(x_n, w).
\end{aligned}$$

Hence,

$$\begin{aligned}
&d^2(x_{n+1}, w)\\
&= d^2(P_C((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n), w)\\
&\leq d^2((1-\alpha_n)Tx_n \oplus \alpha_n Ty_n, w)\\
&\leq (1-\alpha_n)d^2(Tx_n, w) + \alpha_n d^2(Ty_n, w) - \alpha_n(1-\alpha_n)d^2(Tx_n, Ty_n)\\
&\leq (1-\alpha_n)d^2(x_n, w) + \alpha_n d^2(y_n, w) - \alpha_n(1-\alpha_n)d^2(Tx_n, Ty_n)\\
&\leq d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(Tx_n, Ty_n)\\
&\leq d^2(x_n, w).
\end{aligned}$$

Hence, $\lim_{n\to\infty} d(x_n, w)$ exists, and $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Besides, we know that

$$\lim_{n\to\infty} \alpha_n(1-\alpha_n)d^2(Tx_n, Ty_n) = 0.$$

This implies that $\lim_{n\to\infty} d(Tx_n, Ty_n) = 0$. So,

$$\limsup_{n\to\infty} d(x_{n+1}, Tx_n) \leq \lim_{n\to\infty} d(Tx_n, Ty_n) = 0.$$

And this implies that $\lim_{n\to\infty} d(x_{n+1}, Tx_n) = 0$. Furthermore, we also have:

$$0 \leq \alpha_n(1-\alpha_n)d^2(Tx_n, Ty_n) \leq d^2(x_n, w) - d^2(x_{n+1}, w) + \alpha_n[d^2(y_n, w) - d^2(x_n, w)].$$

So,

$$\alpha_n(1-\alpha_n)[d^2(x_n, w) - d^2(y_n, w)] \leq d^2(x_n, w) - d^2(x_{n+1}, w)$$

This implies that $\lim_{n\to\infty}(d^2(y_n, w) - d^2(x_n, w)) = 0$. So,

$$\beta_n(1-\beta_n)d^2(x_n, Tx_n) \leq d^2(x_n, w) - d^2(y_n, w).$$

This implies that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By Lemma 4.1, there exists $\bar{x} \in C$ such that $\omega_w(\{x_n\}) = \{\bar{x}\} \subseteq F(T)$. So, $\Delta - \lim_n x_n = \bar{x}$ and the proof is completed. □

Remark 4.2. If $0 < a < b < 1$ and $\{\alpha_n\}$ is a sequence in [0, 1] with $a \leq \alpha_n \leq b$ for all $n \in \mathbb{N}$, then $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) \geq a(1-b) > 0$. Furthermore, the class of generalized hybrid mappings contains the class of nonexpansive mappings in CAT(0) spaces. Hence, Theorem 4.2 generalizes Theorem 1 in [22].

Corollary 4.1. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to X$ be any one of nonspreading mapping, nonexpansive mapping, hybrid mapping, TJ-1 mapping, and TJ-2 mapping. Let P_C be the metric projection from H onto C. Suppose that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0, 1]. Let $\{x_n\}$ be defined as

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)Tx_n + \alpha_n Ty_n), \\ y_n := P_C((1-\beta_n)x_n + \beta_n Tx_n). \end{cases}$$

Assume that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Then, $\{x_n\}$ converges weakly to a point x of $F(T)$.

Proof. For each $x, y \in H$, let $d(x, y) := ||x - y||$. Clearly, H is a CAT(0) space, and C is a nonempty closed convex subset of H. Furthermore, $tx \oplus (1 - t)y = tx + (1 - t)y$ for all $x, y \in C$ and $t \in [0, 1]$. Since T is any one of nonspreading mapping, nonexpansive mapping, hybrid mapping, TJ-1 mapping, and TJ-2 mapping, $k_1(w) = k_2(w) = 0$ for all $w \in F(T)$. By Theorem 4.2, $\{x_n\}$ is a bounded sequence, and $\{x_n\}$ Δ-converges to a point x of $F(T)$.

Next, we want to show that $\{x_n\}$ converges to x. If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\{x_{n_k}\}$ converges weakly to $u \in C$, then $x_{n_k} \rightharpoonup u$ and $A(\{x_{n_k}\}) = \{u\}$. Since $\{x_n\}$ Δ-converges to x, $u = x$. Then, every weakly convergent subsequence of $\{x_n\}$ has the same limit. So, $\{x_n\}$ converges weakly to x, and the proof is completed. □

Lemma 4.2. Let X be a CAT(0) space. Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in X with $\lim_{n\to\infty} d(x_n, y_n) = 0$. If $\Delta - \lim_n x_n = x$, then Δ - $\lim_n y_n = x$.

Proof. Since ??, we know that

$$r(\{x_n\}) = r(x, \{x_{n_k}\}) = \limsup_{k\to\infty} d(x_{n_k}, x)$$

for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Now, take any subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and let $\{y_{n_k}\}$ be fixed. Then, there exists $y \in X$ such that $A(\{y_{n_k}\}) = \{y\}$. Hence,

$$\begin{aligned} \limsup_{n\to\infty} d(y_{n_k}, y) &\le \limsup_{n\to\infty} d(y_{n_k}, x) \\ &\le \limsup_{n\to\infty} d(y_{n_k}, x_{n_k}) + \limsup_{n\to\infty} d(x_{n_k}, x) \\ &= \limsup_{n\to\infty} d(x_{n_k}, x) \\ &= r(\{x_n\}) \\ &\le \limsup_{n\to\infty} d(x_{n_k}, y) \\ &\le \limsup_{n\to\infty} d(x_{n_k}, y). \end{aligned}$$

So, $\limsup_{n\to\infty} d(y_{n_k}, y) = \limsup_{n\to\infty} d(y_{n_k}, x)$. And this implies that $x \in A(\{y_{n_k}\})$. Since $A(\{y_{n_k}\}) = \{y\}$, $x = y$. So, $A(\{y_{n_k}\}) = \{x\}$ for every subsequence $\{y_{n_k}\}$ of $\{y_n\}$. Therefore, $\Delta - \lim_n y_n = x$. □

Theorem 4.3. Let C be a nonempty closed convex subset of a complete CAT(0) space X. Let $T, S : C \to X$ be two generalized hybrid mapping with $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0, 1]. Let $\{x_n\}$ be defined as

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)x_n \oplus \alpha_n T y_n), \\ y_n := P_C((1-\beta_n)x_n \oplus \beta_n S x_n). \end{cases}$$

Assume that:

(i) $k_2^T(w) = 0$ for all $w \in F(T) \cap F(S)$;

(ii) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n[(1-\beta_n) - \frac{k_2^S(w)}{1-a_3^S(w)}] > 0$.

Then, $\{x_n\}$ Δ-converges to a common fixed point of S and T.

Proof. Take any $w \in F(T) \cap F(S)$ and let w be fixed. Then, $d(Tx, w) \leq d(x, w)$ for all $x \leq C$. By Lemma 2.1, we have:

$$\begin{aligned} & d^2(y_n, w) \\ &= d^2(P_C((1-\beta_n)x_n \oplus \beta_n S x_n), w) \\ &\leq d^2((1-\beta_n)x_n \oplus \beta_n S x_n, w) \\ &\leq (1-\beta_n)d^2(x_n, w) + \beta_n d^2(Sx_n, w) - \beta_n(1-\beta_n)d^2(x_n, Sx_n) \\ &\leq (1-\beta_n)d^2(x_n, w) + \beta_n[d^2(x_n, w) + \frac{k_2^S(w)}{1-a_3^S(w)}d^2(Sx_n, x_n)] - \beta_n(1-\beta_n)d^2(x_n, Sx_n) \\ &\leq d^2(x_n, w) - \beta_n[(1-\beta_n) - \frac{k_2^S(w)}{1-a_3^S(w)}]d^2(Sx_n, x_n). \end{aligned}$$

By (ii), there exists $\delta > 0$ and $M \in \mathbb{N}$ such that

$$\beta_n[(1-\beta_n) - \frac{k_2^S(w)}{1-a_3^S(w)}] \geq \delta > 0$$

for all $n \geq M$. Without loss of generality, we may assume that

$$\beta_n[(1-\beta_n) - \frac{k_2^S(w)}{1-a_3^S(w)}] > 0$$

for all $n \in \mathbb{N}$. Hence, we know that $d(y_n, w) \leq d(x_n, w)$, and

$$\begin{aligned} & d^2(x_{n+1}, w) \\ &= d^2(P_C((1-\alpha_n)x_n \oplus \alpha_n T y_n), w) \\ &\leq d^2((1-\alpha_n)x_n \oplus \alpha_n T y_n, w) \\ &\leq (1-\alpha_n)d^2(x_n, w) + \alpha_n d^2(Ty_n, w) - \alpha_n(1-\alpha_n)d^2(x_n, Ty_n) \\ &\leq (1-\alpha_n)d^2(x_n, w) + \alpha_n d^2(y_n, w) - \alpha_n(1-\alpha_n)d^2(x_n, Ty_n) \\ &\leq d^2(x_n, w) - \alpha_n(1-\alpha_n)d^2(x_n, Ty_n) \\ &\leq d^2(x_n, w). \end{aligned}$$

Hence, $\lim_{n\to\infty} d(x_n, w)$ exists, and $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Besides, we know that

$$\lim_{n\to\infty} \alpha_n(1-\alpha_n)d^2(x_n, Ty_n) = 0.$$

This implies that $\lim_{n\to\infty} d(x_n, Ty_n) = 0$. Furthermore, we also have:

$$\alpha_n(1-\alpha_n)d^2(x_n, Ty_n) \le d^2(x_n, w) - d^2(x_{n+1}, w) + \alpha_n[d^2(y_n, w) - d^2(x_n, w)].$$

Then,

$$\alpha_n[d^2(x_n, w) - d^2(y_n, w)] \le d^2(x_n, w) - d^2(x_{n+1}, w).$$

And this implies that $\lim_{n\to\infty}[d^2(x_n, w) - d^2(y_n, w)] = 0$. Besides, we have:

$$\beta_n[(1-\beta_n) - \frac{k_2^S(w)}{1-a_3^S(w)}]d^2(x_n, Sx_n) \le d^2(x_n, w) - d^2(y_n, w).$$

This implies that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$. Hence,

$$\limsup_{n\to\infty} d(y_n, x_n) = \limsup_{n\to\infty} \beta_n d(x_n, Sx_n) \le \limsup_{n\to\infty} d(x_n, Sx_n) = 0.$$

So, $\lim_{n\to\infty} d(y_n, x_n) = 0$, and $\lim_{n\to\infty} d(y_n, Ty_n) = 0$. By Lemma 4.1, there exist $\bar{x}, \bar{y} \in C$ such that $\omega_w(\{x_n\}) = \{\bar{x}\} \subseteq F(S)$ and $\omega_w(\{y_n\}) = \{\bar{y}\} \subseteq F(T)$. So, $\Delta - \lim_n x_n = \bar{x}$ and $\Delta - \lim_n y_n = \bar{y}$. By Lemma 4.2, $\bar{x} = \bar{y}$.

Remark 4.3. Theorem 4.3 generalizes Theorem 4 in [22].

Following the same argument as in the proof of Corollary 4.1, we have the following result from Theorem 4.3.

Corollary 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T, S : C \to X$ be any two of nonspreading mapping, hybrid mapping, TJ-1 mapping, TJ-2 mapping. Suppose that $F(T) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0, 1]. Let P_C be the metric projection from H onto C. Let $\{x_n\}$ be defined as

$$\begin{cases} x_1 \in C \text{ chosen arbitrary,} \\ x_{n+1} := P_C((1-\alpha_n)x_n + \alpha_n Ty_n), \\ y_n := P_C((1-\beta_n)x_n + \beta_n Sx_n). \end{cases}$$

Assume that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $\liminf_{n\to\infty} \beta_n[(1-\beta_n) - \frac{k_2^S(w)}{1-a_3^S(w)}] > 0$. Then, $\{x_n\}$ Δ-converges to a common fixed point of S and T.

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