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Strong convergence theorem for a generalized equilibrium problem and system of variational inequalities problem and infinite family of strict pseudo-contractions

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Abstract

In this article, we introduce a new mapping generated by an infinite family of κ_i-strict pseudo-contractions and a sequence of positive real numbers. By using this mapping, we consider an iterative method for finding a common element of the set of a generalized equilibrium problem of the set of solution to a system of variational inequalities, and of the set of fixed points of an infinite family of strict pseudo-contractions. Strong convergence theorem of the purposed iteration is established in the framework of Hilbert spaces.

Keywords: nonexpansive mappings, strongly positive operator, generalized equilibrium problem, strict pseudo-contraction, fixed point

1 Introduction

Let C be a closed convex subset of a real Hilbert space H, and let $G : C \times C \to \mathbb{R}$ be a bifunction. We know that the equilibrium problem for a bifunction G is to find $x \in C$ such that

$$G(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(G)$. Given a mapping $T : C \to H$, let $G(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in$. Then, $z \in EP(G)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Let $A : C \to H$ be a nonlinear mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au\rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. Now, we consider the following generalized equilibrium problem:

$$\text{Find } z \in C \text{ such that } G(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of such $z \in C$ is denoted by $EP(G, A)$, i.e.,

$$EP(G, A) = \{z \in C : G(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C.$$

In the case of $A \equiv 0$, $EP(G, A)$ is denoted by $EP(G)$. In the case of $G \equiv 0$, $EP(G, A)$ is also denoted by $VI(C, A)$. Numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and economics reduce to find a solution of (1.3) (see, for instance, [1]-[3]).

A mapping A of C into H is called *inverse-strongly monotone* (see [4]), if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha||Ax - Ay||^2$$

for all $x, y \in C$.

A mapping T with domain $D(T)$ and range $R(T)$ is called *nonexpansive* if

$$||Tx - Ty|| \leq ||x - y|| \tag{1.4}$$

for all $x, y \in D(T)$ and T is said to be κ*-strict pseudo-contration* if there exist $\kappa \in [0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + \kappa||(I - T)x - (I - T)y||^2, \quad \forall x, y \in D(T). \tag{1.5}$$

We know that the class of κ-strict pseudo-contractions includes class of nonexpansive mappings. If $\kappa = 1$, then T is said to be *pseudo-contractive*. T is *strong pseudo-contraction* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. In a real Hilbert space H (1.5) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 - \frac{1-\kappa}{2}||(I - T)x - (I - T)y||^2 \quad \forall x, y \in D(T). \tag{1.6}$$

T is pseudo-contractive if and only if

$$\langle Tx - Ty, x - y\rangle \leq ||x - y||^2 \quad \forall x, y \in D(T).$$

Then, T is strongly pseudo-contractive, if there exists a positive constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \leq (1 - \lambda)\|x - y\|^2, \quad \forall x, y \in D(T).$$

The class of κ-strict pseudo-contractions fall into the one between classes of nonexpansive mappings and pseudo-contractions, and the class of strong pseudo-contractions is independent of the class of κ-strict pseudo-contractions.

We denote by $F(T)$ the set of fixed points of T. If $C \subset H$ is bounded, closed and convex and T is a nonexpansive mapping of C into itself, then $F(T)$ is nonempty; for instance, see [5]. Recently, Tada and Takahashi [6] and Takahashi and Takahashi [7] considered iterative methods for finding an element of $EP(G) \cap F(T)$. Browder and Petryshyn [8] showed that if a κ-strict pseudo-contraction T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula:

$$x_{n+1} = \alpha x_n + (1 - \alpha)Tx_n, \tag{1.7}$$

where α is a constant such that $0 < \alpha < 1$, converges weakly to a fixed point of T. Marino and Xu [9] extended Browder and Petryshyn's above mentioned result by proving that the sequence $\{x_n\}$ generated by the following Manns algorithm [10]:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n \tag{1.8}$$

converges weakly to a fixed point of T provided the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ satisfies the conditions that $\kappa < \alpha_n < 1$ for all n and $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1-\alpha_n) = \infty$.

Recently, in 2009, Qin et al. [11] introduced a general iterative method for finding a common element of $EP(F, T)$, $F(S)$, and $F(D)$. They defined $\{x_n\}$ as follows:

$$\begin{cases} x_1, \ u \in C, \\ F\langle u_n, \ y\rangle + \langle Tx_n, \ y-u_n\rangle + \dfrac{1}{r}\langle y-u_n, \ u_n - x_n\rangle, \quad \forall y \in C, \\ y_n = P_C(x_n - \eta Bx_n), \\ v_n = P_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(\mu_1 S_k x_n + \mu_2 u_n + \mu_3 v_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.9}$$

where the mapping $D : C \to C$ is defined by $D(x) = P_C(P_C(x - \eta Bx) - \lambda AP_C(x - \eta Bx))$, S_k is the mapping defined by $S_k x = kx + (1 - k)Sx$, $\forall x \in C$, $S : C \to C$ is a κ-strict pseudo-contraction, and A, $B : C \in H$ are a-inverse-strongly monotone mapping and b-inverse-strongly monotone mappings, respectively. Under suitable conditions, they proved strong convergence of $\{x_n\}$ defined by (1.9) to $z = P_{EP(F,\ T)\cap F(S)\ \cap F(D)}u$.

Let C be a nonempty convex subset of a real Hilbert space. Let T_i, $i = 1, 2, \ldots$ be mappings of C into itself. For each $j = 1, 2, \ldots$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I = [0, 1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. For every $n \in \mathbb{N}$, we define the mapping $S_n : C \to C$ as follows:

$$\begin{aligned} U_{n,n+1} &= I \\ U_{n,n} &= \alpha_1^n T_n U_{n,n+1} + \alpha_2^n U_{n,n+1} + \alpha_3^n I \\ U_{n,n-1} &= \alpha_1^{n-1} T_{n-1} U_{n,n} + \alpha_2^{n-1} U_{n,n} + \alpha_3^{n-1} I \\ &\vdots \\ U_{n,k+1} &= \alpha_1^{k+1} T_{k+1} U_{n,k+2} + \alpha_2^{k+1} U_{n,k+2} + \alpha_3^{k+1} I \\ U_{n,k} &= \alpha_1^{k} T_{k} U_{n,k+1} + \alpha_2^{k} U_{n,k+1} + \alpha_3^{k} I \\ &\vdots \\ U_{n,2} &= \alpha_1^{2} T_{2} U_{n,1} + \alpha_2^{2} U_{n,1} + \alpha_3^{2} I \\ S_n = U_{n,1} &= \alpha_1^{1} T_{1} U_{n,2} + \alpha_2^{1} U_{n,2} + \alpha_3^{1} I. \end{aligned}$$

This mapping is called *S-mapping* generated by $T_n, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$.

Question. How can we define an iterative method for finding an element in $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, \ A_i) \bigcap \bigcap_{i=1}^{N} F(G_i)$?

In this article, motivated by Qin et al. [11], by using S-mapping, we introduce a new iteration method for finding a common element of the set of a generalized equilibrium problem of the set of solution to a system of variational inequalities, and of the set of fixed points of an infinite family of strict pseudo-contractions. Our iteration scheme is define as follows.

For u, $x_1 \in C$, let $\{x_n\}$ be generated by

$$\begin{cases} F_i\langle v_n^i, \ v\rangle + \langle A_i x_n, \ v - v_n^i\rangle + \dfrac{1}{r_i}\langle v - v_n^i, \ v_n^i - x_n\rangle, \quad \forall v \in C, \ i = 1, 2, \ldots, N. \\ y_n = \Sigma_{i=1}^{N}\delta_i v_n^i \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(a_n S_n x_n + b_n Bx_n + c_n y_n), \quad \forall n \in \mathbb{N}. \end{cases}$$

For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be bifunction, $A_i : C \to H$ be α_i-inverse strongly monotone and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$, $\forall y \in C$ with

(0, 1] ⊂ (0, 2 α_i) such that $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i,\ A_i) \bigcap \bigcap_{i=1}^{N} F(G_i) \neq \emptyset$, where B is the K-mapping generated by $G_1, G_2, ..., G_N$ and $\beta_1, \beta_2, ..., \beta_N$.

We prove a strong convergence theorem of purposed iterative sequence $\{x_n\}$ to a point $z \in \mathbb{F}$ and z is a solution of (1.10)

$$\begin{cases} \langle x - z,\ A_1 z\rangle \geq 0 \\ \langle x - z,\ A_2 z\rangle \geq 0 \\ \vdots \\ \langle x - z,\ A_N z\rangle \geq 0, \quad \forall x \in C \text{ and } \lambda_i \in\ (0,\ 1]\ i = 1, 2, \dots, N. \end{cases} \tag{1.10}$$

2 Preliminaries

In this section, we collect and provide some useful lemmas that will be used for our main result in the next section.

Let C be a closed convex subset of a real Hilbert space H, and let P_C be the metric projection of H onto C i.e., so that for $x \in\ H$, $P_C x$ satisfies the property:

$$|x - P_C x|| = \min_{y \in C} ||x - y||.$$

The following characterizes the projection P_C.

Lemma 2.1 [5]. *Given* $x \in\ H$ *and* $y \in\ C$. *Then,* $P_C x = y$ *if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C.$$

Lemma 2.2 [12]. *Let* $\{s_n\}$ *be a sequence of nonnegative real number satisfying*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n \beta_n, \quad \forall n \geq 0$$

where $\{\alpha_n\}$, $\{\beta_n\}$ *satisfy the conditions*

$$(1)\ \{\alpha_n\} \subset [0, 1],\ \ \sum_{n=1}^{\infty} \alpha_n = \infty;$$

$$(2)\ \limsup_{n\to\infty} \beta_n \leq 0\ \textit{or}\ \ \sum_{n=1}^{\infty} |\alpha_n \beta_n| < \infty.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 [13]. *Let* C *be a closed convex subset of a strictly convex Banach space* E. *Let* $\{T_n : n \in\ \mathbb{N}\}$ *be a sequence of nonexpansive mappings on* C. *Suppose* $\bigcap_{n=1}^{\infty} F(T_n)$*is nonempty. Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\Sigma_{n=1}^{\infty} \lambda_n = 1$. *Then, a mapping* S *on* C *defined by*

$$S(x) = \Sigma_{n=1}^{\infty} \lambda_n T_n x_n$$

for $x \in\ C$ *is well defined, nonexpansive and* $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$*hold.*

Lemma 2.4 [14]. *Let* E *be a uniformly convex Banach space,* C *be a nonempty closed convex subset of* E *and* $S : C \to C$ *be a nonexpansive mapping. Then,* I - S *is demi-closed at zero.*

Lemma 2.5 [15]. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\beta_n\}$ *be a sequence in 0*[1]*with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) z_n$$

for all integer $n \geq 0$ *and*

$$\limsup_{n\to\infty} (||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then $\lim_{n\to\infty} ||x_n - z_n|| = 0$.

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

($A1$) $F(x, x) = 0\ \forall x \in C$;
($A2$) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;
($A3$) $\forall x, y, z \in C$,

$$\lim_{t\to 0^+} F(tz + (1 - t)x, y) \leq F(x, y);$$

($A4$) $\forall x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [1].

Lemma 2.6 [1]. *Let C be a nonempty closed convex subset of H, and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* ($A1$) - ($A4$). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \tag{2.1}$$

for all $x \in C$.

Lemma 2.7 [16]. *Assume that $F : C \times C \to \mathbb{R}$ satisfies* ($A1$) - ($A4$). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows.*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \ \forall y \in C\}.$$

for all $z \in H$. Then, the following hold.

(1) T_r is single-valued,

(2) T_r is firmly nonexpansive i.e

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle \quad \forall x, y \in H;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Definition 2.1 [17]. *Let C be a nonempty convex subset of real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpanxive mappings of C into itself, and let $\lambda_1, ..., \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i = 1, ..., N$. We define a mapping $K : C \to C$ as follows.*

$$\begin{aligned} U_1 &= \lambda_1 T_1 + (1 - \lambda_1) I, \\ U_2 &= \lambda_2 T_2 U_1 + (1 - \lambda_2) U_1, \\ U_3 &= \lambda_3 T_3 U_2 + (1 - \lambda_3) U_2, \\ &\vdots \\ U_{N-1} &= \lambda_{N-1} T_{N-1} U_{N-2} + (1 - \lambda_{N-1}) U_{N-2}, \\ K = U_N &= \lambda_N T_N U_{N-1} + (1 - \lambda_N) U_{N-1}. \end{aligned} \tag{2.3}$$

Such a mapping K is called the *K-mapping* generated by $T_1, ..., T_N$ and $\lambda_1, ..., \lambda_N$.

Lemma 2.8 [17]. *Let C be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{T_i\}_{i=1}^N$ *be a finite family of nonexpanxive mappings of C into itself with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $\lambda_1, ..., \lambda_N$ *be real numbers such that* $0 < \lambda_i < 1$ *for every* $i = 1, ..., N - 1$ *and* $0 < \lambda_N \leq 1$. *Let K be the K-mapping generated by* $T_1, ..., T_N$ *and* $\lambda_1, ..., \lambda_N$. *Then* $F(K) = \bigcap_{i=1}^N F(T_i)$.

Lemma 2.9 [9]. *Let C be a nonempty closed convex subset of a real Hilbert space H and* $S : C \to C$ *be a self-mapping of C. If S is a* κ*-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition.*

$$||Sx - Sy|| \leq \frac{1+\kappa}{1-\kappa}||x - y||, \quad \forall x, y \in C.$$

Lemma 2.10. *Let C be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^N$ *be* κ_i*-strict pseudo-contraction mappings of C into itself with* $\bigcap_{i=1}^\infty F(T_i) \neq \emptyset$ *and* $\kappa = \sup_i \kappa_i$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j \leq b < 1, \alpha_1^j + \alpha_2^j \leq b < 1$, *and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ *for all* $j = 1, 2, ...$ *For every* $n \in \mathbb{N}$, *let* S_n *be S-mapping generated by* $T_n, ..., T_1$ *and* $\alpha_n, \alpha_{n-1}, ..., \alpha_1$. *Then, for every* $x \in C$ *and* $k \in \mathbb{N}$, $\lim_{n\to\infty} U_{n,k}x$ *exists.*

Proof. Let $x \in C$ and $y \in \bigcap_{i=1}^\infty F(T_i)$. Fix $k \in \mathbb{N}$, then for every $n \in \mathbb{N}$ with $n \geq k$, we have

$$\begin{aligned}
\|U_{n+1,k}x - U_{n,k}x\|^2 &= \|\alpha_1^k T_k U_{n+1,k+1}x + \alpha_2^k U_{n+1,k+1}x + \alpha_3^k x - \alpha_1^k T_k U_{n,k+1}x \\
&\quad -\alpha_2^k U_{n,k+1}x - \alpha_3^k x\|^2 \\
&= \|\alpha_1^k (T_k U_{n+1,k+1}x - T_k U_{n,k+1}x) + \alpha_2^k (U_{n+1,k+1}x - U_{n,k+1}x)\|^2 \\
&\leq \alpha_1^k \|T_k U_{n+1,k+1}x - T_k U_{n,k+1}x\|^2 + \alpha_2^k \|U_{n+1,k+1}x - U_{n,k+1}x\|^2 \\
&\quad -\alpha_1^k \alpha_2^k \|T_k U_{n+1,k+1}x - T_k U_{n,k+1}x - U_{n+1,k+1}x + U_{n,k+1}x\|^2 \\
&\leq \alpha_1^k (\|U_{n+1,k+1}x - U_{n,k+1}x\|^2 + \kappa\|(I - T_k)U_{n+1,k+1}x \\
&\quad -(I - T_k)U_{n,k+1}x\|^2) + \alpha_2^k \|U_{n+1,k+1}x - U_{n,k+1}x\|^2 \\
&\quad -\alpha_1^k \alpha_2^k \|(I - T_k)U_{n,k+1}x - (I - T_k)U_{n+1,k+1}x\|^2 \\
&\leq (1 - \alpha_3^k)\|U_{n+1,k+1}x - U_{n,k+1}x\|^2 \\
&\;\;\vdots \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)\|U_{n+1,n+1}x - U_{n,n+1}x\|^2 \\
&= \Pi_{j=k}^n (1 - \alpha_3^j)\|\alpha_1^{n+1} T_{n+1} U_{n+1,n+2}x + \alpha_2^{n+1} U_{n+1,n+2}x + \alpha_3^{n+1}x - x\|^2 \\
&= \Pi_{j=k}^n (1 - \alpha_3^j)\|\alpha_1^{n+1} T_{n+1}x + (1 - \alpha_1^{n+1})x - x\|^2 \\
&= \Pi_{j=k}^n (1 - \alpha_3^j)\|\alpha_1^{n+1}(T_{n+1}x - x)\|^2 \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)(\|T_{n+1}x - y\| + \|y - x\|)^2 \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)\left(\frac{1+\kappa}{1-\kappa}\|x - y\| + \|y - x\|\right)^2 \\
&\leq \Pi_{j=k}^n (1 - \alpha_3^j)\left(\frac{2}{1-\kappa}\|x - y\|\right)^2 \\
&\leq b^{n-(k-1)}\left(\frac{2}{1-\kappa}\|x - y\|\right)^2.
\end{aligned}$$

It follows that

$$||U_{n+1,k}x - U_{n,k}x|| \leq b^{\frac{n-(k-1)}{2}}\left(\frac{2}{1-\kappa}||x - y||\right)$$

$$
\begin{aligned}
&= \frac{b^{\frac{n}{2}}}{b^{\frac{k-1}{2}}}\left(\frac{2}{1-\kappa}||x-y||\right)\\
&= \frac{a^n}{a^{k-1}}M,
\end{aligned} \tag{2.4}
$$

where $a = b^{\frac{1}{2}} \in (0, 1)$ and $M = \dfrac{2}{1-\kappa}||x-y||$

For any k, n, $p \in \mathbb{N}$, $p > 0$, $n \geq k$, we have

$$
\begin{aligned}
\|U_{n+p,k}x - U_{n,k}x\| &\leq \|U_{n+p,k}x - U_{n+p-1,k}x\| + \|U_{n+p-1,k}x - U_{n+p-2,k}x\| + \ \ldots\\
&\quad + \|U_{n+1,k}x - U_{n,k}x\|\\
&= \Sigma_{j=n}^{n+p-1}\|U_{j+1,k}x - U_{j,k}x\|\\
&\leq \Sigma_{j=n}^{n+p-1}\frac{a^j}{a^{k-1}}M\\
&\leq \frac{a^n}{(1-a)a^{k-1}}M.
\end{aligned} \tag{2.5}
$$

Since $a \in (0, 1)$, we have $\lim_{n\to\infty} a^n = 0$. From (2.5), we have that $\{U_{n,k}x\}$ is a Cauchy sequence. Hence $\lim_{n\to\infty} U_{n,k}x$ exists. □

For every $k \in \mathbb{N}$ and $x \in C$, we define mapping $U_{\infty,k}$ and $S : C \in C$ as follows:

$$
\lim_{n\to\infty} U_{n,k}x = U_{\infty,k}x \tag{2.6}
$$

and

$$
\lim_{n\to\infty} S_n x = \lim_{n\to\infty} U_{n,1}x = Sx \tag{2.7}
$$

Such a mapping S is called S-mapping generated by T_n, T_{n-1}, ... and α_n, α_{n-1}, ...

Remark 2.11. For each $n \in \mathbb{N}$, S_n is nonexpansive and $\lim_{n\to\infty} sup_{x\in D}||S_n x - Sx|| = 0$ for every bounded subset D of C. To show this, let x, $y \in C$ and D be a bounded subset of C. Then, we have

$$
\begin{aligned}
\|S_n x - S_n y\|^2 &= \|\alpha_1^1(T_1U_{n,2}x - T_1U_{n,2}y) + \alpha_2^1(U_{n,2}x - U_{n,2}y) + \alpha_3^1(x-y)\|^2\\
&\leq \alpha_1^1\|T_1U_{n,2}x - T_1U_{n,2}y\|^2 + \alpha_2^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2\\
&\quad -\alpha_1^1\alpha_2^1\|T_1U_{n,2}x - T_1U_{n,2}y - U_{n,2}x + U_{n,2}y\|^2\\
&\leq \alpha_1^1(\|U_{n,2}x - U_{n,2}y\|^2 + \kappa\|(I-T_1)U_{n,2}x - (I-T_1)U_{n,2}y\|^2)\\
&\quad +\alpha_2^1\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2 - \alpha_1^1\alpha_2^1\|(I-T_1)U_{n,2}y - (I-T_1)U_{n,2}x\|^2\\
&\leq (1-\alpha_3^1)\|U_{n,2}x - U_{n,2}y\|^2 + \alpha_3^1\|x-y\|^2\\
&\leq (1-\alpha_3^1)((1-\alpha_3^2)\|U_{n,3}x - U_{n,3}y\|^2 + \alpha_3^2\|x-y\|^2) + \alpha_3^1\|x-y)\|^2\\
&= (1-\alpha_3^1)(1-\alpha_3^2)\|U_{n,3}x - U_{n,3}y\|^2 + \alpha_3^2(1-\alpha_3^1)\|x-y\|^2 + \alpha_3^1\|x-y)\|^2\\
&= \Pi_{j=1}^2(1-\alpha_3^j)||U_{n,3}x - U_{n,3}y||^2 + (1 - \Pi_{j=1}^2(1-\alpha_3^j))||x-y||^2\\
&\ \vdots\\
&\leq \Pi_{j=1}^n(1-\alpha_3^j)||U_{n,n+1}x - U_{n,n+1}y||^2 + (1 - \Pi_{j=1}^n(1-\alpha_3^j))||x-y||^2\\
&= ||x-y||^2.
\end{aligned}
$$

Then, we have that $S : C \to C$ is also nonexpansive indeed, observe that for each x, $y \in C$

$$
|Sx - Sy|| = \lim_{n\to\infty}||S_n x - S_n y|| \leq ||x-y||.
$$

By (2.8), we have

$$
\begin{aligned}
||S_{n+1}x - S_n x|| &= ||U_{n+1,1}x - U_{n,1}x||\\
&\leq a^n M.
\end{aligned}
$$

This implies that for $m > n$ and $x \in D$,

$$\begin{aligned} ||S_m x - S_n x|| &\leq \Sigma_{j=n}^{m-1} ||S_{j+1}x - S_j x|| \\ &\leq \Sigma_{j=n}^{m-1} a^j M \\ &\leq \frac{a^n}{1-a} M. \end{aligned}$$

By letting $m \to \infty$, for any $x \in D$, we have

$$||Sx - S_n x|| \leq \frac{a^n}{1-a} M. \tag{2.8}$$

It follows that

$$\lim_{n\to\infty} sup_{x\in D} ||S_n x - Sx|| = 0. \tag{2.9}$$

Lemma 2.12. *Let C be a nonempty closed convex subset of a real Hilbert space. Let* $\{T_i\}_{i=1}^{\infty}$*be* κ_i*-strict pseudo-contraction mappings of C into itself with* $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$*and* $\kappa = \sup_{i\in} \kappa_i$ *and let* $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$*, where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$*and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$*for all j = 1, ... For every* $n \in \mathbb{N}$*, let* S_n *and S be S-mappings generated by* T_n*, ...,* T_1 *and* α_n*,* α_{n-1}*, ...,* α_1 *and* T_n*,* T_{n-1}*, ..., and* α_n*,* α_{n-1}*, ..., respectively. Then* $F(S) = \bigcap_{i=1}^{\infty} F(T_i)$.

Proof. It is evident that $\bigcap_{i=1}^{\infty} F(T_i) \subseteq F(S)$. For every $n, k \in \mathbb{N}$, *with* $n \geq k$, let $x_0 \in F(S)$ and $x^* \in \bigcap_{i=1}^{\infty} F(T_i)$, we have

$$\begin{aligned}
\|S_n x_0 - x^*\|^2 &= \|\alpha_1^1(T_1U_{n,2}x_0 - x^*) + \alpha_2^1(U_{n,2}x_0 - x^*) + \alpha_3^1(x_0 - x^*)\|^2 \\
&\leq \alpha_1^1\|T_1U_{n,2}x_0 - x^*\|^2 + \alpha_2^1\|U_{n,2}x_0 - x^*\|^2 + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\leq \alpha_1^1(\|U_{n,2}x_0 - x^*\|^2 + \kappa\|(I - T_1)U_{n,2}x_0\|^2) + \alpha_2^1\|U_{n,2}x_0 - x^*\|^2 \\
&\quad +\alpha_3^1\|x_0 - x^*\|^2 - \alpha_1^1\alpha_2^1\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= (1 - \alpha_3^1)\|U_{n,2}x_0 - x^*\|^2 + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\leq (1 - \alpha_3^1)((1 - \alpha_3^2)\|U_{n,3}x_0 - x^*\|^2 + \alpha_3^2\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2\|U_{n,3}x_0 - x_0\|^2) + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= (1 - \alpha_3^1)(1 - \alpha_3^2)\|U_{n,3}x_0 - x^*\|^2 + \alpha_3^2(1 - \alpha_3^1)\|x_0 - x^*\|^2 + \alpha_3^1\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= \Pi_{j=1}^2(1 - \alpha_3^j)\|U_{n,3}x_0 - x^*\|^2 + (1 - \Pi_{j=1}^2(1 - \alpha_3^j))\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\leq \Pi_{j=1}^2(1 - \alpha_3^j)((1 - \alpha_3^3)\|U_{n,4}x_0 - x^*\|^2 + \alpha_3^3\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^3(\alpha_2^3 - \kappa)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 - \alpha_2^3\alpha_3^3\|U_{n,4}x_0 - x_0\|^2) \\
&\quad +(1 - \Pi_{j=1}^2(1 - \alpha_3^j))\|x_0 - x^*\|^2 - \alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 \\
&\quad -\alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 - \alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 \\
&\quad -\alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= \Pi_{j=1}^2(1 - \alpha_3^j)(1 - \alpha_3^3)\|U_{n,4}x_0 - x^*\|^2 + \alpha_3^3\Pi_{j=1}^2(1 - \alpha_3^j)\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^3(\alpha_2^3 - \kappa)\Pi_{j=1}^2(1 - \alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 \\
&\quad -\alpha_2^3\alpha_3^3\Pi_{j=1}^2(1 - \alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 + (1 - \Pi_{j=1}^2(1 - \alpha_3^j))\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 - \alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 \\
&\quad -\alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 - \alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&= \Pi_{j=1}^3(1 - \alpha_3^j)\|U_{n,4}x_0 - x^*\|^2 + (1 - \Pi_{j=1}^3(1 - \alpha_3^j))\|x_0 - x^*\|^2 \\
&\quad -\alpha_1^3(\alpha_2^3 - \kappa)\Pi_{j=1}^2(1 - \alpha_3^j)\|T_3U_{n,4}x_0 - U_{n,4}x_0\|^2 \\
&\quad -\alpha_2^3\alpha_3^3\Pi_{j=1}^2(1 - \alpha_3^j)\|U_{n,4}x_0 - x_0\|^2 - \alpha_1^2(\alpha_2^2 - \kappa)(1 - \alpha_3^1)\|T_2U_{n,3}x_0 - U_{n,3}x_0\|^2 \\
&\quad -\alpha_2^2\alpha_3^2(1 - \alpha_3^1)\|U_{n,3}x_0 - x_0\|^2 - \alpha_1^1(\alpha_2^1 - \kappa)\|T_1U_{n,2}x_0 - U_{n,2}x_0\|^2 \\
&\quad -\alpha_2^1\alpha_3^1\|U_{n,2}x_0 - x_0\|^2 \\
&\ \ \vdots
\end{aligned} \tag{2.10}$$

$$
\begin{aligned}
&\vdots\\
\leq\ & \Pi_{j=1}^{k+1}(1-\alpha_3^j)||U_{n,k+2}x_0-x^*||^2+\ (1-\Pi_{j=1}^{k+1}(1-\alpha_3^j)||x_0-x^*||^2\\
&-\alpha_1^{k+1}(\alpha_2^{k+1}-\kappa)\Pi_{j=1}^{k}(1-\alpha_3^j)||T_{k+1}U_{n,k+2}x_0-U_{n,k+2}x_0||^2\\
&-\alpha_2^{k+1}\alpha_3^{k+1}\Pi_{j=1}^{k}(1-\alpha_3^j)||U_{n,k+2}x_0-x_0||^2\\
&-\alpha_1^{k}(\alpha_2^{k}-\kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)||T_kU_{n,k+1}x_0-U_{n,k+1}x_0||^2\\
&-\alpha_2^{k}\alpha_3^{k}\Pi_{j=1}^{k-1}(1-\alpha_3^j)||U_{n,k+1}x_0-x_0||^2\\
\vdots\\
&-\alpha_1^3(\alpha_2^3-\kappa)\Pi_{j=1}^{2}(1-\alpha_3^j)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2-\alpha_2^3\alpha_3^3\Pi_{j=1}^{2}(1-\alpha_3^j)\|U_{n,4}x_0-x_0\|^2\\
&-\alpha_1^2(\alpha_2^2-\kappa)(1-\alpha_3^1)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2-\alpha_2^2\alpha_3^2(1-\alpha_3^1)\|U_{n,3}x_0-x_0\|^2\\
&-\alpha_1^1(\alpha_2^1-\kappa)\|T_1U_{n,2}x_0-U_{n,2}x_0\|^2-\alpha_2^1\alpha_3^1\|U_{n,2}x_0-x_0\|^2
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
\vdots\\
\leq\ & \Pi_{j=1}^{n}(1-\alpha_3^j)\|U_{n,n+1}x_0-x^*\|^2+(1-\Pi_{j=1}^{n}(1-\alpha_3^j)\|x_0-x^*\|^2\\
&-\alpha_1^n(\alpha_2^n-\kappa)\Pi_{j=1}^{n-1}(1-\alpha_3^j)\|T_nU_{n,n+1}x_0-U_{n,n+1}x_0\|^2\\
&-\alpha_2^n\alpha_3^n\Pi_{j=1}^{n-1}(1-\alpha_3^j)\|U_{n,n+1}x_0-x_0\|^2\\
\vdots\\
&-\alpha_1^{k+1}(\alpha_2^{k+1}-\kappa)\Pi_{j=1}^{k}(1-\alpha_3^j)\|T_{k+1}U_{n,k+2}x_0-U_{n,k+2}x_0\|^2\\
&-\alpha_2^{k+1}\alpha_3^{k+1}\Pi_{j=1}^{k}(1-\alpha_3^j)\|U_{n,k+2}x_0-x_0\|^2\\
&-\alpha_1^{k}(\alpha_2^{k}-\kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|T_kU_{n,k+1}x_0-U_{n,k+1}x_0\|^2\\
&-\alpha_2^{k}\alpha_3^{k}\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|U_{n,k+1}x_0-x_0\|^2\\
&-\alpha_1^{k-1}(\alpha_2^{k-1}-\kappa)\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|T_{k-1}U_{n,k}x_0-U_{n,k}x_0\|^2\\
&-\alpha_2^{k-1}\alpha_3^{k-1}\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|U_{n,k}x_0-x_0\|^2\\
\vdots\\
&-\alpha_1^3(\alpha_2^3-\kappa)\Pi_{j=1}^{2}(1-\alpha_3^j)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2-\alpha_2^3\alpha_3^3\Pi_{j=1}^{2}(1-\alpha_3^j)\|U_{n,4}x_0-x_0\|^2\\
&-\alpha_1^2(\alpha_2^2-\kappa)(1-\alpha_3^1)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2-\alpha_2^2\alpha_3^2(1-\alpha_3^1)\|U_{n,3}x_0-x_0\|^2\\
&-\alpha_1^1(\alpha_2^1-\kappa)\|T_1U_{n,2}x_0-U_{n,2}x_0\|^2-\alpha_2^1\alpha_3^1\|U_{n,2}x_0-x_0\|^2
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
=\ & \|x_0-x^*\|^2\\
&-\alpha_1^n(\alpha_2^n-\kappa)\Pi_{j=1}^{n-1}(1-\alpha_3^j)\|T_nU_{n,n+1}x_0-U_{n,n+1}x_0\|^2\\
\vdots\\
&-\alpha_1^{k+1}(\alpha_2^{k+1}-\kappa)\Pi_{j=1}^{k}(1-\alpha_3^j)\|T_{k+1}U_{n,k+2}x_0-U_{n,k+2}x_0\|^2\\
&-\alpha_2^{k+1}\alpha_3^{k+1}\Pi_{j=1}^{k}(1-\alpha_3^j)\|U_{n,k+2}x_0-x_0\|^2\\
&-\alpha_1^{k}(\alpha_2^{k}-\kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|T_kU_{n,k+1}x_0-U_{n,k+1}x_0\|^2\\
&-\alpha_2^{k}\alpha_3^{k}\Pi_{j=1}^{k-1}(1-\alpha_3^j)\|U_{n,k+1}x_0-x_0\|^2\\
&-\alpha_1^{k-1}(\alpha_2^{k-1}-\kappa)\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|T_{k-1}U_{n,k}x_0-U_{n,k}x_0\|^2\\
&-\alpha_2^{k-1}\alpha_3^{k-1}\Pi_{j=1}^{k-2}(1-\alpha_3^j)\|U_{n,k}x_0-x_0\|^2\\
\vdots\\
&-\alpha_1^3(\alpha_2^3-\kappa)\Pi_{j=1}^{2}(1-\alpha_3^j)\|T_3U_{n,4}x_0-U_{n,4}x_0\|^2-\alpha_2^3\alpha_3^3\Pi_{j=1}^{2}(1-\alpha_3^j)\|U_{n,4}x_0-x_0\|^2\\
&-\alpha_1^2(\alpha_2^2-\kappa)(1-\alpha_3^1)\|T_2U_{n,3}x_0-U_{n,3}x_0\|^2-\alpha_2^2\alpha_3^2(1-\alpha_3^1)\|U_{n,3}x_0-x_0\|^2\\
&-\alpha_1^1(\alpha_2^1-\kappa)\|T_1U_{n,2}x_0-U_{n,2}x_0\|^2-\alpha_2^1\alpha_3^1\|U_{n,2}x_0-x_0\|^2.
\end{aligned}
\tag{2.13}
$$

For $k \in \mathbb{N}$ and (2.12), we have

$$\alpha_2^{k-1}\alpha_3^{k-1}\Pi_{j=1}^{k-2}(1-\alpha_3^j)||U_{n,k}x_0 - x_0||^2 \leq ||x_0 - x^*||^2 - ||S_n x_0 - x^*||^2, \qquad (2.14)$$

as $n \to \infty$. This implies that $U_{\infty,k}x_0 = x_0$, $\forall k \in \mathbb{N}$.

Again by (2.12), we have

$$\alpha_1^k(\alpha_2^k - \kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)||T_k U_{n,k+1}x_0 - U_{n,k+1}x_0||^2 \leq ||x_0 - x^*||^2 - ||S_n x_0 - x^*||^2, \qquad (2.15)$$

as $n \to \infty$. Hence

$$\alpha_1^k(\alpha_2^k - \kappa)\Pi_{j=1}^{k-1}(1-\alpha_3^j)||T_k U_{\infty,k+1}x_0 - U_{\infty,k+1}x_0||^2 \leq 0. \qquad (2.16)$$

From $U_{\infty,k}x_0 = x_0$, $\forall k \in \mathbb{N}$, and (2.15), we obtain that $T_k x_0 = x_0$, $\forall k \in \mathbb{N}$. This implies that $x_0 \in \bigcap_{i=1}^{\infty} F(T_i)$. □

Lemma 2.13. *Let C be a closed convex subset of Hilbert space H. Let $A_i : C \to H$ be mappings and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$ with $\lambda_i > 0$, $\forall_i = 1, 2, \ldots N$. Then $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$ if and only if $x^* \in \bigcap_{i=1}^{N} F(G_i)$.*

Proof. For given $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$, we have $x^* \in VI(C, A_i)$, $\forall_i = 1, 2, \ldots, N$. Since $\langle A_i x^*, x - x^*\rangle \geq 0$, we have $\langle \lambda_i A_i x^*, x - x^*\rangle \geq 0$, $\forall \lambda_i > 0$, $i = 1, 2, \ldots, N$. It follows that

$$\langle x^* - (I - \lambda_i A_i)x^*, x - x^*\rangle = \langle \lambda_i A_i x^*, x - x^*\rangle \geq 0, \quad \forall x \in C, i = 1, 2, \ldots, N. \qquad (2.17)$$

Hence, $x^* = P_C(I - \lambda_i A_i)x^* = G_i(x^*)$, $\forall x \in C$, $i = 1, 2, \ldots, N$. Therefore, we have $x^* \in \bigcap_{i=1}^{N} F(G_i)$. For the converse, let $x^* \in \bigcap_{i=1}^{N} F(G_i)$; then, we have for every $i = 1, \ldots, N$, $x^* = G_i(x^*) = P_C(I - \lambda_i A_i)x^*$, $\forall \lambda_i > 0$, $i = 1, 2, \ldots, N$. It implies that

$$\langle x^* - (I - \lambda_i A_i)x^*, x - x^*\rangle = \langle \lambda_i A_i x^*, x - x^*\rangle \geq 0, \quad \forall i = 1, 2, \ldots, N, \quad \forall x \in C. \qquad (2.18)$$

Hence, $\langle A_i x^*, x - x^*\rangle \geq 0$, $\forall x \in C$, so $x^* \in VI(C, A_i)$, $\forall i = 1, 2, \ldots, N$. Hence, $x^* \in \bigcap_{i=1}^{N} VI(C, A_i)$.

□

3 Main results

Theorem 3.1. *Let C be a closed convex subset of Hilbert space H. For every $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1) - (A_4), let $A_i : C \to H$ be α_i-inverse strongly monotone and let $G_i : C \to C$ be defined by $G_i(y) = P_C(I - \lambda_i A_i)y$, $\forall y \in C$ with $\lambda_i \in (0, 1] \subset (0, 2\alpha_i)$. Let $B : C \to C$ be the K-mapping generated by $G_1, G_2, \ldots, G_N$ and $\beta_1, \beta_2, \ldots, \beta_N$ where $\beta_i \in (0, 1)$, $\forall i = 1, 2, 3, \ldots, N - 1$, $\beta_N \in (0, 1]$ and let $\{T_i\}_{i=1}^{\infty}$ be κ_i-strict pseudo-contraction mappings of C into itself with $\kappa = \sup_i \kappa_i$ and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ for all $j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S are S-mapping generated by $T_n, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$ and $T_n, T_{n-1}, \ldots$, and $\rho_n, \rho_{n-1}, \ldots$, respectively. Assume that $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, A_i) \bigcap \bigcap_{i=1}^{N} F(G_i) \neq \emptyset$. For every $n \in \mathbb{N}$, $i = 1, 2, \ldots, N$, let $\{x_n\}$ and $\{v_n^i\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F_i\langle v_n^i, v\rangle + \langle A_i x_n, v - v_n^i\rangle + \dfrac{1}{r_i}\langle v - v_n^i, v_n^i - x_n\rangle \geq 0, \quad \forall v \in C, \\ i = 1, 2, \ldots, N. \\ y_n = \Sigma_{i=1}^{N}\delta_i v_n^i \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(a_n S_n x_n + b_n B x_n + c_n y_n), \quad \forall n \in \mathbb{N}, \end{cases} \qquad (3.1)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = \Sigma_{i=1}^{N}\delta_i = 1$, *and* $\{r_i\}_{i=1}^{N} \subset (\varsigma, \tau) \subset (0, 2\alpha_i)$, *satisfy the following conditions:*

(*i*) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(*ii*) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$,
(*iii*) $\lim_{n\to\infty}a_n = a$, $\lim_{n\to\infty}b_n = b$, $\lim_{n\to\infty}c_n = c$, *with* $a, b, c \in (0, 1)$.

Then, the sequence $\{x_n\}$, $\{y_n\}$, $\{v_n^i\}$, $\forall i = 1, 2, ..., N$, *converge strongly to* $z = P_{\Im}u$ *and* z *is a solution of (1.10).*

Proof. First, we show that $(I - \lambda_i A_i)$ is nonexpansive mapping for every $i = 1, 2, ..., N$. For $x, y \in C$, we have

$$\begin{aligned}\left\|(I-\lambda_iA_i)x-(I-\lambda_iA_i)y\right\|^2 &= ||x-y-\lambda_i(A_ix-A_iy)||^2\\ &= ||x-y||^2 - 2\lambda_i\langle x-y, A_ix-A_iy\rangle + \lambda_i^2||A_ix-A_iy||^2\\ &\le ||x-y||^2 - 2\alpha_i\lambda_i||A_ix-A_iy||^2 + \lambda_i^2||A_ix-A_iy||^2 \\ &= ||x-y||^2 + \lambda_i(\lambda_i-2\alpha_i)||A_ix-A_iy||^2\\ &\le ||x-y||^2.\end{aligned} \tag{3.2}$$

Thus, $(I - \lambda_i A_i)$ is nonexpansive, and so are B and G_i, for all $i = 1, 2, ..., N$.

Now, we shall divide our proof into five steps.

Step 1. We shall show that the sequence $\{x_n\}$ is bounded. Since

$$F(v_n^i, v) + \langle A_ix_n, v - v_n^i\rangle + \frac{1}{r_i}\langle v - v_n^i, v_n^i - x_n\rangle \ge 0, \quad \forall v \in C,\ i = 1, 2, \ldots, N, \tag{3.3}$$

we have

$$F(v_n^i, v) + \frac{1}{r_i}\langle v - v_n^i, v_n^i - (I - r_iA_i)x_n\rangle \ge 0, \quad \forall v \in C,\ i = 1, 2, \ldots, N.$$

By Lemma 2.7, we have $v_n^i = T_{r_i}(I - r_iA_i)x_n$.

Let $z = \Im$. Then $F(z, y) + \langle y - z, A_iz\rangle \ge 0\ \forall y \in C$, so we have

$$F(z, y) + \frac{1}{r_i}\langle y - z, z - z + r_iA_iz\rangle \ge 0, \quad \forall i = 1, 2, \ldots, N.$$

Again by Lemma 2.7, we have $z = T_{r_i}(I - r_iA_i)z$, $\forall i = 1, 2, ..., N$. Since B is K-mapping generated by $G_1, G_2, ..., G_N$ and $\beta_1, \beta_2, ..., \beta_N$ and $\bigcap_{i=1}^{N} F(G_i) \neq \emptyset$. By Lemma 2.8, we have $\bigcap_{i=1}^{N} F(G_i) = F(B)$. Since $z = \Im$, we have $z \in F(B)$. Setting $e_n = a_nS_nx_n + b_nBx_n + c_ny_n$, $\forall n \in \mathbb{N}$, we have

$$\begin{aligned}\|x_{n+1}-z\| &= ||\alpha_n(u-z) + \beta_n(x_n-z) + \gamma_n(e_n-z)||\\ &\le \alpha_n||u-z|| + \beta_n||x_n-z|| + \gamma_n||e_n-z||\\ &= \alpha_n||u-z|| + \beta_n||x_n-z|| + \gamma_n||a_n(S_nx_n-z) + b_n(Bx_n-z) + c_n(y_n-z)||\\ &\le \alpha_n||u-z|| + \beta_n||x_n-z|| + \gamma_n((1-c_n)||x_n-z|| + c_n||y_n-z||)\\ &= \alpha_n||u-z|| + \beta_n||x_n-z|| + \gamma_n((1-c_n)||x_n-z|| + c_n||\Sigma_{i=1}^{N}\delta_i(v_n^i-z)||) \\ &\le \alpha_n||u-z|| + \beta_n||x_n-z|| + \gamma_n((1-c_n)||x_n-z|| + c_n\Sigma_{i=1}^{N}\delta_i||v_n^i-z||)\\ &\le \alpha_n||u-z|| + \beta_n||x_n-z|| + \gamma_n((1-c_n)||x_n-z|| + c_n||x_n-z||)\\ &= \alpha_n||u-z|| + (1-\alpha_n)||x_n-z||,\\ &\le \max\{||u-z||,\ ||x_n-z||\}.\end{aligned} \tag{3.4}$$

By induction, we can prove that $\{x_n\}$ is bounded, and so are $\{v_n^i\}$, $\{y_n\}$, $\{Bx_n\}$ $\{S_nx_n\}$, $\{e_n\}$.

Step 2. We will show that $\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0$. Let $d_n = \dfrac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, and then we have

$$x_{n+1} = (1-\beta_n)d_n + \beta_n x_n, \quad \forall n \in \mathbb{N}. \tag{3.5}$$

From definition of d_n, we have

$$\begin{aligned}
\|d_{n+1} - d_n\| &= ||\frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}|| \\
&= ||\frac{\alpha_{n+1}u + \gamma_{n+1}e_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + \gamma_n e_n}{1-\beta_n}|| \\
&= ||\frac{\alpha_{n+1}u + (1-\beta_{n+1}-\alpha_{n+1})e_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\beta_n-\alpha_n)e_n}{1-\beta_n}|| \\
&= ||\frac{\alpha_{n+1}}{1-\beta_{n+1}}(u - e_{n+1}) - \frac{\alpha_n}{1-\beta_n}(u-e_n) + e_{n+1} - e_n|| \\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u-e_{n+1}|| + \frac{\alpha_n}{1-\beta_n}||u-e_n|| \\
&\quad +||e_{n+1} - e_n||.
\end{aligned} \tag{3.6}$$

By definition of e_n, we have

$$\begin{aligned}
\|e_{n+1} - e_n\| &= \|a_{n+1}S_{n+1}x_{n+1} + b_{n+1}Bx_{n+1} + c_{n+1}y_{n+1} - a_nS_nx_n - b_nBx_n - c_ny_n\| \\
&= \|a_{n+1}S_{n+1}x_{n+1} - a_nS_{n+1}x_{n+1} + a_nS_{n+1}x_{n+1} + b_{n+1}Bx_{n+1} - b_nBx_{n+1} \\
&\quad +b_nBx_{n+1} + c_{n+1}y_{n+1} - c_ny_{n+1} + c_ny_{n+1} - a_nS_nx_n - b_nBx_n - c_ny_n\| \\
&= \|(a_{n+1} - a_n)S_{n+1}x_{n+1} + a_n(S_{n+1}x_{n+1} - S_nx_n) + (b_{n+1} - b_n)Bx_{n+1} \\
&\quad +b_n(Bx_{n+1} - Bx_n) + (c_{n+1} - c_n)y_{n+1} + c_n(y_{n+1} - y_n)\| \\
&\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n\|S_{n+1}x_{n+1} - S_nx_n\| + |b_{n+1} - b_n|\|Bx_{n+1}\| \\
&\quad +b_n\|Bx_{n+1} - Bx_n\| + |c_{n+1} - c_n|\|y_{n+1}\| + c_n\|y_{n+1} - y_n\| \\
&\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n(\|S_{n+1}x_{n+1} - S_{n+1}x_n\| + \|S_{n+1}x_n - S_nx_n\|) \\
&\quad +|b_{n+1} - b_n|\|Bx_{n+1}\| + b_n\|Bx_{n+1} - Bx_n\| + |c_{n+1} - c_n|\|y_{n+1}\| \\
&\quad +c_n\Sigma_{i=1}^N \delta_i\|T_{r_i}(I - r_iA_i)x_{n+1} - T_{r_i}(I - r_iA_i)x_n\| \\
&\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n(\|x_{n+1} - x_n\| + \|S_{n+1}x_n - S_nx_n\|) \\
&\quad +|b_{n+1} - b_n|\|Bx_{n+1}\| + b_n\|x_{n+1} - x_n\| + |c_{n+1} - c_n|\|y_{n+1}\| \\
&\quad +c_n\|x_{n+1} - x_n\| \\
&\le |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + a_n\|x_{n+1} - x_n\| + \|S_{n+1}x_n - S_nx_n\| \\
&\quad +|b_{n+1} - b_n|\|Bx_{n+1}\| + b_n\|x_{n+1} - x_n\| + |c_{n+1} - c_n|\|y_{n+1}\| \\
&\quad +c_n\|x_{n+1} - x_n\| \\
&= \|x_{n+1} - x_n\| + |a_{n+1} - a_n|\|S_{n+1}x_{n+1}\| + \|S_{n+1}x_n - S_nx_n\| \\
&\quad +|b_{n+1} - b_n|\|Bx_{n+1}\| + |c_{n+1} - c_n|\|y_{n+1}\|.
\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we have

$$\begin{aligned}
||d_{n+1} - d_n|| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u - e_{n+1}|| + \frac{\alpha_n}{1-\beta_n}||u-e_n|| \\
&\quad + ||e_{n+1} - e_n||
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\leq \quad & \frac{\alpha_{n+1}}{1-\beta_{n+1}}||u-e_{n+1}|| + \frac{\alpha_n}{1-\beta_n}||u-e_n|| \\
& + ||x_{n+1}-x_n|| + |a_{n+1}-a_n|\ ||S_{n+1}x_{n+1}|| + ||S_{n+1}x_n - S_nx_n|| \\
& + |b_{n+1}-b_n|\ ||Bx_{n+1}|| + |c_{n+1}-c_n|\ ||\gamma_{n+1}||.
\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}
\|d_{n+1}-d_n\| - \|x_{n+1}-x_n\| \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-e_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-e_n\| \\
& +|a_{n+1}-a_n|\|S_{n+1}x_{n+1}\| + \|S_{n+1}x_n - S_nx_n\| \\
& +|b_{n+1}-b_n|\|Bx_{n+1}\| + |c_{n+1}-c_n|\|\gamma_{n+1}\| \\
\leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-e_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-e_n\|
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
& +|a_{n+1}-a_n|\ ||S_{n+1}x_{n+1}|| + ||S_{n+1}x_n - Sx_n|| + ||Sx_n - S_nx_n|| \\
& +|b_{n+1}-b_n|\ ||Bx_{n+1}|| + |c_{n+1}-c_n|\ ||\gamma_{n+1}||.
\end{aligned} \tag{3.11}$$

From Remark 2.11 and conditions (i)-(iii), we have

$$\limsup_{n\to\infty}(||d_{n+1}-d_n|| - ||x_{n+1}-x_n||) \leq 0. \tag{3.12}$$

From (3.5), (3.12) and Lemma 2.5, we have

$$\lim_{n\to\infty} ||d_n - x_n|| = 0. \tag{3.13}$$

We can rewrite (3.5) as

$$x_{n+1} - x_n = (1-\beta_n)(d_n - x_n), \quad \forall n \in \mathbb{N}. \tag{3.14}$$

By (3.13) and (3.14), we have

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.15}$$

Step. 3. Show that $\lim_{n\to\infty} ||x_n - e_n|| = 0$. From (3.1), we have

$$x_{n+1} - x_n + \alpha_n(x_n - u) = \gamma_n(e_n - x_n).$$

It implies that

$$\gamma_n||e_n - x_n|| \leq ||x_{n+1} - x_n|| + \alpha_n||x_n - u||.$$

By conditions (i), (ii), and (3.15), we have

$$\lim_{n\to\infty} ||e_n - x_n|| = 0. \tag{3.16}$$

Step. 4. We show that $\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \leq 0$, where $z = P_{\mathfrak{F}}u$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z,\ x_n - z\rangle = \limsup_{j\to\infty}\langle u - z,\ x_{n_j} - z\rangle \tag{3.17}$$

Without loss of generality, we may assume that $\{x_{n_j}\}$ converges weakly to some q in H. Next, we will show that

$$q \in \mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i,\ A_i) \bigcap \bigcap_{i=1}^{N} F(G_i). \tag{3.18}$$

First, we define a mapping $A : C \to C$ by

$$Ax = \Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x, \quad \forall x \in C.$$

Since $F(T_{r_i}(I - r_i A_i)) = EF(F_i,\ A_i), \ \forall i = 1,\ 2,\ \ldots,\ N,$ we have $\bigcap_{i=1}^{N} F(T_{r_i}(I - r_i A_i)) = \bigcap_{i=1}^{N} EF(F_i,\ A_i) \neq \emptyset$. By Lemma 2.3, we have $F(A) = \bigcap_{i=1}^{N} F(T_{r_i}(I - r_i A_i))$.

Next, we define $Q : C \to C$ by

$$Qx = aSx + bBx + cAx \quad \forall x \in C. \tag{3.19}$$

Again, by Lemma 2.3, we have

$$F(Q) = F(S) \bigcap F(B) \bigcap F(A) = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} F(G_i) \bigcap \bigcap_{i=1}^{N} EF(F_i, A_i).$$

By (3.19), we have

$$\begin{aligned} \|Qx_n - e_n\| &= \|aSx_n + bBx_n + cAx_n - a_n S_n x_n - b_n Bx_n - c_n y_n\| \\ &= \|aSx_n - aS_n x_n + aS_n x_n + bBx_n + c\Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x_n - a_n S_n x_n \\ &\quad - b_n Bx_n - c_n \Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x_n\| \\ &= \|a(Sx_n - S_n x_n) + (a - a_n)S_n x_n + (b - b_n)Bx_n + (c - c_n)\Sigma_{i=1}^{N} \delta_i T_{r_i}(I - r_i A_i)x_n\| \\ &\leq a\|Sx_n - S_n x_n\| + |a - a_n|\|S_n x_n\| + |b - b_n|\|Bx_n\| \\ &\quad + |c - c_n|\Sigma_{i=1}^{N} \delta_i \|T_{r_i}(I - r_i A_i)x_n\|. \end{aligned} \tag{3.20}$$

By condition (iii), (3.20), and (2.11), we have

$$\lim_{n\to\infty} ||Qx_n - e_n|| = 0. \tag{3.21}$$

Since

$$||Qx_n - x_n|| \leq ||Qx_n - e_n|| + ||e_n - x_n||.$$

by (3.16) and (3.21), we have

$$\lim_{n\to\infty} ||Qx_n - x_n|| = 0. \tag{3.22}$$

From, (3.22), we have

$$\lim_{j\to\infty} ||Qx_{n_j} - x_{n_j}|| = 0. \tag{3.23}$$

By Lemma 2.4, we obtain that

$$q \in F(Q) = \mathfrak{F}. \tag{3.24}$$

From (3.17)

$$\limsup_{n\to\infty} \langle u - z,\ x_n - z\rangle = \limsup_{j\to\infty} \langle u - z,\ x_{n_j} - z\rangle = \langle u - z,\ q - z\rangle \leq 0. \tag{3.25}$$

Step. 5. Finally, we show that $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathfrak{F}} u$.

By nonexpansiveness of S_n and B, we can show that $||e_n - z|| \leq ||x_n - z||$. Then,

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n(e_n - z)\|^2 \\ &= \alpha_n\langle u - z,\ x_{n+1} - z\rangle + \beta_n\langle x_n - z,\ x_{n+1} - z\rangle + \gamma_n\langle e_n - z,\ x_{n+1} - z\rangle \\ &\leq \alpha_n\langle u - z,\ x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|e_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle u - z,\ x_{n+1} - z\rangle + \beta_n\|x_n - z\|\|x_{n+1} - z\| + \gamma_n\|x_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle u - z,\ x_{n+1} - z\rangle + (1 - \alpha_n)\|x_n - z\|\|x_{n+1} - z\| \\ &\leq \alpha_n\langle u - z,\ x_{n+1} - z\rangle + \frac{(1 - \alpha_n)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2). \end{aligned}$$

It follows that

$$\|x_{n+1} - z\|^2 \leq 2\alpha_n \langle u - z,\ x_{n+1} - z\rangle + (1 - \alpha_n)\|x_n - z\|^2. \tag{3.26}$$

From Step 4, (3.26), and Lemma 2.2, we have $\lim_{n\to\infty} x_n = z$, where $z = P_{\mathfrak{F}}u$. The proof is complete. □

4 Applications

From Theorem 3.1, we obtain the following strong convergence theorems in a real Hilbert space:

Theorem 4.1. *Let C be a closed convex subset of Hilbert space H. For every* $i = 1, 2, \ldots, N$, *let* $F_i : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A_1) - (A_4) *and let* $\{T_i\}_{i=1}^{\infty}$ *be* κ_i*-strict pseudo-contraction mappings of C into itself with* $\kappa = \sup_i \kappa_i$ *and let* $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, *where* $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, *and* $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (\kappa, 1)$ *for all* $j = 2, \ldots$. *For every* $n \in \mathbb{N}$, *let* S_n *and S are S-mappings generated by* $T_n, \ldots, T_1$ *and* $\rho_n, \rho_{n-1}, \ldots, \rho_1$ *and* $T_n, T_{n-1}, \ldots$, *and* $\rho_n, \rho_{n-1}, \ldots$, *respectively. Assume that* $\mathfrak{F} = \bigcap_{i=1}^{\infty} F(T_i) \bigcap \bigcap_{i=1}^{N} EF(F_i) \neq \emptyset$. *For every* $n \in \mathbb{N}$, $i = 1, 2, \ldots, N$, *let* $\{x_n\}$ *and* $\{v_n^i\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{cases} F_i\langle v_n^i,\ v\rangle + \dfrac{1}{r_i}\langle v - v_n^i,\ v_n^i - x_n\rangle \geq 0, & \forall v \in C,\ i = 1,\ 2,\ \ldots,\ N. \\ y_n = \Sigma_{i=1}^{N}\delta_i v_n^i \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(a_n S_n x_n + b_n x_n + c_n y_n), & \forall n \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = \Sigma_{i=1}^{N}\delta_i = 1$, *and* $\{r_i\}_{i=1}^{N} \subset (\varsigma, \tau) \subset (0, 2\alpha_i)$, *satisfy the following conditions:*

(*i*) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(*ii*) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,
(*iii*) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} c_n = c$, *with* $a, b, c \in (0, 1)$,

Then, the sequence $\{x_n\}, \{y_n\}, \{v_n^i\}$, $\forall i = 1, 2, \ldots, N$, *converge strongly to* $z = P_{\mathfrak{F}}u$, *and z is solution of (1.10)*

Proof. From Theorem 3.1, let $A_i \equiv 0$; then we have $G_i(y) = P_{Cy} = y\ \forall y \in C$. Then, we get $Bx_n = x_n\ \forall n \in \mathbb{N}$. Then, from Theorem 3.1, we obtain the desired conclusion. □

Next theorem is derived from Theorem 3.1, and we modify the result of [11] as follows:

Theorem 4.2. *Let C be a closed convex subset of Hilbert space H and let* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A_1)-(A_4), *let* $A : C \to H$ *be* α*-inverse strongly monotone mapping, and let T be* κ*-strict pseudo-contraction mappings of C into itself. Define a mapping* T_κ *by* $T_\kappa x = \kappa x + (1 - \kappa)Tx$, $\forall x \in C$. *Assume that* $\mathfrak{F} = F(T) \bigcap EF(F, A) \bigcap VI(C, A) \neq \emptyset$. *For every* $n \in \mathbb{N}$, *let* $\{x_n\}$ *and* $\{v_n\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{cases} F\langle v_n,\ v\rangle + \langle Ax_n,\ v - v_n\rangle + \dfrac{1}{r}\langle v - v_n,\ v_n - x_n\rangle \geq 0, & \forall v \in C \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n(aT_\kappa x_n + bP_C(I - \lambda A)x_n + cv_n), & \forall n \in \mathbb{N}, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a, b, c\} \subset (0, 1)$, $\alpha_n + \beta_n + \gamma_n = a + b + c = 1$, *and* $\{r, \lambda\} \subset (\varsigma, \tau) \subset (0, 2\alpha)$ *satisfy the following conditions:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(*ii*) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$,

Then, the sequence $\{x_n\}$ *and* $\{v_n\}$ *converge strongly to* $z = P_{\Im}u$.

Proof. From Theorem 3.1, choose $N = 1$ and let $A_1 = A$, $\lambda_1 = \lambda$. Then, we have $B(y) = G_1(y) = P_C(I - \lambda A)y$, $\forall y \in C$. Choose $v_n^1 = v_n$, $a = a_n$, $b = b_n$, $c = c_n$ for all $n \in \mathbb{N}$, and let $T_\kappa \equiv S_1 : C \to C$ be S-mapping generated by T_1 and ρ_1 with $T_1 = T$ and $\alpha_1^1 = \kappa$, and then we obtain the desired result from Theorem 3.1 □

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The author declares that they have no competing interests.

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