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Ergodicity of the implicit midpoint rule for nonexpansive mappings

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Abstract
We prove a mean ergodic theorem for the implicit midpoint rule for nonexpansive mappings in a Hilbert space. We obtain weak convergence for the general case and strong convergence for certain special cases.
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1 Introduction

The first mean ergodic theorem for nonlinear noncompact operators was proved by Baillon [1]. Let C be a closed convex subset of a Hilbert space H and let $T: C \to C$ be a nonexpansive mapping (*i.e.*, $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$) with fixed points. Then, for each $x \in C$, the Cesàro means

$$S_n(x) := \frac{1}{n}\sum_{k=0}^{n-1} T^k x, \quad n \ge 1,$$

converge weakly to a fixed point of T. This mean ergodic theorem was extended by Bruck [2] to the setting of Banach spaces that are uniformly convex and have a Fréchet differentiable norm. Baillon and Clement [3] also investigated ergodicity of the nonlinear Volterra integral equations in Hilbert spaces.

It is quite natural to consider ergodic convergence of iterative algorithms in the case where the sequences generated by the algorithms either are not guaranteed to converge or not convergent at all. For instance, the double-backward method of Passty [4] generates a sequence $\{x_n\}$ in the recursive manner:

$$x_{n+1} = \left(J_B^{\lambda_{n+1}} \circ J_A^{\lambda_{n+1}}\right)x_n, \quad n \ge 0, \tag{1.1}$$

where A and B are maximal monotone operators in a Hilbert space such that $A + B$ is also maximal monotone and the inclusion $0 \in (A + B)x$ is solvable, and J_A^λ and J_B^λ are the resolvents of A and B, respectively, that is, $J_A^\lambda = (I + \lambda A)^{-1}$ and $J_B^\lambda = (I + \lambda B)^{-1}$. It is well known [5] that the sequence $\{x_n\}$ generated by the double-backward method (1.1) fails to converge weakly, in general. However, Passty [4] showed that if the sequence of parameters, $\{\lambda_n\}$, is in $\ell^2 \setminus \ell^1$, then the averages

$$z_n := \frac{\sum_{i=1}^n \lambda_i x_i}{\sum_{i=1}^n \lambda_i}, \quad n = 1, 2, \ldots, \tag{1.2}$$

converge weakly to a solution to the inclusion $0 \in (A + B)x$.

The implicit midpoint rule (IMR) for nonexpansive mappings in a Hilbert space H, inspired by the IMR for ordinary differential equations [6–12], was introduced in [13]. This rule generates a sequence $\{x_n\}$ via the semi-implicit procedure:

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{1.3}$$

where the initial guess $x_0 \in C$ is arbitrarily chosen, $t_n \in (0,1)$ for all n, and $T : C \to C$ is a nonexpansive mapping with fixed points.

The IMR (1.3) is proved to converge weakly [13] in the Hilbert space setting provided the sequence $\{t_n\}$ satisfies the two conditions:

(C1) $t_{n+1}^2 \leq a t_n$ for all $n \geq 0$ and some $a > 0$, and
(C2) $\liminf_{n\to\infty} t_n > 0$.

However, this algorithm may fail to converge weakly without the assumption (C2). We therefore turn our attention to the ergodic convergence of the algorithm. We will show that for any sequence $\{t_n\}$ in the interval $(0,1)$, the mean averages $\{z_n\}$ as defined by (1.2) will always converge weakly to a fixed point of T as long as $\{x_n\}$ is an approximate fixed point of T (*i.e.*, $\|x_n - Tx_n\| \to 0$). We will also show that under certain additional conditions the means $\{z_n\}$ can converge in norm to a fixed point of T. This paper is organized as follows. In the next section we introduce the concept of nearest point projections and properties of nonexpansive mappings. The main results of this paper (*i.e.*, weak and strong ergodicity of the IMR (1.3)) are presented in Section 3.

2 Preliminaries

Let C be a nonempty closed convex subset of a Hilbert space H. Recall that the nearest point projection from H to C, P_C, is defined by

$$P_C x := \arg\min_{y\in C} \|x - y\|^2, \quad x \in H. \tag{2.1}$$

We need the following characterization of projections.

Lemma 2.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if any one of the following properties is satisfied*:

(i) $\|x - z\| \leq \|x - y\|$ *for all* $y \in C$;
(ii) $\langle x - z, y - z\rangle \leq 0$ *for all* $y \in C$;
(iii) $\|x - z\|^2 \leq \|x - y\|^2 - \|z - y\|^2$ *for all* $y \in C$.

Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in C.$$

A point $x \in C$ such that $Tx = x$ is said to be a fixed point of T. The set of all fixed points of T is denoted by $\mathrm{Fix}(T)$, namely,

$$\mathrm{Fix}(T) = \{x \in C : Tx = x\}.$$

In the rest of this paper we always assume $\mathrm{Fix}(T) \neq \emptyset$.

We need the demiclosedness principle of nonexpansive mappings as described below.

Lemma 2.2 [14] *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then the mapping $I - T$ is demiclosed in the sense that, for any sequence $\{x_n\}$ of C, the following implication holds*:

$$x_n \to x \text{ weakly and } (I - T)x_n \to 0 \text{ in norm} \quad \Longrightarrow \quad (I - T)x = 0.$$

Next we need the following lemma (not hard to prove).

Lemma 2.3 [15] *For each integer $n \geq 2$, $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ such that $\lambda_1 + \cdots + \lambda_n = 1$, points $u_1, \dots, u_n \in C$, and any nonexpansive mapping $T : C \to C$, we have*

$$\left\| T\left(\sum_{j=1}^{n} \lambda_j u_j\right) - \sum_{j=1}^{n} \lambda_j T u_j \right\|^2 \leq \sum_{i<j} \lambda_i \lambda_j \big(\|u_i - u_j\|^2 - \|Tu_i - Tu_j\|^2\big). \tag{2.2}$$

Recall also that the implicit midpoint rule (IMR) [13] generates a sequence $\{x_n\}$ by the recursion process

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0, \tag{2.3}$$

where $t_n \in (0,1)$ for all n, and $T : C \to C$ is a nonexpansive mapping.

The following properties of the IMR (2.3) are proved in [13].

Lemma 2.4 *Let $\{t_n\}$ be any sequence in* $(0,1)$ *and let $\{x_n\}$ be the sequence generated by the IMR* (2.3). *Then*

(i) $\|x_{n+1} - p\| \leq \|x_n - p\|$ *for all $n \geq 0$ and $p \in \mathrm{Fix}(T)$. In particular, $\{x_n\}$ is bounded, and moreover, we have*

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists for every } p \in \mathrm{Fix}(T). \tag{2.4}$$

(ii) $\sum_{n=1}^{\infty} t_n \|x_n - x_{n+1}\|^2 < \infty$.

(iii) $\sum_{n=1}^{\infty} t_n(1 - t_n)\|x_n - T(\frac{x_n + x_{n+1}}{2})\|^2 < \infty$.

The convergence of the IMR (2.3) is proved in [13].

Theorem 2.5 *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Assume $\{x_n\}$ is generated by the IMR* (2.3) *where the sequence* $\{t_n\}$ *of parameters satisfies the conditions* (C1) *and* (C2) *in the Introduction. Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

3 Ergodicity

In this section we discuss the ergodic convergence of the sequence $\{x_n\}$ generated by the IMR (2.3), that is, the convergence of the means

$$z_n := \frac{1}{s_n} \sum_{k=1}^{n} a_k x_k, \quad n = 1, 2, \ldots, \tag{3.1}$$

where $\{a_n\}$ is a sequence of positive numbers such that

$$s_n := \sum_{k=1}^{n} a_k \to \infty \quad (\text{as } n \to \infty). \tag{3.2}$$

Set $F = \mathrm{Fix}(T)$ and let P_F be the nearest point projection from H to F.

Lemma 3.1 *The sequence* $\{P_F x_n\}$ *is convergent in norm.*

Proof First observe that

$$\lim_{n\to\infty} \|x_n - P_F x_n\| \text{ exists.} \tag{3.3}$$

As a matter of fact, we get for $n > m$, by Lemma 2.1(i) and Lemma 2.4(i),

$$\|x_n - P_F x_n\| \le \|x_n - P_F x_m\| \le \|x_m - P_F x_m\|.$$

That is, $\{\|x_n - P_F x_n\|\}$ is decreasing and (3.3) is proven.

Applying the inequality (Lemma 2.1(iii))

$$\|P_F v - u\|^2 \le \|v - u\|^2 - \|P_F v - v\|^2, \quad v \in H, u \in F \tag{3.4}$$

to the case where $v = x_n$ and $u = P_F x_m$ (with $n > m$) together with Lemma 2.4(i), we get

$$\begin{aligned} \|P_F x_n - P_F x_m\|^2 &\le \|x_n - P_F x_m\|^2 - \|P_F x_n - x_n\|^2 \\ &\le \|x_m - P_F x_m\|^2 - \|P_F x_n - x_n\|^2. \end{aligned}$$

The strong convergence of $\{P_F x_n\}$ follows immediately from the fact (3.3). □

Remark 3.2 The limit of $\{P_F x_n\}$, which we denote by $\hat{p}$, can also be identified as the asymptotic center of the sequence $\{x_n\}$ with respect to the fixed point set F of T. In other words,

$$\hat{p} = \arg\min_{x\in F} f(p) := \limsup_{n\to\infty} \|x_n - p\|^2. \tag{3.5}$$

As a matter of fact, by (3.4) we get, for any $p \in F$,

$$\|P_F x_n - x_n\|^2 \le \|x_n - p\|^2 - \|P_F x_n - p\|^2.$$

Upon taking limsup we immediately obtain

$$f(\hat{p}) \le f(p) - \|p - \hat{p}\|^2, \quad p \in F.$$

Hence, (3.5) holds.

Theorem 3.3 *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F \equiv \mathrm{Fix}(T) \neq \emptyset$. Assume $\{t_n\}$ is any sequence of positive numbers in the unit interval* $(0,1)$ *and let $\{x_n\}$ be the sequence generated by the IMR* (2.3). *Define the means $\{z_n\}$ by* (3.1), *where the weights $\{a_n\}$ are all positive and satisfy the condition* (3.2). *Assume, in addition,* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. *Then $\{z_n\}$ converges weakly to a point z, where $z = \lim_{n\to\infty} P_F x_n$ (in norm).*

Proof Let $z = \lim_{n\to\infty} P_F x_n$ which is well defined by Lemma 3.1. By Lemma 2.1(ii), we have, for each k,

$$\langle x_k - P_F x_k, u - P_F x_k \rangle \le 0, \quad u \in F.$$

It turns out that, for $u \in F$,

$$\langle x_k - P_F x_k, u - z \rangle \le -\langle x_k - P_F x_k, z - P_F x_k \rangle \le M\|z - P_F x_k\|.$$

(Here M is a constant such that $M \ge \|x_k - P_F x_k\|$ for all k.)

By multiplying by a_k and then summing up from $k = 1$ to n, we conclude

$$\left\langle z_n - \frac{1}{s_n}\sum_{k=1}^{n} a_k P_F x_k, u - z \right\rangle \le \frac{M}{s_n}\sum_{k=1}^{n} a_k \|z - P_F x_k\|. \tag{3.6}$$

We now claim that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.7}$$

Consequently, by Lemma 2.2, each weak cluster point of $\{z_n\}$ falls in F.

To see (3.7), we will prove that

$$\|Tz_n - z_n\| < \delta(\varepsilon) \tag{3.8}$$

for all n big enough, where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. For the sake of simplicity, we may, due to the assumption $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, assume that

$$\|x_n - Tx_n\| < \varepsilon \tag{3.9}$$

for all n.

Let $h(t) = \sqrt{t}$ for $t \geq 0$ and let M be a constant such that $M \geq 2 \cdot \operatorname{diam}(x_n)$. For each n, we put $\lambda_j^{(n)} = \frac{a_j}{s_n}$ for $1 \leq j \leq n$ and apply (2.2) to get

$$\begin{aligned}\left\| T(z_n) - \sum_{j=1}^{n} \lambda_j^{(n)} Tx_j \right\| &\leq h\left(M \sum_{i<j} \lambda_i^{(n)} \lambda_j^{(n)} \left(\|x_i - x_j\| - \|Tx_i - Tx_j\| \right) \right) \\ &\leq h\left(M \sum_{i<j} \lambda_i^{(n)} \lambda_j^{(n)} \left(\|x_i - Tx_i\| + \|x_j - Tx_j\| \right) \right) \\ &\leq h\left(2\varepsilon M \sum_{i<j} \lambda_i^{(n)} \lambda_j^{(n)} \right) \\ &\leq h(\varepsilon M). \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we derive that

$$\begin{aligned}\|T(z_n) - z_n\| &\leq \left\| T(z_n) - \sum_{j=1}^{n} \lambda_j^{(n)} Tx_j \right\| + \left\| \sum_{j=1}^{n} \lambda_j^{(n)} (Tx_j - x_j) \right\| \\ &\leq h(\varepsilon M) + \sum_{j=1}^{n} \lambda_j^{(n)} \|Tx_j - x_j\| \\ &\leq h(\varepsilon M) + \varepsilon. \end{aligned} \tag{3.11}$$

It turns out that (3.8) with $\delta(\varepsilon) = \sqrt{\varepsilon M} + \varepsilon$.

Now since $P_F x_n \to z$ in norm, we see that the means $\frac{1}{s_n} \sum_{k=1}^{n} a_k P_F x_k \to z$ in norm, as well. Consequently, if $\{z_{n_j}\}$ is a subsequence weakly converging to some point z^*, it follows from (3.6) that

$$\langle z^* - z, u - z \rangle \leq 0, \quad u \in F. \tag{3.12}$$

This together with the fact that $z^* \in F$ implies that $z = P_F z^* = z^*$. That is, z is the only weak cluster point of the sequence $\{z_n\}$ and therefore, we must have $z_n \to z$ weakly. □

Remark 3.4 In Theorem 3.3 we assumed that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. This assumption is guaranteed if the sequence $\{t_n\}$ satisfies the condition (C2) in the Introduction, that is, $\liminf_{n\to\infty} t_n > 0$. Indeed, by (C2) and Lemma 2.4(ii), we find

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}$$

Since the definition of IMR (2.3) yields

$$\|x_{n+1} - x_n\| = t_n \left\| x_n - T\left(\frac{x_n + x_{n+1}}{2} \right) \right\|,$$

we also have

$$\lim_{n\to\infty} \left\| x_n - T\left(\frac{x_n + x_{n+1}}{2} \right) \right\| = 0. \tag{3.14}$$

Combining (3.13) and (3.14), we infer that

$$\begin{aligned}\|x_n - Tx_n\| &\le \left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \left\|Tx_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| \\ &\le \left\|x_n - T\left(\frac{x_n + x_{n+1}}{2}\right)\right\| + \frac{1}{2}\|x_n - x_{n+1}\| \to 0.\end{aligned}$$

Remark 3.5 If we assume (3.9) holds for all $n > N$, then we need some more delicate technicalities dealing with (3.10). We may proceed as follows. Decompose z_n (for $n > N$) as

$$z_n = \frac{s_N}{s_n} z_N + \frac{s_n^N}{s_n} z_n^N,$$

where

$$s_n^N = \sum_{j=N+1}^{n} a_j, \qquad z_n^N = \sum_{j=N+1}^{n} \lambda_j^{(n,N)} x_j, \qquad \lambda_j^{(n,N)} = \frac{a_j}{s_n^N}, \quad n > N.$$

As $s_n \to \infty$, we may assume $\frac{s_N}{s_n}\|z_N\| < \varepsilon$. Repeating the argument for (3.10) and (3.11), we get

$$\|T(z_n^N) - z_n^N\| \le h(\varepsilon M) + \varepsilon.$$

Let $M_1 = \sup_{n\ge 0}\|x_n\|$. We finally obtain, for $n > N$,

$$\begin{aligned}\|T(z_n) - z_n\| &\le \|T(z_n) - T(z_n^N)\| + \|T(z_n^N) - z_n^N\| + \|z_n^N - z_n\| \\ &\le \|T(z_n^N) - z_n^N\| + 2\|z_n^N - z_n\| \\ &= \|T(z_n^N) - z_n^N\| + 2\frac{s_N}{s_n}\|z_N - z_n^N\| \\ &\le h(\varepsilon M) + \varepsilon + 4M_1\varepsilon.\end{aligned}$$

Next we show that in some circumstances, the sequence $\{z_n\}$ can converge strongly.

Theorem 3.6 *Let the assumptions of Theorem* 3.3 *holds. Then the sequence* $\{z_n\}$ *converges in norm to the point* $z = \lim_{n\to\infty} P_F x_n$ *if, in addition, any one of the following conditions is satisfied*:

(i) *The fixed point set F of T has nonempty interior.*

(ii) *T is a contraction, that is,*

$$\|Tx - Ty\| \le \rho\|x - y\|, \quad x, y \in C,$$

where $\rho \in [0,1)$ is a constant. In this case, the sequence $\{x_n\}$ generated by the IMR (2.3) *converges in norm to the unique fixed point of T.*

(iii) *T is compact, namely, T maps bounded sets to relatively norm-compact sets.*

Proof (i) By assumption, we have $x_0 \in F$ and $\delta > 0$ such that

- $x_0 + \delta w \in F$ for all $w \in H$ such that $\|w\| \le 1$.

Therefore, upon substituting $x_0 + \delta w$ for u in (3.6) we obtain

$$\begin{aligned}\delta\left\langle z_n - \frac{1}{s_n}\sum_{k=1}^{n} a_k P_F x_k, w\right\rangle \\ \le -\left\langle z_n - \frac{1}{s_n}\sum_{k=1}^{n} a_k P_F x_k, x_0 - z\right\rangle + \frac{M}{s_n}\sum_{k=1}^{n} a_k \|z - P_F x_k\|, \end{aligned} \tag{3.15}$$

for all $w \in H$ such that $\|w\| \le 1$.

Taking the supremum in (3.15) over $w \in H$ such that $\|w\| \le 1$ immediately yields

$$\begin{aligned}\delta\left\| z_n - \frac{1}{s_n}\sum_{k=1}^{n} a_k P_F x_k\right\| \\ \le -\left\langle z_n - \frac{1}{s_n}\sum_{k=1}^{n} a_k P_F x_k, x_0 - z\right\rangle + \frac{M}{s_n}\sum_{k=1}^{n} a_k \|z - P_F x_k\| \to 0.\end{aligned}$$

This verifies that $z_n \to z$ in norm.

(ii) Since T is a contraction, T has a unique fixed point which is denoted by p. By (2.3) we deduce that (noticing $\|x_{n+1} - p\| \le \|x_n - p\|$)

$$\begin{aligned}\|x_{n+1} - p\| &\le (1 - t_n)\|x_n - p\| + t_n\rho\left\|\frac{1}{2}(x_n + x_{n+1}) - p\right\| \\ &\le (1 - t_n)\|x_n - p\| + \frac{1}{2}\rho t_n\bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr) \\ &\le \bigl(1 - (1 - \rho)t_n\bigr)\|x_n - p\|.\end{aligned}$$

It turns out that

$$(1 - \rho)t_n\|x_n - p\| \le \|x_n - p\| - \|x_{n+1} - p\|$$

and hence

$$\sum_{n=1}^{\infty} t_n\|x_n - p\| < \infty.$$

Since $\sum_{n=1}^{\infty} t_n = \infty$, we must have $\liminf_{n\to\infty} \|x_n - p\| = 0$. However, since the sequence $\{\|x_n - p\|\}$ is decreasing, we must have $\lim_{n\to\infty} \|x_n - p\| = 0$. Namely, $x_n \to p$ in norm, and so $z_n \to p$ in norm.

(iii) Since T is compact and since $\{z_n\}$ is weakly convergent, $\{Tz_n\}$ is relatively norm-compact. This together with (3.7) evidently implies that $\{z_n\}$ is relatively norm-compact. Therefore, $\{z_n\}$ must converge in norm to $z = \lim_{n\to\infty} P_F x_n$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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