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Fixed point and weak convergence theorems for point-dependent λ-hybrid mappings in Banach spaces

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Abstract

The purpose of this article is to study the fixed point and weak convergence problem for the new defined class of point-dependent λ-hybrid mappings relative to a Bregman distance D_f in a Banach space. We at first extend the Aoyama-Iemoto-Kohsaka-Takahashi fixed point theorem for λ-hybrid mappings in Hilbert spaces in 2010 to this much wider class of nonlinear mappings in Banach spaces. Secondly, we derive an Opial-like inequality for the Bregman distance and apply it to establish a weak convergence theorem for this new class of nonlinear mappings. Some concrete examples in a Hilbert space showing that our extension is proper are also given.

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1 Introduction

Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to H$ is said to be

(1.1) nonexpansive if $||Tx - Ty|| \leq ||x - y||$, $\forall x, y \in C$, cf. [1,2];

(1.2) nonspreading if $||Tx - Ty||^2 \leq ||x - y||^2 + 2 \langle x - Tx, y - Ty\rangle$, $\forall x, y \in C$, cf. [3-5];

(1.3) hybrid if $||Tx - Ty||^2 \leq ||x - y||^2 + \langle x - Tx, y - Ty\rangle$, $\forall x, y \in C$, cf. [3,5-7].

As shown in [3], (1.2) is equivalent to

$$2||Tx - Ty||^2 \leq ||Tx - y||^2 + ||x - Ty||^2$$

for all $x, y \in C$.

In 1965, Browder [1] established the following

Browder fixed point Theorem. *Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a nonexpansive mapping. Then, the following are equivalent:*

(a) *There exists $x \in C$ such that $\{T^n x\}_{n \in \mathbb{N}}$ is bounded;*

(b) *T has a fixed point.*

The above result is still true for nonspreading mappings which was shown in Kohsaka and Takahashi [4]. (We call it the Kohsaka-Takahashi fixed point theorem.)

Recently, Aoyama et al. [8] introduced a new class of nonlinear mappings in a Hilbert space containing the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings. For $\lambda \in \mathbb{R}$, they call a mapping $T : C \to H$

(1.4) λ-hybrid if $||Tx - Ty||^2 \le ||x - y||^2 + \lambda \langle x - Tx, y - Ty\rangle, \forall x, y \in C$.

And, among other things, they establish the following

Aoyama-Iemoto-Kohsaka-Takahashi fixed point Theorem. [8]*Let C be a nonempty closed convex subset of a Hilbert space H, and let $T : C \to C$ be a λ-hybrid mapping. Then, the following are equivalent:*

(a) *There exists $x \in C$ such that $\{T^n x\}_{n \in \mathbb{N}}$ is bounded;*

(b) *T has a fixed point.*

Obviously, T is nonexpansive if and only if it is 0-hybrid; T is nonspreading if and only if it is 2-hybrid; T is hybrid if and only if it is 1-hybrid.

Motivated by the above works, we extend the concept of λ-hybrid from Hilbert spaces to Banach spaces in the following way:

Definition 1.1. *For a nonempty subset C of a Banach space X, a Gâteaux differentiable convex function $f : X \to (-\infty,\infty]$ and a function $\lambda : C \to \mathbb{R}$, a mapping $T : C \to X$ is said to be point-dependent λ-hybrid relative to D_f if*

(1.5) $D_f(Tx, Ty) \le D_f(x, y) + \lambda(y)\langle x - Tx, f'(y) - f'(Ty)\rangle, \forall x, y \in C$,

where D_f is the Bregman distance associated with f and $f'(x)$ denotes the Gâteaux derivative of f at x.

In this article, we study the fixed point and weak convergence problem for mappings satisfying (1.5). This article is organized in the following way: Section 2 provides preliminaries. We investigate the fixed point problem for point-dependent λ-hybrid mappings in Section 3, and we give some concrete examples showing that even in the setting of a Hilbert space, our fixed point theorem generalizes the Aoyama-Iemoto-Kohsaka-Takahashi fixed point theorem properly in Section 4. Section 5 is devoting to studying the weak convergence problem for this new class of nonlinear mappings.

2 Preliminaries

In what follows, X will be a real Banach space with topological dual X^* and $f : X \to (-\infty,\infty]$ will be a convex function. $\mathcal{D}$ denotes the domain of f, that is,

$$\mathcal{D} = \{x \in X : f(x) < \infty\},$$

and $\mathcal{D}^\circ$ denotes the algebraic interior of $\mathcal{D}$, i.e., the subset of $\mathcal{D}$ consisting of all those points $x \in \mathcal{D}$ such that, for any $y \in X \setminus \{x\}$, there is z in the open segment (x, y) with $[x, z] \subseteq \mathcal{D}$. The topological interior of $\mathcal{D}$, denoted by $\mathrm{Int}(\mathcal{D})$, is contained in $\mathcal{D}^\circ$. f is said to be proper provided that $\mathcal{D} \neq \emptyset$. f is called lower semicontinuous (l.s.c.) at $x \in X$ if $f(x) \le \liminf_{y \to x} f(y)$. f is strictly convex if

$$f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha)f(y)$$

for all $x, y \in X$ and $\alpha \in (0, 1)$.

The function $f : X \to (-\infty, \infty]$ is said to be Gâteaux differentiable at $x \in X$ if there is $f'(x) \in X^*$ such that

$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t} = \langle y, f'(x)\rangle$$

for all $y \in X$.

The Bregman distance D_f associated with a proper convex function f is the function $D_f : \mathcal{D} \times \mathcal{D} \to [0,\infty]$ defined by

$$D_f(y,x) = f(y) - f(x) + f^{\circ}(x, x-y), \tag{1}$$

where $f^{\circ}(x, x-y) = \lim_{t \to 0^+} \frac{f(x+t(x-y)) - f(x)}{t}$. $D_f(y, x)$ is finite valued if and only if $x \in \mathcal{D}^{o}$, cf. Proposition 1.1.2 (iv) of [9]. When f is Gâteaux differentiable on D, (1) becomes

$$D_f(y,x) = f(y) - f(x) - \langle y - x, f'(x) \rangle, \tag{2}$$

and then the modulus of total convexity is the function $\nu_f : \mathcal{D}^{\circ} \times [0,\infty) \to [0,\infty]$ defined by

$$\nu_f(x,t) = \inf\{D_f(y,x) : y \in \mathcal{D}, ||y-x|| = t\}.$$

It is known that

$$\nu_f(x,ct) \geq c\nu_f(x,t) \tag{3}$$

for all $t \geq 0$ and $c \geq 1$, cf. Proposition 1.2.2 (ii) of [9]. By definition it follows that

$$D_f(y,x) \geq \nu_f(x, ||y-x||). \tag{4}$$

The modulus of uniform convexity of f is the function $\delta_f : [0, \infty) \to [0, \infty]$ defined by

$$\delta_f(t) = \inf\left\{f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) : x, y \in \mathcal{D}, ||x-y|| \geq t\right\}.$$

The function f is called uniformly convex if $\delta_f(t) > 0$ for all $t > 0$. If f is uniformly convex then for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)}{2} + \frac{f(y)}{2} - \delta \tag{5}$$

for all $x, y \in \mathcal{D}$ with $||x - y|| \geq \varepsilon$.

Note that for $y \in \mathcal{D}$ and $x \in \mathcal{D}^{\circ}$, we have

$$\begin{aligned} & f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \\ &= f(y) - f(x) - \frac{f(x + \frac{y-x}{2}) - f(x)}{\frac{1}{2}} \\ &\leq f(y) - f(x) - f^{\circ}(x, y-x) \leq D_f(y,x), \end{aligned}$$

where the first inequality follows from the fact that the function $t \to f(x + tz) - f(x)/t$ is nondecreasing on $(0, \infty)$. Therefore,

$$\nu_f(x,t) \geq \delta_f(t) \tag{6}$$

whenever $x \in \mathcal{D}^{\circ}$ and $t \geq 0$. For other properties of the Bregman distance D_f, we refer readers to [9].

The normalized duality mapping J from X to 2^{X^*} is defined by

$$Jx = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\}$$

for all $x \in X$.

When $f(x) = ||x||^2$ in a smooth Banach space X, it is known that $f'(x) = 2J(x)$ for $x \in X$, cf. Corollaries 1.2.7 and 1.4.5 of [10]. Hence, we have

$$\begin{aligned} D_f(y,\ x) &= \ ||y||^2 - ||x||^2 - \ \langle y - x,\ f'(x)\rangle \\ &= \ ||y||^2 - ||x||^2 - 2\langle y - x,\ Jx\rangle \\ &= \ ||y||^2 + ||x||^2 - 2\langle y,\ Jx\rangle. \end{aligned}$$

Moreover, as the normalized duality mapping J in a Hilbert space H is the identity operator, we have

$$D_f(y,x) = \ ||y||^2 + ||x||^2 - 2\langle y,x\rangle = \ ||y-x||^2.$$

Thus, in case λ is a constant function and $f(x) = ||x||^2$ in a Hilbert space, (1.5) coincides with (1.4). However, in general, they are different.

A function $g : X \to (-\infty,\infty]$ is said to be subdifferentiable at a point $x \in X$ if there exists a linear functional $x^* \in X^*$ such that

$$g(y) - g(x) \geq \langle y - x, x^*\rangle, \qquad \forall y \in X.$$

We call such x^* the subgradient of g at x. The set of all subgradients of g at x is denoted by $\partial g(x)$ and the mapping $\partial g : X \to 2^{X^*}$ is called the subdifferential of g. For a l.s.c. convex function f, ∂f is bounded on bounded subsets of $\mathrm{Int}(\mathcal{D})$ if and only if f is bounded on bounded subsets there, cf. Proposition 1.1.11 of [9]. A proper convex l.s.c. function f is Gâteaux differentiable at $x \in \mathrm{Int}(\mathcal{D})$ if and only if it has a unique subgradient at x; in such case $\partial f(x) = f'(x)$, cf. Corollary 1.2.7 of [10].

The following lemma will be quoted in the sequel.

Lemma 2.1. (Proposition 1.1.9 of [9]) *If a proper convex function $f : X \to (-\infty, \infty]$ is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$ in a Banach space X, then the following statements are equivalent:*

(a) *The function f is strictly convex on $\mathrm{Int}(\mathcal{D})$.*

(b) *For any two distinct points $x, y \in \mathrm{Int}(\mathcal{D})$, one has $D_f\,(y, x) > 0$.*

(c) *For any two distinct points $x, y \in \mathrm{Int}(\mathcal{D})$, one has*

$$\langle x - y, f'(x) - f'(y)\rangle > 0.$$

Throughout this article, $F(T)$ will denote the set of all fixed points of a mapping T.

3 Fixed point theorems

In this section, we apply Lemma 2.1 to study the fixed point problem for mappings satisfying (1.5).

Theorem 3.1. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$ and is bounded on bounded subsets of $\mathrm{Int}(\mathcal{D})$. Suppose $C \subseteq \mathrm{Int}(\mathcal{D})$ is a nonempty closed convex subset of X and $T : C \to C$ is point-dependent λ-hybrid relative to D_f for some function $\lambda : C \to \mathbb{R}$. For $x \in C$ and any $n \in \mathbb{N}$ define*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x,$$

where T^0 is the identity mapping on C. If $\{T^n x\}_{n\in\mathbb{N}}$ is bounded, then every weak cluster point of $\{S_n x\}_{n\in\mathbb{N}}$ is a fixed point of T.

Proof. Since T is point-dependent λ-hybrid relative to D_f, we have, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned}
0 &\le D_f(T^k x, y) - D_f(T^{k+1}x, Ty) + \lambda(y)\langle T^k x - T^{k+1}x, f'(y) - f'(Ty)\rangle \\
&= f(T^k x) - f(y) - \langle T^k x - y, f'(y)\rangle - f(T^{k+1}x) + f(Ty) + \langle T^{k+1}x - Ty, f'(Ty)\rangle \\
&\quad + \lambda(y)\langle T^k x - T^{k+1}x, f'(y) - f'(Ty)\rangle \\
&= [f(T^k x) - f(T^{k+1}x)] + [f(Ty) - f(y)] + \langle \lambda(y)(T^k x - T^{k+1}x) - T^k x + y, f'(y)\rangle \\
&\quad + \langle T^{k+1}x - Ty - \lambda(y)(T^k x - T^{k+1}x), f'(Ty)\rangle.
\end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n - 1$, we get

$$\begin{aligned}
0 &\le [f(x) - f(T^n x)] + n[f(Ty) - f(y)] + \langle \lambda(y)(x - T^n x) + ny - nS_n x, f'(y)\rangle \\
&\quad + \langle (n+1)S_{n+1}x - x - nTy - \lambda(y)(x - T^n x), f'(Ty)\rangle.
\end{aligned}$$

Dividing the above inequality by n, we have

$$\begin{aligned}
0 &\le \frac{f(x) - f(T^n x)}{n} + [f(Ty) - f(y)] + \left\langle \frac{\lambda(y)(x - T^n x)}{n} + y - S_n x, f'(y)\right\rangle \\
&\quad + \left\langle \frac{n+1}{n} S_{n+1}x - \frac{x}{n} - Ty - \frac{\lambda(y)(x - T^n x)}{n}, f'(Ty)\right\rangle.
\end{aligned} \tag{7}$$

Since $\{T^n x\}_{n\in\mathbb{N}}$ is bounded, $\{S_n x\}_{n\in\mathbb{N}}$ is bounded, and so, in view of X being reflexive, it has a subsequence $\{S_{n_i}x\}_{i\in\mathbb{N}}$ so that $S_{n_i}x$ converges weakly to some $v \in C$ as $n_i \to \infty$. Replacing n by n_i in (7), and letting $n_i \to \infty$, we obtain from the fact that $\{T^n x\}_{n\in\mathbb{N}}$ and $\{f(T^n x)\}_{n\in\mathbb{N}}$ are bounded that

$$0 \le f(Ty) - f(y) + \langle y - v, f'(y)\rangle + \langle v - Ty, f'(Ty)\rangle. \tag{8}$$

Putting $y = v$ in (8), we get

$$0 \le f(Tv) - f(v) + \langle v - Tv, f'(Tv)\rangle,$$

that is,

$$0 \le -D_f(v, Tv),$$

from which follows that $D_f(v, Tv) = 0$. Therefore $Tv = v$ by Lemma 2.1. □

The following theorem comes from Theorem 3.1 immediately.

Theorem 3.2. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$ and is bounded on bounded subsets of $\mathrm{Int}(\mathcal{D})$. Suppose $C \subseteq \mathrm{Int}(\mathcal{D})$ is a nonempty closed convex subset of X and $T : C \to C$ is point-dependent λ-hybrid relative to D_f for some function $\lambda : C \to \mathbb{R}$. Then, the following two statements are equivalent:*

(a) *There is a point $x \in C$ such that $\{T^n x\}_{n\in\mathbb{N}}$ is bounded.*

(b) $F(T) \neq \emptyset$.

Taking $\lambda(x) = \lambda$, a constant real number, for all $x \in C$ and noting the function $f(x) = ||x||^2$ in a Hilbert space H satisfies all the requirements of Theorem 3.2, the corollary below follows immediately.

Corollary 3.3. [8]*Let C be a nonempty closed convex subset of Hilbert space H and suppose $T : C \to C$ is λ-hybrid. Then, the following two statements are equivalent:*

(a) *There exists $x \in C$ such that $\{T^n(x)\}_{n\in\mathbb{N}}$ is bounded.*

(b) *T has a fixed point.*

We now show that the fixed point set $F(T)$ is closed and convex under the assumptions of Theorem 3.2.

A mapping $T : C \to X$ is said to be quasi-nonexpansive with respect to D_f if $F(T) \neq \emptyset$ and $D_f(v, Tx) \le D_f(v, x)$ for all $x \in C$ and all $v \in F(T)$.

Lemma 3.4. *Let $f : X \to (-\infty,\infty]$ be a proper strictly convex function on a Banach space X so that it is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$, and let $C \subseteq \mathrm{Int}(\mathcal{D})$ be a nonempty closed convex subset of X. If $T : C \to C$ is quasi-nonexpansive with respect to D_f, then $F(T)$ is a closed convex subset.*

Proof. Let $x \in \overline{F(T)}$ and choose $\{x_n\}_{n\in\mathbb{N}} \subseteq F(T)$ such that $x_n \to x$ as $n \to \infty$. By the continuity of $D_f(\cdot, Tx)$ and $D_f(x_n, T_x) \le D_f(x_n, x)$, we have

$$D_f(x, Tx) = \lim_{n\to\infty} D_f(x_n, Tx) \le \lim_{n\to\infty} D_f(x_n, x) = D_f(x, x) = 0.$$

Thus, due to the strict convexity of f, it follows from Lemma 2.2 that $Tx = x$. This shows $F(T)$ is closed. Next, let $x, y \in F(T)$ and $\alpha \in [0, 1]$. Put $z = \alpha x + (1 - \alpha)y$. We show that $Tz = z$ to conclude $F(T)$ is convex. Indeed,

$$\begin{aligned}
&D_f(z,\ Tz)\\
&= f(z) - f(Tz) - \langle z - Tz,\ f'(Tz)\rangle\\
&= f(z) + [\alpha f(x) + (1-\alpha)f(y)] - f(Tz) - \langle z - Tz,\ f'(Tz)\rangle - [\alpha f(x) + (1-\alpha)f(y)]\\
&= f(z) + \alpha[f(x) - f(Tz) - \langle x - Tz,\ f'(Tz)\rangle]\\
&\quad + (1-\alpha)[f(y) - f(Tz) - \langle y - Tz,\ f'(Tz)\rangle] - [\alpha f(x) + (1-\alpha)f(y)]\\
&= f(z) + \alpha D_f(x, Tz) + (1-\alpha)D_f(y,\ Tz) - [\alpha f(x) + (1-\alpha)f(y)]\\
&\le f(z) + \alpha D_f(x,\ z) + (1-\alpha)D_f(y,\ z) - [\alpha f(x) + (1-\alpha)f(y)]\\
&= f(z) + \alpha[f(x) - f(z) - \langle x - z,\ f'(z)\rangle] + (1-\alpha)[f(y) - f(z) - \langle y - z,\ f'(z)\rangle]\\
&\quad - [\alpha f(x) + (1-\alpha)f(y)]\\
&= f(z) + \alpha f(x) - \alpha f(z) - \langle \alpha x - \alpha z,\ f'(z)\rangle + (1-\alpha)f(y) - (1-\alpha)f(z)\\
&\quad - \langle (1-\alpha)y - (1-\alpha)z,\ f'(z)\rangle - [\alpha f(x) + (1-\alpha)f(y)]\\
&= -\langle \alpha x + (1-\alpha)y - (\alpha z + (1-\alpha)z),\ f'(z)\rangle\\
&= -\langle 0,\ f'(z)\rangle = 0.
\end{aligned}$$

Therefore, $Tz = z$ by the strictly convex of f. This completes the proof. □

Proposition 3.5. *Let $f : X \to (-\infty,\infty]$ be a proper strictly convex function on a reflexive Banach space X so that it is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$ and is bounded on bounded subsets of Int(D), and let $C \subseteq \mathrm{Int}(\mathcal{D})$ be a nonempty closed convex subset of X. Suppose $T : C \to C$ is point-dependent λ-hybrid relative to D_f for some function $\lambda : C \to \mathbb{R}$ and has a point $x_0 \in C$ such that $\{T^n(x_0)\}_{n\in\mathbb{N}}$ is bounded. Then, T is quasi-nonexpansive with respect to D_f, and therefore, $F(T)$ is a nonempty closed convex subset of C.*

Proof. In view of Theorem 3.2, $F(T) \neq \emptyset$. Now, for any $v \in F(T)$ and any $y \in C$, as T is point-dependent λ-hybrid relative to D_f, we have

$$\begin{aligned} D_f(v, Ty) &= D_f(Tv, Ty) \\ &\leq D_f(v, y) + \lambda(y)\, \langle v - Tv, f'(y) - f'(Ty) \rangle \\ &= D_f(v, y) \end{aligned}$$

for all $y \in C$, so T is quasi-nonexpansive with respect to D_f, and hence, $F(T)$ is a nonempty closed convex subset of C by Lemma 3.4. □

For the remainder of this section, we establish a common fixed point theorem for a commutative family of point-dependent λ-hybrid mappings relative to D_f.

Lemma 3.6. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on $Int(\mathcal{D})$ and is bounded on bounded subsets of $Int(\mathcal{D})$. Suppose $C \subseteq Int(\mathcal{D})$ is a nonempty bounded closed convex subset of X and $\{T_1, T_2,..., T_N\}$ is a commutative finite family of point-dependent λ-hybrid mappings relative to D_f for some function $\lambda : C \to \mathbb{R}$ from C into itself. Then $\{T_1, T_2,..., T_N\}$ has a common fixed point.*

Proof. We prove this lemma by induction with respect to N. To begin with, we deal with the case that $N = 2$. By Proposition 3.5, we see that $F(T_1)$ and $F(T_2)$ are nonempty bounded closed convex subsets of X. Moreover, $F(T_1)$ is T_2-invariant. Indeed, for any $v \in F(T_1)$, it follows from $T_1T_2 = T_2T_1$ that $T_1T_2v = T_2T_1v = T_2v$, which shows that $T_2v \in F(T_1)$. Consequently, the restriction of T_2 to $F(T_1)$ is point-dependent λ-hybrid relative to D_f, and hence by Theorem 3.2, T_2 has a fixed point $u \in F(T_1)$, that is, $u \in F(T_1) \cap F(T_2)$.

By induction hypothesis, assume that for some $n \geq 2$, $E = \cap_{k=1}^{n} F(T_k)$ is nonempty. Then, E is a nonempty closed convex subset of X and the restriction of T_{n+1} to E is a point-dependent λ-hybrid mapping relative to D_f from E into itself. By Theorem 3.2, T_{n+1} has a fixed point in X. This shows that $E \cap F(T_{n+1}) \neq \emptyset$, that is, $\cap_{k=1}^{n+1} F(T_k) \neq \emptyset$, completing the proof. □.

Theorem 3.7. *Let X be a reflexive Banach space and let $f : X \to (-\infty,\infty]$ be a l.s.c. strictly convex function so that it is Gâteaux differentiable on $Int(\mathcal{D})$. Suppose $C \subseteq Int(\mathcal{D})$ is a nonempty bounded closed convex subset of X and $\{T_i\}_{i \in I}$ is a commutative family of point-dependent λ-hybrid mappings relative to D_f for some function $\lambda : C \to \mathbb{R}$ from C into itself. Then, $\{T_i\}_{i \in I}$ has a common fixed point.*

Proof. Since C is a nonempty bounded closed convex subset of the reflexive Banach space X, it is weakly compact. By Proposition 3.5, each $F(T_i)$ is a nonempty weakly compact subset of C. Therefore, the conclusion follows once we note that $\{F(T_i)\}_{i \in I}$ has the finite intersection property by Lemma 3.6. □.

4 Examples

In this section, we give some concrete examples for our fixed point theorem. At first, we need a lemma.

Lemma 4.1. *Let h and k be two real numbers in* [0, 1]. *Then, the following two statements are true.*

(a) $(h^2 - k^2)^2 - (h - k)^2 \geq 0$, *if* $\frac{h+k}{2} > 0.5$.

(b) $(h^2 - k^2)^2 - (h - k)^2 \leq 0$, *if* $\frac{h+k}{2} \leq 0.5$.

Proof. First, we represent h and k by

$$h = 0.5 + a, \quad \text{where} \ -0.5 \le a \le 0.5,$$

and

$$k = 0.5 + b, \quad \text{where} \ -0.5 \le b \le 0.5.$$

Then, we have

$$(h^2 - k^2)^2 - (h - k)^2 = (a - b)^2(a + b)\ (a + b + 2).$$

If $\frac{h+k}{2} > 0.5$, then $a + b > 0$, and so through the above equation, we obtain that $(h^2 - k^2)^2 - (h - k)^2 \ge 0$. On the other hand, $\frac{h+k}{2} \le 0.5$ implies $a + b \le 0$, and hence, $(h^2 - k^2)^2 - (h - k)^2 \le 0$.

Example 4.2. *Let* $C = \{x \in l^2(\mathbb{N}) : x = (x_1, x_2, \ldots, x_n, \ldots), 0 \le x_i \le 1 - \frac{1}{i+1}\}$ *and* δ *be a positive number so small that* $\sqrt{\delta} < 0.5$. *Define a mapping* $T : C \to C$ *by*

$$Tx = (Tx_1, Tx_2, \ldots, Tx_n, \ldots) : Tx_i = \begin{cases} x_i^2, \ if \sqrt{\delta} < x_i \le 1 - \frac{1}{i+1}; \\ \delta, \ \ if \delta < x_i \le \sqrt{\delta}; \\ x_i, \ \ if 0 \le x_i \le \delta. \end{cases}$$

Then for any $\lambda \in \mathbb{R}$, T *is not* λ*-hybrid. However, for each* $x \in C$, *if we let* $n_x = \min\{n : \sum_{i=n+1}^{\infty} x_i^2 \le \delta^2\}$ *and define* $\lambda : C \to \mathbb{R}$ *by*

$$\lambda(x) = \frac{1}{\left(\frac{1}{n_x+1} - \frac{1}{(n_x+1)^2}\right)^2},$$

then T is point-dependent λ*-hybrid, that is,*

$$||Tx - Ty||^2 \le \ ||x - y||^2 + \lambda(y)\ \langle x - Tx, y - Ty\rangle \tag{9}$$

for all $x, y \in C$. *Therefore, we can apply Theorem* 3.2 *to conclude that T has a fixed point, while the Aoyama-Iemoto-Kohsaka-Takahashi fixed point theorem fails to give us the desired conclusion.*

Proof. Let x and y be two elements from C so that the m^{th} coordinate of x is $1 - \frac{1}{m+1}$ the m^{th} coordinate of y is 0.5 and the rest coordinates of x and y are zero. We have

$$\begin{aligned} & ||Tx - Ty||^2 - ||x - y||^2 - m\ \langle x - Tx, y - Ty\rangle \\ = & \left[\left(1 - \frac{1}{m+1}\right)^2 - (0.5)^2\right]^2 - \left[\left(1 - \frac{1}{m+1}\right) - 0.5\right]^2 \\ & -m\left[\left(1 - \frac{1}{m+1}\right) - \left(1 - \frac{1}{m+1}\right)^2\right][0.5 - (0.5)^2] \\ = & \frac{9}{16} - \frac{2}{m+1} + \frac{9}{2(m+1)^2} - \frac{4}{(m+1)^3} + \frac{1}{(m+1)^4} - \frac{m^2}{4(m+1)^2} \\ & \to \frac{5}{16} \text{ as } m \to \infty. \end{aligned}$$

Since the value of above equality is always positive as m is large enough, we conclude that there is no constant λ to satisfy the inequality:

$$||Tx - Ty||^2 \leq ||x - y||^2 + \lambda \langle x - Tx, y - Ty\rangle$$

for all $x, y \in C$.

It remains to show that T satisfies the inequality (9). We can rewrite the inequality as

$$\sum_{i=1}^{\infty}(Tx_i - Ty_i)^2 \leq \sum_{i=1}^{\infty}(x_i - y_i)^2 + \sum_{i=1}^{\infty}\lambda(y)(x_i - Tx_i)(y_i - Ty_i).$$

Thus, if we can show that for all $i \in \mathbb{N}$,

$$(Tx_i - Ty_i)^2 \leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i), \tag{10}$$

then the assertion follows. We prove inequality (10) holds for all $i \in \mathbb{N}$ by considering the following two cases: (I) $i > \min\{n_x, n_y\}$ and (II) $i \leq \min\{n_x, n_y\}$.

• Case (I). $i > \min\{n_x, n_y\}$.

In this case, at least one of x_i and y_i is less than or equal to δ. Suppose that $0 \leq x_i \leq \delta$. There are three subcases to discuss.

(I-1): If $\sqrt{\delta} < y_i \leq 1 - \frac{1}{i+1}$, then we have

$$\begin{aligned}(Tx_i - Ty_i)^2 = (x_i - y_i^2)^2 &\leq (x_i - y_i)^2 \\ &\leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).\end{aligned}$$

(I-2): $\delta < y_i \leq \sqrt{\delta}$, then we have

$$\begin{aligned}(Tx_i - Ty_i)^2 = (x_i - \delta)^2 &\leq (x_i - y_i)^2 \\ &\leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).\end{aligned}$$

(I-3): If $0 \leq y_i \leq \delta$, then we have

$$(Tx_i - Ty_i)^2 = (x_i - y_i)^2 \leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).$$

The case that $0 \leq y_i \leq \delta$ can be proved in the same manner.

• Case (II). $i \leq \min\{n_x, n_y\}$.

In this case, there are 9 subcases to discuss.

(II-1): $\sqrt{\delta} < x_i \leq 1 - \frac{1}{i+1}$ and $\sqrt{\delta} < y_i \leq 1 - \frac{1}{i+1}$.

If $\frac{x_i+y_i}{2} \leq 0.5$, it follows from Lemma 4.1 that

$$\begin{aligned}(Tx_i - Ty_i)^2 = (x_i^2 - y_i^2)^2 &\leq (x_i - y_i)^2 \\ &\leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).\end{aligned}$$

If $\frac{x_i+y_i}{2} > 0.5$, then both x_i and y_i are greater than $\frac{1}{i+1}$, and so by considering the graph of the function $g(z) = z - z^2$ in $\mathbb{R}$, which is symmetric to the line $L : x = 0.5$, we have

$$(x_i - x_i^2) \geq \left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2 \geq \left(\frac{1}{n_y+1}\right) - \left(\frac{1}{n_y+1}\right)^2$$

and

$$(y_i - y_i^2) \geq \left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2 \geq \left(\frac{1}{n_y+1}\right) - \left(\frac{1}{n_y+1}\right)^2.$$

Consequently, we obtain

$$(Tx_i - Ty_i)^2 = (x_i^2 - y_i^2)^2 \le 1 \le \frac{1}{\left(\frac{1}{n_y+1} - \frac{1}{(n_y+1)^2}\right)^2}(x_i - x_i^2)(y_i - y_i^2)$$
$$\le (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).$$

(II-2): $\delta < x_i \le \sqrt{\delta}$ and $\sqrt{\delta} < y_i \le 1 - \frac{1}{i+1}$.

If $y_i \le 0.5$, then $\frac{x_i+y_i}{2} < 0.5$. Thus, from Lemma 4.1, we have

$$\begin{aligned}(Tx_i - Ty_i)^2 = (\delta - y_i^2)^2 &\le (x_i^2 - y_i^2)^2 \\ &\le (x_i - y_i)^2 \\ &\le (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).\end{aligned}$$

If y_i >0.5, we have either

$$\delta < x_i \le \delta + \left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2$$

or

$$\delta + \left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2 < x_i \le \sqrt{\delta}.$$

When $\delta < x_i \le \delta + (\frac{1}{i+1}) - (\frac{1}{i+1})^2$, by considering the graph of the function $g(z) = z - z^2$ in $\mathbb{R}$, we have

$$y_i - y_i^2 \ge \left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2 \ge x_i - \delta.$$

and thus, we obtain

$$y_i - x_i \ge y_i^2 - \delta > 0.$$

Therefore,

$$\begin{aligned}(Tx_i - Ty_i)^2 &= (\delta - y_i^2)^2 \\ &\le (x_i - y_i)^2 \le (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).\end{aligned}$$

When $\delta + \left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2 < x_i \le \sqrt{\delta}$, both of x_i-δ and $y_i - y_i^2$ are greater than $\left(\frac{1}{i+1}\right) - \left(\frac{1}{i+1}\right)^2$ and thus also greater than $\left(\frac{1}{n_y+1}\right) - \left(\frac{1}{n_y+1}\right)^2$.

Therefore,

$$(Tx_i - Ty_i)^2 = (\delta - y_i^2)^2 \le 1 \le \frac{1}{\left(\frac{1}{n_y+1} - \frac{1}{(n_y+1)^2}\right)^2}(x_i - \delta)(y_i - y_i^2)$$
$$\le (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).$$

Likely, we can prove the case:

(II-3): $\sqrt{\delta} < x_i \le 1 - \frac{1}{i+1}$ and $\delta < y_i \le \sqrt{\delta}$.

(II-4): $0 \le x_i \le \delta$ and $\sqrt{\delta} < y_i \le 1 - \frac{1}{i+1}$.

Then, we have

$$\begin{aligned}(Tx_i - Ty_i)^2 = (x_i - y_i^2)^2 &\leq (x_i - y_i)^2 \\ &\leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).\end{aligned}$$

Similarly, we can prove the case:

(II-5): $\sqrt{\delta} < x_i \leq 1 - \frac{1}{i+1}$ and $0 \leq y_i \leq \delta$.

(II-6): $\delta < x_i \leq \sqrt{\delta}$ and $\delta < y_i \leq \sqrt{\delta}$.

In this case, we have

$$(Tx_i - Ty_i)^2 = (\delta - \delta)^2 = 0 \leq (x_i - y_i)^2 + \lambda(y)(x_i - Tx_i)(y_i - Ty_i).$$

(II-7): $0 \leq x_i \leq \delta$ and $\delta < y_i \leq \sqrt{\delta}$.

This case can be treated as (I-2).

(II-8): $0 \leq x_i \leq \delta$ and $0 \leq y_i \leq \delta$.

This case can be treated as (I-3).

(II-9): $\delta < x_i \leq \sqrt{\delta}$ and $0 \leq y_i \leq \delta$.

This case can be treated as (I-2). □

To end this section, we give another example which shows that the concept of a nonspreading mapping in the sense of (1.2) is generally different from that of a 2-hybrid mapping relative to some D_f in Hilbert spaces.

Example 4.3. *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = x^{10}$ *for all* $x \in \mathbb{R}$, *and define* $T : [0, 0.85] \to [0, 0.85]$ *by* $Tx = x^2$ *for all* $x \in [0, 0.85]$. *Then, T is neither nonexpansive nor non-spreading, but it is* λ*-hybrid relative to* D_f *for any* $\lambda \geq 0$. *Thus, we can apply Theorem 3.2 to conclude T has a fixed point, while both of the Browder Fixed Point Theorem and the Kohsaka-Takahashi fixed point theorem fail.*

Proof. It is easy to check that T is not nonexpansive. As for not nonspreading, taking $x = 0.85$ and $y = 0.5$, we have

$$||Tx - Ty||^2 = (x^2 - y^2)^2 = [(0.85)^2 - (0.5)^2]^2 = 0.22325625$$

while

$$\begin{aligned}&||x - y||^2 + 2\langle x - Tx, y - Ty\rangle \\ &= (x - y)^2 + 2(x - x^2)(y - y^2) \\ &= (0.85 - 0.5)^2 + 2[0.85 - (0.85)^2][0.5 - (0.5)^2] = 0.18625.\end{aligned}$$

Hence, T is not nonspreading in the sense of (1.2). It remains to show that for any $\lambda \geq 0$, T is λ-hybrid relative to D_f. Note at first that, for all $\lambda \geq 0$ and for all $x, y \in [0, 0.85]$,

$$\begin{aligned}&\lambda\langle x - Tx, f'(y) - f'(Ty)\rangle \\ &= \lambda(x - x^2)(10y^9 - 10y^{18}) \geq 0.\end{aligned}$$

Hence, it suffices to prove that T is 0-hybrid relative to D_f, that is, to show that

$$D_f(Tx, Ty) - D_f(x, y) \leq 0, \quad \forall x, y \in [0, 0.85].$$

Fixed any $x \in [0, 0.85]$, let $h(y) = D_f(T_x, T_y) - D_f(x, y)$. Then

$$\begin{aligned} h(y) &= f(Tx) - f(Ty) - \langle Tx - Ty, f'(Ty)\rangle - [f(x) - f(y) - \langle x - y, f'(y)\rangle] \\ &= x^{20} + 9y^{20} - 10x^2y^{18} - x^{10} - 9y^{10} + 10xy^9. \end{aligned}$$

We have

$$\begin{aligned} h'(y) &= 180y^{19} - 180x^2y^{17} - 90y^9 + 90xy^8 \\ &= 90y^8(2y^{11} - 2x^2y^9 - y + x) \\ &= 90y^8[2y^9(y^2 - x^2) - (y - x)] \\ &= 90y^8[2y^9(y + x)(y - x) - (y - x)] \\ &= 90y^8(y - x)[2y^9(y + x) - 1]. \end{aligned}$$

Since y and x are in [0, 0.85], one has

$$2y^9(y + x) - 1 < 2(0.85)^9(0.85 + 0.85) - 1 < 0,$$

and hence

$$h'(y) \begin{cases} \geq 0 & , \text{if } y \leq x; \\ \leq 0 & , \text{if } y > x. \end{cases}$$

Moreover, we know $h(y) = 0$ if $x = y$. Therefore, $h(y)$ is always less than or equal to zero and we have proved that $D_f(Tx, Ty) - D_f(x, y) \leq 0$ for all $x, y \in [0, 0.85]$. □

5 Weak convergence theorems

In this section, we discuss the demiclosedness and the weak convergence problem of point-dependent λ-hybrid relative to D_f. We denote the weak convergence and strong convergence of a sequence $\{x_n\}$ to v in a Banach space by $x_n \rightharpoonup v$ and $x_n \to v$, respectively. For a nonempty closed convex subset C of a Banach space X, a mapping $T : C \to X$ is demiclosed if for any sequence $\{x_n\}$ in C with $x_n \rightharpoonup v$ and $x_n - Tx_n \to 0$, one has $Tv = v$.

We first derive an Opial-like inequality for the Bregman distance. For the Opial's inequality, we refer readers to Lemma 1 of [11].

Lemma 5.1. *Suppose $f : X \to (-\infty,\infty]$ is a proper strictly convex function so that it is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$ in a Banach space X and $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{D}$ such that $x_n \rightharpoonup v$ for some $v \in \mathrm{Int}(\mathcal{D})$. Then*

$$\liminf_{n\to\infty} D_f(x_n, v) < \liminf_{n\to\infty} D_f(x_n, y), \quad \forall y \in \mathrm{Int}(\mathcal{D}) \text{ with } y \neq v.$$

Proof. Since

$$\begin{aligned} & D_f(x_n, v) - D_f(x_n, y) \\ &= f(x_n) - f(v) - \langle x_n - v, f'(v)\rangle - [f(x_n) - f(y) - \langle x_n - y, f'(y)\rangle] \\ &= f(x_n) - f(v) - \langle x_n - v, f'(v)\rangle - f(x_n) + f(y) + \langle x_n - y, f'(y)\rangle] \\ &\quad + \langle x_n - v, f'(y)\rangle - \langle x_n - v, f'(y)\rangle \\ &= -\ [f(v) - f(y) - \langle v - y, f'(y)\rangle] + \langle x_n - v, f'(y) - f'(v)\rangle \\ &= -\ D_f(v, y) + \langle x_n - v, f'(y) - f'(v)\rangle \end{aligned}$$

and $x_n \rightharpoonup v$, we have

$$\lim_{n\to\infty}[D_f(x_n, v) - D_f(x_n, y)] = -D_f(v, y).$$

Consequently,

$$\begin{aligned}\liminf_{n\to\infty} D_f(x_n, v) &= \liminf_{n\to\infty}[(D_f(x_n, v) - D_f(x_n, y)) + D_f(x_n, y)] \\ &= \lim_{n\to\infty}(D_f(x_n, v) - D_f(x_n, y)) + \liminf_{n\to\infty} D_f(x_n, y) \\ &= -D_f(v, y) + \liminf_{n\to\infty} D_f(x_n, y),\end{aligned}$$

and hence in view of $D_f(v, y) > 0$ for $y \neq v$ we obtain

$$\liminf_{n\to\infty} D_f(x_n, v) < \liminf_{n\to\infty} D_f(x_n, y).$$

□

Proposition 5.2. *Let $f : X \to (-\infty,\infty]$ be a strictly convex function so that it is Gâteaux differentiable on $\mathrm{Int}(\mathcal{D})$ and is bounded on bounded subsets of $\mathrm{Int}(\mathcal{D})$. Suppose C is a closed convex subset of $\mathrm{Int}(\mathcal{D})$ and $T : C \to C$ is point-dependent λ-hybrid relative to D_f for some $\lambda : C \to \mathbb{R}$. Then T is demiclosed.*

Proof. Let $\{x_n\}$ be any sequence in C with $x_n \rightharpoonup v$ and $x_n - Tx_n \to 0$. We have to show that $Tv = v$. Since f is bounded on bounded subsets, by Proposition 1.1.11 of [9] there exists a constant $M > 0$ such that

$$\max\{\sup\{||f'(x_n)|| \ : n \in \mathbb{N}\},\ ||\lambda(v)||,\ ||f'(Tv)||,\ ||f'(v)||\} \leq M.$$

Rewrite $D_f(x_n, Tv)$ as

$$\begin{aligned}D_f(x_n, Tv) &= f(x_n) - f(Tv) - \langle x_n - Tv, f'(Tv)\rangle \\ &= f(x_n) + f(Tx_n) - f(Tx_n) - f(Tv) - \langle x_n - Tv, f'(Tv)\rangle \\ &\quad + \langle Tx_n - Tv, f'(Tv)\rangle - \langle Tx_n - Tv, f'(Tv)\rangle \\ &= [f(Tx_n) - f(Tv) - \langle Tx_n - Tv, f'(Tv)\rangle] + f(x_n) - f(Tx_n) \\ &\quad + \langle Tx_n - x_n, f'(Tv)\rangle \\ &= D_f(Tx_n, Tv) + f(x_n) - f(Tx_n) + \langle Tx_n - x_n, f'(Tv)\rangle.\end{aligned} \tag{11}$$

Noting $f(x_n) - f(Tx_n) \leq \langle x_n - Tx_n, f'(x_n)\rangle$ and T is point-dependent λ-hybrid relative to D_f, we have from (11) that

$$\begin{aligned}&D_f(x_n, Tv) \\ &\leq D_f(Tx_n, Tv) + \langle x_n - Tx_n, f'(x_n)\rangle - \langle x_n - Tx_n, f'(Tv)\rangle \\ &\leq D_f(x_n, v) + \lambda(v)\langle x_n - Tx_n, f'(v) - f'(Tv)\rangle + \langle x_n - Tx_n, f'(x_n) - f'(Tv)\rangle \\ &\leq D_f(x_n, v) + [|\lambda(v)|(||f'(v)|| + ||f'(Tv)||) + (||f'(x_n)|| + ||f'(Tv)||)]||x_n - Tx_n|| \\ &\leq D_f(x_n, v) + 2M(M + 1)||x_n - Tx_n||.\end{aligned} \tag{12}$$

If $Tv \neq v$, then Lemma 5.1 and (12) imply that

$$\begin{aligned}&\liminf_{n\to\infty} D_f(x_n,\ v) \\ &< \liminf_{n\to\infty} D_f(x_n,\ Tv) \\ &\leq \liminf_{n\to\infty}[D_f(x_n,\ v) + 2M(M + 1)||x_n - Tx_n||] = \liminf_{n\to\infty} D_f(x_n, v),\end{aligned}$$

a contradiction. This completes the proof. □

A mapping $T : C \to C$ is said to be asymptotically regular if, for any $x \in C$, the sequence $\{T^{n+1}x - T^n x\}$ tends to zero as $n \to \infty$.

Theorem 5.3. *Suppose the following conditions hold:*

(5.3.1) *$f : X \to (-\infty,\infty]$ is l.s.c. uniformly convex function so that it is Gâteaux differentiable on $Int(\mathcal{D})$ and is bounded on bounded subsets of $Int(\mathcal{D})$ in a reflexive Banach space X.*

(5.3.2) *$C \subseteq Int(\mathcal{D})$ is a closed convex subset of X.*

(5.3.3) *$T : C \to C$ is point-dependent λ-hybrid relative to D_f for some $\lambda : C \to \mathbb{R}$ and is asymptotically regular with a bounded sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ for some $x_0 \in C$.*

(5.3.4) *The mapping $x \to f(x)$ for $x \in X$ is weak-to-weak* continuous.*

Then for any $x \in C$, $\{T^n x\}_{n\in\mathbb{N}}$ is weakly convergent to an element $v \in F(T)$.

Proof. Let $v \in F(T)$ and $x \in C$. If $\{T^n x\}_{n\in\mathbb{N}}$ is not bounded, then there is a subsequence $\{T^{n_i}x\}_{i\in\mathbb{N}}$ such that $||v - T^{n_i}x|| \geq 1$ for all $i \in \mathbb{N}$ and $||v - T^{n_i}x|| \to \infty$ as $i \to \infty$. From (5.3.3), for any $n \in \mathbb{N}$, we have

$$\begin{aligned} D_f(v, T^{n+1}x) &= D_f(Tv, T^{n+1}x) \\ &\leq D_f(v, T^n x) + \lambda(T^n x)\,\{v - Tv,\ f'(T^n x) - f'(T^{n+1}x)\} = D_f(v, T^n x) \\ &\leq D_f(v, x), \end{aligned}$$

which in conjunction with (3), (4), and (6) implies that

$$\begin{aligned} D_f(v, x) \geq D_f(v, T^{n_i}x) &\geq \nu_f(T^{n_i}x, ||v - T^{n_i}x||) \\ &\geq ||v - T^{n_i}x||\nu_f(T^{n_i}x, 1) \\ &\geq ||v - T^{n_i}x||\delta_f(1) \to \infty, \quad \text{as } i \to \infty, \end{aligned}$$

a contradiction. Therefore, for any $x \in X$, $\{T^n x\}_{n\in\mathbb{N}}$ is bounded, and so it has a subsequence $\{T^{n_j}x\}_{j\in\mathbb{N}}$ which is weakly convergent to w for some $w \in C$. As $T^{n_j}x - T^{n_j+1}x \to 0$, it follows from the demiclosedness of T that $w \in F(T)$. It remains to show that $T^n x \rightharpoonup w$ as $n \to \infty$. Let $\{T^{n_k}x\}_{n\in\mathbb{N}}$ be any subsequence of $\{T^n x\}_{n\in\mathbb{N}}$ so that $T^{n_k}x \rightharpoonup u$ for some $u \in C$. Then $u \in F(T)$. Since both of $\{D_f(w, T^n x)\}_{n\in\mathbb{N}}$ and $\{D_f(u, T^n x)\}_{n\in\mathbb{N}}$ are decreasing, we have

$$\lim_{n\to\infty}[D_f(w, T^n x) - D_f(u, T^n x)] = \lim_{n\to\infty}[f(w) - f(u) - \langle w - u,\ f'(T^n x)\rangle] = a$$

for some $a \in \mathbb{R}$. Particularly, from (5.3.4) we obtain

$$a = \lim_{n_j\to\infty}[f(w) - f(u) - \langle w - u, f'(T^{n_j}x)\rangle] = f(w) - f(u) - \langle w - u, f'(w)\rangle$$

and

$$a = \lim_{n_k\to\infty}[f(w) - f(u) - \langle w - u, f'(T^{n_k}x)\rangle] = f(w) - f(u) - \langle w - u, f'(u)\rangle.$$

Consequently, $\langle w - u, f'(w) - f'(u)\rangle = 0$, and hence $w = u$ by the strict convexity of f. This shows that $T^n x \rightharpoonup w$ for some $w \in F(T)$.□

Adopting the technique of [8], we have the following ergodic theorem for point-dependent λ-hybrid mappings in Hilbert spaces.

Theorem 5.4. *Suppose*

(5.4.1) *C is nonempty closed convex subset of a Hilbert space H.*

(5.4.2) *$T : C \to C$ is a point-dependent λ-hybrid mapping for some function $\lambda : C \to \mathbb{R}$, that is,*

$$||Tx - Ty||^2 \leq ||x - y||^2 + \lambda(y)\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

(5.4.3) $F(T) \neq \emptyset$.

Then for any $x \in C$, the sequence $\{S_n(x)\}_{n \in \mathbb{N}}$ defined by

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to some point $v \in F(T)$.

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