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The fixed point theorems of 1-set-contractive operators in Banach space

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Abstract

In this paper, we obtain some new fixed point theorems and existence theorems of solutions for the equation $Ax = \mu x$ using properties of strictly convex (concave) function and theories of topological degree. Our results and methods are different from the corresponding ones announced by many others.

MSC: 47H09, 47H10

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1 Introduction

For convenience, we first recall the topological degree of 1-set-contractive fields due to Petryshyn [1].

Let E be a real Banach space, $p \in E$, Ω be a bounded open subset of E. Suppose that $A : \overline{\Omega} \to E$ is a 1-set-contractive operator such that

$$||(I - A)x - p|| \geq \delta > 0, \quad \forall x \in \partial\Omega$$

In addition, if there exists a k-set-contractive operator $(k < 1)$ $W : \overline{D} \to E$ such that

$$||Ax - Wx|| \leq \frac{\delta}{3}, \quad \forall x \in \partial D,$$

then $(I - W)x \neq p$, $\forall x \in \partial D$, and so it is easy to see that $\deg(I - W, D, p)$ is well defined and independent of W. Therefore, we are led to define the topological degree as follows:

$$\deg(I - A, D, p) = \deg(I - W, D, p).$$

Without loss of generality, we set $p = \theta$ in the above definition.

Let $A : \overline{\Omega} \to E$ be a 1-set-contractive operator. A is said to be a semi-closed 1-set-contractive operator, if $I - A$ is closed operator (see [2]).

It should be noted that this class of operators, as special cases, includes completely continuous operators, strict set-contractive operators, condensing operators, semi-compact 1-set-contractive operators and others (see [2]).

Petryshyn [1] and Nussbaum [3] first introduced the topological degree of 1-set-contractive fields, studied its basic properties and obtained fixed point theorems of 1-set-contractive operators. Amann [4] and Nussbaum [5] have introduced the fixed point indices of k-set contractive operators ($0 \leq k < 1$) and condensing operators to derive some fixed point theorems. As a complement, Li [2] has defined the fixed point index of 1-set-contractive operators and obtained some fixed point theorems of 1-set-contractive operators. Recently, Li [6] obtained some fixed point theorems for 1-set-contractive operators and existence theorems of solutions for the equation $Ax = \mu x$. Very recently, Xu [7] extended the results of Li [6] and obtained some fixed point theorems. In this paper, we continue to investigate boundary conditions, under which the topological degree of 1-set contractive fields, $\deg(I - A, \Omega, p)$, is equal to unity or zero. Consequently, we obtain some new fixed point theorems and existence theorems of solutions for the equation $Ax = \mu x$ using properties of strictly convex (concave) functions. Our results and methods are different from the corresponding ones announced by many others (e.g., Li [6], Xu [7]).

We need the following concepts and lemmas for the proof of our main results.

Suppose that $A : \overline{\Omega} \to E$ is a semi-closed 1-set-contractive operator and $\theta \notin (I - A)\partial\Omega$, then, by the standard method, we can easily see that the topological degree has the basic properties as follows:

(a) (Normalization) $\deg(I, \Omega, p) = 1$, when $p \in \Omega$; $\deg(I, \Omega, p) = 0$, when $p \notin \Omega$;

(b) (Solution property) If $\deg(I - A, \Omega, \theta) \neq 0$, then A has at least one fixed point in Ω.

(c) (Additivity) For every pair of disjoint open subsets Ω_1, Ω_2 of Ω such that $\{x \in \Omega \mid (I - A)x = 0\} \subset \Omega_1 \cup \Omega_2$, we have

$$\deg(I - A, \Omega, \theta) = \deg(I - A, \Omega_1, \theta) + \deg(I - A, \Omega_2, \theta).$$

(d) (Homotopy invariance) Let $H(t, x) = H_t(x) : [0, 1] \times \overline{\Omega} \to E$ be a continuous operator such that

$$||x - H_t(x)|| \geq \delta > 0, \quad \text{for } (t, x) \in [0, 1] \times \partial\Omega$$

and the measure of non-compactness $\gamma(H([0, 1] \times Q)) \leq \gamma(Q)$ for every $Q \subset \overline{\Omega}$. Then $\deg(I - H_t, \Omega, \theta) = const$, for any $t \in [0, 1]$.

(e) Let B be an open ball with center θ, $A : \bar{B} \to E$ a semi-closed 1-set-contractive operator and $(I - A)x \neq 0$ for all $x \in \partial B$. Suppose that A is odd on ∂B (i.e., $A(-x) = Ax$, for $x \in \partial B$), then $\deg(I - A, B, \theta) \neq 0$.

(f) (Change of base) Let $p \neq \theta$, then $\deg(I - A, \Omega, p) = \deg(I - A - p, \Omega, \theta)$.

Lemma 1.1. [7]. *Let E be a real Banach space, Ω a bounded open subset of E and $\theta \in \Omega$. $A : \overline{\Omega} \to E$ is a semi-closed 1-set-contractive operator and satisfies the Leray-Schauder boundary condition*

$$Ax \neq tx, \quad \text{for all } x \in \partial\Omega, \quad \text{and } t \geq 1, \qquad (\mathrm{L-S})$$

then $\deg(I - A, \Omega, \theta) = 1$ *and so A has a fixed point in Ω.*

Definition 1.2. Let D be a nonempty subset of R. If $\phi : D \to R$ is a real function such that

$$\varphi[tx + (1 - t)y] < t\varphi(x) + (1 - t)\varphi(y), \quad \forall x, y \in D,\ x \neq y,\ t \in (0, 1),$$

then ϕ is called strictly convex function on D. If $\phi : D \to R$ is a real function such that

$$\varphi[tx + (1-t)y] > t\varphi(x) + (1-t)\varphi(y), \quad \forall x, y \in D, \ x \neq y, \ t \in (0,1),$$

then ϕ is called strictly concave function on D.

2 Main results

We are now in the position to apply the topological degree and properties of strictly convex (concave) function to derive some new fixed point theorems for semi-closed 1-set-contractive operators and existence theorems of solutions for the equation $Ax = \mu x$ which generalize a great deal of well-known results and relevant recent ones.

Theorem 2.1. *Let E, Ω, A be the same as in Lemma 1.1. Moreover, if there exist strictly convex function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ and real function $\varphi : R^+ \to R$ with $\varphi(t) \geq 1$, for all $t > 1$, such that*

$$\varphi(||Ax - x||) \geq \varphi(||Ax||)\phi(||Ax|| \cdot ||x||^{-1}) - \varphi(||x||), \quad \forall x \in \partial\Omega, \tag{1}$$

then $\deg(I - A, \Omega, \theta) = 1$ *if A has no fixed point on $\partial\Omega$, and so A has at least one fixed point in $\overline{\Omega}$.*

Proof. If the operator A has a fixed point on $\partial\Omega$, then A has at least one fixed point in $\overline{\Omega}$. Now suppose that A has no fixed points on $\partial\Omega$. Next we shall prove that the condition (L-S) is satisfied.

Suppose this is not true. Then there exists $x_0 \in \partial\Omega$, $t_0 \geq 1$ such that $Ax_0 = t_0x_0$, i.e., $x_0 = t_0^{-1}Ax_0$. It is easy to see that $||Ax_0|| \neq 0$ and $t_0 > 1$.

From (1), we have

$$\varphi(||Ax_0 - t_0^{-1}Ax_0||) \geq \varphi(||Ax_0||)\phi(||Ax_0|| \cdot ||t_0^{-1}Ax_0||^{-1}) - \varphi(||t_0^{-1}Ax_0||),$$

which implies

$$\varphi[(1 - t_0^{-1})||Ax_0||] + \varphi(t_0^{-1}||Ax_0||) \geq \varphi(||Ax_0||)\phi(t_0). \tag{2}$$

By strict convexity of ϕ and $\phi(0) = 0$, we obtain

$$\begin{aligned}\varphi[(1 - t_0^{-1})||Ax_0||] + \varphi(t_0^{-1}||Ax_0||) &= \varphi[(1 - t_0^{-1})||Ax_0|| + t_0^{-1}||\theta||] + \varphi[t_0^{-1}||Ax_0|| + (1 - t_0^{-1})||\theta||] \\ &< (1 - t_0^{-1})\varphi(||Ax_0||) + t_0^{-1}\varphi(0) + t_0^{-1}\varphi(||Ax_0||) + (1 - t_0^{-1})\varphi(0) \\ &= \varphi(||Ax_0||).\end{aligned} \tag{3}$$

It is easy to see from (2) and (3) that

$$\varphi(||Ax_0||)\phi(t_0) < \varphi(||Ax_0||). \tag{4}$$

Noting that $t_0 > 1$ and $\varphi(t) \geq 1$, for all $t > 1$, we have

$$\varphi(||Ax_0||)\phi(t_0) \geq \varphi(||Ax_0||),$$

which contradicts (4), and so the condition (L-S) is satisfied. Therefore, it follows from Lemma 1.1 that the conclusions of Theorem 2.1 hold. □

Remark 2.2. If there exist convex function $\phi : R^+ \to R^+$, $\phi(0) = 0$ and real function $\varphi : R^+ \to R$, $\varphi(t) > 1$, $\forall t > 1$ satisfied (1), the conclusions of Theorem 2.1 also hold.

Theorem 2.3. *Let E, Ω, A be the same as in Lemma 1.1. Moreover, if there exist strictly concave function $\phi : R^+ \to R^+$ with $\phi(0) = 0$ and real function $\varphi : R^+ \to R$, $\varphi(t) \leq 1$, $\forall t > 1$, such that*

$$\varphi(||Ax - x||) \le \varphi(||Ax||)\phi(||Ax|| \cdot ||x||^{-1}) - \varphi(||x||), \quad \forall x \in \partial\Omega, \tag{5}$$

then deg(I - A, Ω, θ) = 1 *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. If the operator A has a fixed point on $\partial\Omega$, then A has at least one fixed point in $\overline{\Omega}$. Now suppose that A has no fixed points on $\partial\Omega$. Next we shall prove that the condition (L-S) is satisfied.

Suppose this is not true. Then there exists $x_0 \in \partial\Omega$, $t_0 \ge 1$ such that $Ax_0 = t_0 x_0$, i.e., $x_0 = t_0^{-1}Ax_0$. It is easy to see that $||Ax_0|| \ne 0$ and $t_0 > 1$. From (5), we have

$$\varphi(||Ax_0 - t_0^{-1}Ax_0||) \le \varphi(||Ax_0||)\phi(||Ax_0|| \cdot ||t_0^{-1}Ax_0||^{-1}) - \varphi(||t_0^{-1}Ax_0||).$$

This implies that

$$\varphi[(1 - t_0^{-1})||Ax_0||] + \varphi(t_0^{-1}||Ax_0||) \le \varphi(||Ax_0||)\phi(t_0). \tag{6}$$

By strict concavity of ϕ and ϕ (0) = 0, we obtain

$$\begin{aligned}\varphi[(1 - t_0^{-1})||Ax_0||] + \varphi(t_0^{-1}||Ax_0||) &= \varphi[(1 - t_0^{-1})||Ax_0|| + t_0^{-1}||\theta||] + \varphi[t_0^{-1}||Ax_0|| + (1 - t_0^{-1})||\theta||]\\ &> (1 - t_0^{-1})\varphi(||Ax_0||) + t_0^{-1}\varphi(0) + t_0^{-1}\varphi(||Ax_0||) + (1 - t_0^{-1})\varphi(0)\\ &= \varphi(||Ax_0||).\end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\varphi(||Ax_0||)\phi(t_0) > \varphi(||Ax_0||). \tag{8}$$

On the other hand, by $t_0 > 1$ and $\varphi(t) \le 1$, $\forall t > 1$, we have

$$\varphi(||Ax_0||)\phi(t_0) \le \varphi(||Ax_0||),$$

which contradicts (8), and so the condition (L-S) is satisfied. Therefore, it follows from Lemma 1.1 that the conclusions of Theorem 2.3 hold. □

Remark 2.4. If there exist concave function $\phi : R^+ \to R^+$, ϕ (0) = 0 and real function $\varphi : R^+ \to R$, φ (t) < 1, $\forall t > 1$ satisfied (5), the conclusions of Theorem 2.3 also hold.

Corollary 2.5. *Let E,* Ω*, A be the same as in Lemma 1.1. Moreover, if there exist* $\alpha \in (-\infty, 0) \cup (1, +\infty)$ *and* $\beta \ge 0$ *such that*

$$||Ax - x||^{\alpha} \ge ||Ax||^{\alpha+\beta}||x||^{-\beta} - ||x||^{\alpha}, \quad \forall x \in \partial\Omega,$$

then deg(I - A, Ω, θ) = 1 *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. Putting $\phi(t) = t^{\alpha}$, $\varphi(t) = t^{\beta}$, we have ϕ (t) is a strictly convex function with ϕ (0) = 0 and $\varphi(t) \ge 1$, $\forall t > 1$. Therefore, from Theorem 2.1, the conclusions of Corollary 2.5 hold.. □

Remark 2.6. 1. Corollary 2.5 generalizes Theorem 2.2 of Xu [7] from $\alpha > 1$ to $\alpha \in (-\infty, 0) \cup (1, +\infty)$. Moreover, our methods are different from those in many recent works (e.g., Li [6], Xu [7]).

2. Putting $\alpha > 1$, $\beta = 0$ in Corollary 2.5, we can obtain Theorem 5 of Li [6].

Corollary 2.7. *Let E,* Ω*, A be the same as in Lemma 1.1. Moreover, if there exist* $\alpha \in (0, 1)$ *and* $\beta \le 0$ *such that*

$$||Ax - x||^{\alpha} \le ||Ax||^{\alpha+\beta}||x||^{-\beta} - ||x||^{\alpha}, \quad \forall x \in \partial D,$$

then deg(I - A, Ω, θ) = 1 *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. Putting $\phi(t) = t^{\alpha}$, $\varphi(t) = t^{\beta}$, we have $\phi(t)$ is a strictly concave function with $\phi(0) = 0$ and $\varphi(t) \leq 1$, $\forall t > 1$. Therefore, from Theorem 2.3, the conclusions of Corollary 2.7 hold. □

Remark 2.8. Corollary 2.7 extends Theorem 8 of Li [6]. Putting $\beta = 0$ in Corollary 2.7, we can obtain Theorem 8 of Li [6].

Theorem 2.9. *Let E,* Ω, *A be the same as in Lemma 1.1. Moreover, if there exist strictly convex function* $\phi : R^{+} \to R^{+}$ *with* $\phi(0) = 0$ *and real function* $\varphi : R^{+} \to R$ *with* $\varphi(t) \geq 1$, *for all* $t > 1$, *such that*

$$\varphi(||Ax - x||) \geq \varphi(||Ax||)\phi(||Ax + x|| \cdot ||x||^{-1}) - \varphi(||x||), \quad \forall x \in \partial\Omega, \tag{9}$$

then deg(I - A, Ω, θ) = 1 *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. If the operator A has a fixed point on $\partial\Omega$, then A has at least one fixed point in $\overline{\Omega}$. Now suppose that A has no fixed points on $\partial\Omega$. Next we shall prove that the condition (L-S) is satisfied.

Suppose this is not true. Then there exists $x_0 \in \partial\Omega$, $t_0 \geq 1$ such that $Ax_0 = t_0x_0$, i.e., $x_0 = t_0^{-1}Ax_0$. It is easy to see that $||Ax_0|| \neq 0$ and $t_0 > 1$. By virtue of (9), we have

$$\varphi(||Ax_0 - t_0^{-1}Ax_0||) \geq \varphi(||Ax_0||)\phi(||Ax_0 + t_0^{-1}Ax_0|| \cdot ||t_0^{-1}Ax_0||^{-1}) - \varphi(||t_0^{-1}Ax_0||),$$

which implies

$$\varphi[(1 - t_0^{-1})||Ax_0||] + \varphi(t_0^{-1}||Ax_0||) \geq \varphi(||Ax_0||)\phi[(1 + t_0^{-1})t_0]. \tag{10}$$

By strict convexity of ϕ and $\phi(0) = 0$, we obtain (3) holds. From (3) and (10), we have

$$\varphi(||Ax_0||)\phi[(1 + t_0^{-1})t_0] < \varphi(||Ax_0||). \tag{11}$$

Noting that $t_0 > 1$ and $\varphi(t) \geq 1$, for all $t > 1$, we have $(1 + t_0^{-1})t_0 = t_0 + 1 > 1$, and so

$$\varphi(||Ax_0||)\phi[(1 + t_0^{-1})t_0] \geq \varphi(||Ax_0||),$$

which contradicts (11), and so the condition (L-S) is satisfied. Therefore, it follows from Lemma 1.1 that the conclusions of Theorem 2.9 hold. □

Remark 2.10. If there exist convex function $\phi : R^{+} \to R^{+}$, $\phi(0) = 0$ and real function $\varphi : R^{+} \to R$, $\varphi(t) > 1$, $\forall t > 1$ satisfied (9), the conclusions of Theorem 2.9 also hold.

Theorem 2.11. *Let E,* Ω, *A be the same as in Lemma 1.1. Moreover, if there exist strictly concave function* $\phi : R^{+} \to R^{+}$ *with* $\phi(0) = 0$ *and real function* $\varphi : R^{+} \to R$, $\varphi(t) \leq 1$, $\forall t > 1$, *such that*

$$\varphi(||Ax - x||) \leq \varphi(||Ax||)\phi(||Ax + x|| \cdot ||x||^{-1}) - \varphi(||x||), \quad \forall x \in \partial\Omega, \tag{12}$$

then deg(I - A, Ω, θ) = 1 *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. If the operator A has a fixed point on $\partial\Omega$, then A has at least one fixed point in $\overline{\Omega}$. Now suppose that A has no fixed points on $\partial\Omega$. Next we shall prove that the condition (L-S) is satisfied.

Suppose this is not true. Then there exists $x_0 \in \partial\Omega$, $t_0 \geq 1$ such that $Ax_0 = t_0x_0$, i.e., $x_0 = t_0^{-1}Ax_0$. It is easy to see that $||Ax_0|| \neq 0$ and $t_0 > 1$. By (12), we have

$$\varphi(||Ax_0 - t_0^{-1}Ax_0||) \leq \varphi(||Ax_0||)\phi(||Ax_0 + t_0^{-1}Ax_0|| \cdot ||t_0^{-1}Ax_0||^{-1}) - \varphi(||t_0^{-1}Ax_0||),$$

which implies

$$\varphi[(1 - t_0^{-1})||Ax_0||] + \varphi(t_0^{-1}||Ax_0||) \leq \varphi(||Ax_0||)\phi[(1 + t_0^{-1})t_0]. \tag{13}$$

By strict concavity of ϕ and ϕ (0) = 0, we have (7) holds. From (7) and (13), we obtain

$$\varphi(||Ax_0||)\phi[(1 + t_0^{-1})t_0] > \varphi(||Ax_0||). \tag{14}$$

On the other hand, by $t_0 > 1$, we have $(1 + t_0^{-1})t_0 = t_0 + 1 > 1$. Therefore, it follows from $\varphi(t) \leq 1$, $\forall t > 1$ that

$$\varphi(||Ax_0||)\phi[(1 + t_0^{-1})t_0] \leq \varphi(||Ax_0||),$$

which contradicts (14), and so the condition (L-S) is satisfied. Therefore, it follows from Lemma 1.1 that the conclusions of Theorem 2.11 hold. □

Remark 2.12. If there exist convex function $\phi : R^+ \to R^+$, ϕ (0) = 0 and real function $\varphi : R^+ \to R$, φ $(t) > 1$, $\forall t > 1$ satisfied (12), the conclusions of Theorem 2.11 also hold.

Corollary 2.13. *Let E,* Ω, *A be the same as in Lemma 1.1. Moreover, if there exist* $\alpha \in (-\infty, 0) \cup (1, +\infty)$ *and* $\beta \geq 0$ *such that*

$$||Ax - x||^{\alpha}||x||^{\beta} \geq ||Ax||^{\alpha}||Ax + x||^{\beta} - ||x||^{\alpha+\beta}, \quad \forall x \in \partial\Omega, \tag{15}$$

then $\deg(I - A, \Omega, \theta) = 1$ *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. From (15), we have

$$||Ax - x||^{\alpha} \geq ||Ax||^{\alpha}||Ax + x||^{\beta}||x||^{-\beta} - ||x||^{\alpha}, \quad \forall x \in \partial\Omega.$$

Taking $\phi(t) = t^{\alpha}$, $\varphi(t) = t^{\beta}$, we have ϕ (t) is a strictly convex function with ϕ (0) = 0 and $\varphi(t) \geq 1$, $\forall t > 1$. Therefore, from Theorem 2.9, the conclusions of Corollary 2.13 hold. □

Remark 2.14. 1. Corollary 2.13 generalizes Theorem 2.4 of Xu [7] from $\alpha > 1$ to $\alpha \in (-\infty, 0) \cup (1, +\infty)$. Moreover, our methods are different from those in many recent works (e.g., Li [6], Xu [7]).

2. Putting $\alpha > 1$, $\beta = 0$ in Corollary 2.13, we can obtain Theorem 5 of Li [6].

Corollary 2.15. *Let E,* Ω, *A be the same as in Lemma 1.1. Moreover, if there exist* $\alpha \in (0, 1)$ *and* $\beta \leq 0$ *such that*

$$||Ax - x||^{\alpha}||x||^{\beta} \leq ||Ax||^{\alpha}||Ax + x||^{\beta} - ||x||^{\alpha+\beta}, \quad \forall x \in \partial\Omega, \tag{16}$$

then $\deg(I - A, \Omega, \theta) = 1$ *if A has no fixed point on* $\partial\Omega$, *and so A has at least one fixed point in* $\overline{\Omega}$.

Proof. From (16), we have

$$||Ax - x||^{\alpha} \leq ||Ax||^{\alpha}||Ax + x||^{\beta}||x||^{-\beta} - ||x||^{\alpha}, \quad \forall x \in \partial\Omega.$$

Putting $\phi(t) = t^{\alpha}$, $\varphi(t) = t^{\beta}$, we have ϕ (t) is a strictly concave function with ϕ (0) = 0 and $\varphi(t) \leq 1$, $\forall t > 1$. Therefore, from Theorem 2.11, the conclusions of Corollary 2.15 hold. □

Remark 2.16. Corollary 2.15 extends Theorem 8 of Li [6]. Putting $\beta = 0$ in Corollary 2.15, we can obtain Theorem 8 of Li [6].

Theorem 2.17. *Let E, Ω, A be the same as in Lemma 1.1. Moreover, if there exist* $\alpha \in (-\infty, 0) \cup (1, +\infty)$, $\beta \geq 0$ *and* $\mu \geq 1$ *such that*

$$||Ax - \mu x||^{\alpha} \geq ||Ax||^{\alpha+\beta} ||\mu x||^{-\beta} - ||\mu x||^{\alpha}, \quad \forall x \in \partial\Omega,$$

then the equation $Ax = \mu x$ *possesses a solution in* $\overline{\Omega}$.

Proof. Without loss of generality, suppose that $\frac{1}{\mu}A$ has no fixed point on $\partial\Omega$. From (17), we have

$$\frac{1}{\mu^{\alpha}}||Ax - \mu x||^{\alpha} \geq \frac{1}{\mu^{\alpha}}||Ax||^{\alpha+\beta}||\mu x||^{-\beta} - \frac{1}{\mu^{\alpha}}||\mu x||^{\alpha}, \quad \forall x \in \partial\Omega,$$

which implies

$$||\frac{1}{\mu}Ax - x||^{\alpha} \geq ||\frac{1}{\mu}Ax||^{\alpha+\beta}||x||^{-\beta} - ||x||^{\alpha}, \quad \forall x \in \partial\Omega.$$

It is easy to see that $\frac{1}{\mu}A$ is a semi-closed 1-set-contractive operator. It follows from Corollary 2.5 that $\deg(I - \frac{1}{\mu}A, \Omega, \theta) = 1 \neq 0$, and so the equation $Ax = \mu x$ possesses a solution in $\overline{\Omega}$.

Remark 2.18. Similarly, from Corollary 2.7, Corollary 2.13 or Corollary 2.15, we can obtain the equation $Ax = \mu x$ possesses a solution in $\overline{\Omega}$.

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Competing interests
The authors declare that they have no competing interests.

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