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A more accurate half-discrete reverse Hilbert-type inequality with a non-homogeneous kernel

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Abstract

By means of weight functions and the improved Euler-Maclaurin summation formula, a more accurate half-discrete reverse Hilbert-type inequality with the kernel $\frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}$ and a best constant factor is given. Some equivalent forms, the dual forms as well as some related homogeneous cases, are also considered.

MSC: 26D15

Keywords: Hilbert-type inequality; weight function; equivalent form; reverse; dual form

1 Introduction

Assuming that $f,g \in L^2(R_+)$, $\|f\| = \{\int_0^\infty f^2(x)\,dx\}^{\frac{1}{2}} > 0$, $\|g\| > 0$, we have the following Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \|f\|\|g\|, \tag{1}$$

where the constant factor π is best possible. If $a = \{a_n\}_{n=1}^\infty$, $b = \{b_n\}_{n=1}^\infty \in l^2$, $\|a\| = \{\sum_{n=1}^\infty a_n^2\}^{\frac{1}{2}} > 0$, $\|b\| > 0$, then we have the following analogous discrete Hilbert inequality:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \pi \|a\|\|b\|, \tag{2}$$

with the same best constant factor π. Inequalities (1) and (2) are important in analysis and its applications (*cf.* [2–4]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [5] gave an extension of (1). For generalizing the results from [5], Yang [6] gave some best extensions of (1) and (2): If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$ satisfying $k(\lambda_1) = \int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt \in R_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(x) = x^{q(1-\lambda_2)-1}$, $f\ (\geq 0) \in L_{p,\phi}(R_+) = \{f \mid \|f\|_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p\,dx\}^{\frac{1}{p}} < \infty\}$, $g\ (\geq 0) \in L_{q,\psi}(R_+)$, and $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is best possible. Moreover if the value of $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}$ ($k_\lambda(x,y)y^{\lambda_2-1}$) is decreasing for $x > 0$ ($y > 0$), then for $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l_{p,\phi} = \{a | \|a\|_{p,\phi} := \{\sum_{n=1}^\infty \phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\}$, and $b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{4}$$

where the constant $k(\lambda_1)$ is still best value. Clearly, for $p = q = 2$, $\lambda = 1$, $k_1(x,y) = \frac{1}{x+y}$, $\lambda_1 = \lambda_2 = \frac{1}{2}$, (3) reduces to (1), while (4) reduces to (2). The reverses of (3) and (4) as well as the equivalent forms are also considered by [6].

Some other results about integral and discrete Hilbert-type inequalities can be found in [7–15]. On half-discrete Hilbert-type inequalities with the general non-homogeneous kernels, Hardy *et al.* provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are best possible. In 2005, Yang [16] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is best possible. Very recently, Yang [17] and [18] gave the following half-discrete reverse Hilbert inequality with best constant factor: For $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda_1 > 0$, $0 < \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $\theta_\lambda(x) = O(\frac{1}{x^{\lambda_2}}) \in (0,1)$, $\widetilde{\phi}(x) = (1 - \theta_\lambda(x))x^{p(1-\lambda_1)-1}$,

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda}\,dx > B(\lambda_1,\lambda_2)\|f\|_{p,\widetilde{\phi}}\|a\|_{q,\psi}. \tag{5}$$

In this paper, by means of weight functions and the improved Euler-Maclaurin summation formula, a more accurate half-discrete reverse Hilbert-type inequality with the kernel $\frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}$ similar to (5) and a best constant factor is given. Moreover, some equivalent forms, the dual forms as well as some relating homogeneous cases are also considered.

2 Some lemmas

Lemma 1 *If $n_0 \in \mathbf{N}$, $s > n_0$, $g_1(y)$ ($y \in [n_0, s)$), $g_2(y)$ ($y \in [s, \infty)$) are continuous decreasing functions satisfying $g_1(n_0) - g_1(s-0) + g_2(s) > 0$, $g_2(\infty) = 0$, define a function $g(y)$ as follows*:

$$g(y) := \begin{cases} g_1(y), & y \in [n_0, s), \\ g_2(y), & y \in [s, \infty). \end{cases} \tag{6}$$

Then there exists $\varepsilon \in [0,1]$, such that

$$\begin{aligned} &\frac{-1}{8}\bigl[g_1(n_0) + \varepsilon\bigl(g_2(s) - g_1(s-0)\bigr)\bigr] \\ &\quad < \int_{n_0}^\infty \rho(y)g(y)\,dy < \frac{1-\varepsilon}{8}\bigl(g_2(s) - g_1(s-0)\bigr), \end{aligned} \tag{7}$$

where $\rho(y) = y - [y] - \frac{1}{2}$ is the Bernoulli function of the first order. In particular, for $g_1(y) = 0$, $y \in [n_0, s)$, we have $g_2(s) > 0$ and

$$\frac{-1}{8}g_2(s) < \int_s^\infty \rho(y)g(y)\,dy < \frac{1}{8}g_2(s); \tag{8}$$

for $g_2(y) = 0$, $y \in [s,\infty)$, *if* $g_1(s-0) \geq 0$, *then it follows* $g_1(n_0) > 0$ *and*

$$\frac{-1}{8} g_1(n_0) < \int_{n_0}^{s} \rho(y) g_1(y)\,dy < 0. \tag{9}$$

Proof Define a continuous decreasing function $\widetilde{g}(y)$ as follows:

$$\widetilde{g}(y) := \begin{cases} g_1(y) + g_2(s) - g_1(s-0), & y \in [n_0, s), \\ g_2(y), & y \in [s, \infty). \end{cases}$$

Then it follows that

$$\begin{aligned}
\int_{n_0}^{\infty} \rho(y) g(y)\,dy &= \int_{n_0}^{s} \rho(y) g(y)\,dy + \int_{s}^{\infty} \rho(y) g(y)\,dy \\
&= \int_{n_0}^{s} \rho(y)\big(\widetilde{g}(y) - g_2(s) + g_1(s-0)\big)\,dy + \int_{s}^{\infty} \rho(y)\widetilde{g}(y)\,dy \\
&= \int_{n_0}^{\infty} \rho(y)\widetilde{g}(y)\,dy - \big(g_2(s) - g_1(s-0)\big)\int_{n_0}^{s} \rho(y)\,dy,
\end{aligned}$$

$$\begin{aligned}
\int_{n_0}^{s} \rho(y)\,dy &= \int_{n_0}^{[s]} \rho(y)\,dy + \int_{[s]}^{s} \rho(y)\,dy = \int_{[s]}^{s} \left(y - [s] - \frac{1}{2}\right) dy \\
&= \frac{1}{8}\left[4\left(s - [s] - \frac{1}{2}\right)^2 - 1\right] = \frac{\varepsilon - 1}{8} \quad \big(\varepsilon \in [0,1]\big).
\end{aligned}$$

Since $\widetilde{g}(n_0) = g_1(n_0) + g_2(s) - g_1(s-0) > 0$, $\widetilde{g}(y)$ is a non-constant continuous decreasing function with $\widetilde{g}(\infty) = g_2(\infty) = 0$, by the improved Euler-Maclaurin summation formula (*cf.* [6], Theorem 2.2.2), it follows that

$$\frac{-1}{8}\big(g_1(n_0) + g_2(s) - g_1(s-0)\big) = \frac{-1}{8}\widetilde{g}(n_0) < \int_{n_0}^{\infty} \rho(y)\widetilde{g}(y)\,dy < 0,$$

and then in view of the above results and by simple calculation, we have (7). □

Lemma 2 *If* $0 < \alpha + \beta \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\alpha+\beta}{8}(1 + \sqrt{3 + \frac{4}{\alpha+\beta}})$, *and* $\omega(n)$ *and* $\varpi(x)$ *are weight functions given by*

$$\begin{aligned}
\omega(n) &:= \int_{\gamma}^{\infty} \frac{(\min\{1, (x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1, (x-\gamma)(n-\eta)\})^{\alpha}} \frac{(n-\eta)^{\frac{\alpha-\beta}{2}}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}\,dx, \quad n \in \mathbf{N}, \\
\varpi(x) &:= \sum_{n=1}^{\infty} \frac{(\min\{1, (x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1, (x-\gamma)(n-\eta)\})^{\alpha}} \frac{(x-\gamma)^{\frac{\alpha-\beta}{2}}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}, \quad x > \gamma,
\end{aligned} \tag{10}$$

then we have

$$0 < \frac{4}{\alpha+\beta}\big(1 - \theta(x)\big) < \varpi(x) < \omega(n) = \frac{4}{\alpha+\beta}, \tag{11}$$

$$\theta(x) = \begin{cases} \frac{1}{2}(1-\eta)^{\frac{\alpha+\beta}{2}}(x-\gamma)^{\frac{\alpha+\beta}{2}}, & 0 < x - \gamma \leq \frac{1}{1-\eta}, \\ 1 - \frac{1}{2}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}}, & x - \gamma > \frac{1}{1-\eta}. \end{cases} \tag{12}$$

Proof Substituting $t = (x-\gamma)(n-\eta)$ in (10), and by simple calculation, we have

$$\omega(n) = \int_0^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\frac{\alpha-\beta}{2}-1}\,dt = \int_0^1 t^{\beta+\frac{\alpha-\beta}{2}-1}\,dt + \int_1^\infty t^{-\alpha+\frac{\alpha-\beta}{2}-1}\,dt = \frac{4}{\alpha+\beta}.$$

For fixed $x > \gamma$, we find

$$\begin{aligned}
h(x,y) &:= (x-\gamma)^{\frac{\alpha-\beta}{2}} \frac{(\min\{1,(x-\gamma)(y-\eta)\})^\beta}{(\max\{1,(x-\gamma)(y-\eta)\})^\alpha}(y-\eta)^{\frac{\alpha-\beta}{2}-1} \\
&= \begin{cases} (x-\gamma)^{\frac{\alpha+\beta}{2}}(y-\eta)^{\frac{\alpha+\beta}{2}-1}, & \eta < y < \eta + \frac{1}{x-\gamma}, \\ (x-\gamma)^{-\frac{\alpha+\beta}{2}}(y-\eta)^{-\frac{\alpha+\beta}{2}-1}, & y \geq \eta + \frac{1}{x-\gamma}, \end{cases} \\
h'_y(x,y) &= \begin{cases} -(1-\frac{\alpha+\beta}{2})(x-\gamma)^{\frac{\alpha+\beta}{2}}(y-\eta)^{\frac{\alpha+\beta}{2}-2}, & \eta < y < \eta + \frac{1}{x-\gamma}, \\ -(\frac{\alpha+\beta}{2}+1)(x-\gamma)^{-\frac{\alpha+\beta}{2}}(y-\eta)^{-\frac{\alpha+\beta}{2}-2}, & y \geq \eta + \frac{1}{x-\gamma}, \end{cases} \\
\int_\eta^\infty h(x,y)\,dy &\overset{t=(x-\gamma)(y-\eta)}{=} \int_0^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\frac{\alpha-\beta}{2}-1}\,dt = \frac{4}{\alpha+\beta}.
\end{aligned}$$

By the Euler-Maclaurin summation formula (*cf.* [6]), it follows that

$$\begin{aligned}
\varpi(x) &= \sum_{n=1}^\infty h(x,n) = \int_1^\infty h(x,y)\,dy + \frac{1}{2}h(x,1) + \int_1^\infty \rho(y)h'_y(x,y)\,dy \\
&= \int_\eta^\infty h(x,y)\,dy - R(x) = \frac{4}{\alpha+\beta} - R(x), \\
R(x) &:= \int_\eta^1 h(x,y)\,dy - \frac{1}{2}h(x,1) - \int_1^\infty \rho(y)h'_y(x,y)\,dy. \qquad (13)
\end{aligned}$$

(i) For $0 < x-\gamma < \frac{1}{1-\eta}$, we obtain $-\frac{1}{2}h(x,1) = -\frac{1}{2}(x-\gamma)^{\frac{\alpha+\beta}{2}}(1-\eta)^{\frac{\alpha+\beta}{2}-1}$, and

$$\int_\eta^1 h(x,y)\,dy = (x-\gamma)^{\frac{\alpha+\beta}{2}} \int_\eta^1 (y-\eta)^{\frac{\alpha+\beta}{2}-1}\,dy = \frac{2(1-\eta)^{\frac{\alpha+\beta}{2}}}{\alpha+\beta}(x-\gamma)^{\frac{\alpha+\beta}{2}}.$$

Setting $g(y) := -h'_y(x,y)$, wherefrom $g_1(y) = (1-\frac{\alpha+\beta}{2})(x-\gamma)^{\frac{\alpha+\beta}{2}}(y-\eta)^{\frac{\alpha+\beta}{2}-2}$, $g_2(y) = (\frac{\alpha+\beta}{2}+1)(x-\gamma)^{-\frac{\alpha+\beta}{2}}(y-\eta)^{-\frac{\alpha+\beta}{2}-2}$ and

$$\begin{aligned}
&g_2\left(\eta + \frac{1}{x-\gamma}\right) - g_1\left(\left(\eta + \frac{1}{x-\gamma}\right) - 0\right) \\
&\quad = \left(\frac{\alpha+\beta}{2} + 1\right)(x-\gamma)^2 - \left(1 - \frac{\alpha+\beta}{2}\right)(x-\gamma)^2 \\
&\quad = (\alpha+\beta)(x-\gamma)^2 > 0,
\end{aligned}$$

then by (7), we find

$$\begin{aligned}
-\int_1^\infty \rho(y)h'_y(x,y)\,dy &= \int_1^\infty \rho(y)g(y)\,dy \\
&> \frac{-1}{8}\left[g_1(1) + g_2\left(\eta + \frac{1}{x-\gamma}\right) - g_1\left(\left(\eta + \frac{1}{x-\gamma}\right) - 0\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-1}{8}\left[\left(1-\frac{\alpha+\beta}{2}\right)(x-\gamma)^{\frac{\alpha+\beta}{2}}(1-\eta)^{\frac{\alpha+\beta}{2}-2}+(\alpha+\beta)(x-\gamma)^2\right] \\
&> \frac{-1}{8}\left[\left(1-\frac{\alpha+\beta}{2}\right)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}}\right. \\
&\quad \left.+(\alpha+\beta)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^2\right] \\
&= \frac{-1}{8}\left(1+\frac{\alpha+\beta}{2}\right)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}}.
\end{aligned}
$$

In view of (11) and the above results, since for $\eta \le 1-\frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, namely $1-\eta \ge \frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, it follows that

$$
\begin{aligned}
R(x) &> \frac{2}{\alpha+\beta}(1-\eta)^{\frac{\alpha+\beta}{2}}(x-\gamma)^{\frac{\alpha+\beta}{2}}-\frac{1}{2}(x-\gamma)^{\frac{\alpha+\beta}{2}}(1-\eta)^{\frac{\alpha+\beta}{2}-1} \\
&\quad -\frac{1}{8}\left(1+\frac{\alpha+\beta}{2}\right)(1-\eta)^{\frac{\alpha+\beta}{2}-2}(x-\gamma)^{\frac{\alpha+\beta}{2}} \\
&= \left[\frac{2(1-\eta)^2}{\alpha+\beta}-\frac{(1-\eta)}{2}-\frac{2+\alpha+\beta}{16}\right]\frac{(x-\gamma)^{\frac{\alpha+\beta}{2}}}{(1-\eta)^{2-\frac{\alpha+\beta}{2}}} \ge 0.
\end{aligned}
$$

(ii) For $x-\gamma \ge \frac{1}{1-\eta}$, we obtain $-\frac{1}{2}h(x,1) = -\frac{1}{2}(x-\gamma)^{-\frac{\alpha+\beta}{2}}(1-\eta)^{-\frac{\alpha+\beta}{2}-1}$, and

$$
\begin{aligned}
\int_\eta^1 h(x,y)\,dy &= \int_\eta^{\eta+\frac{1}{x-\gamma}}\frac{(x-\gamma)^{\frac{\alpha+\beta}{2}}}{(y-\eta)^{1-\frac{\alpha+\beta}{2}}}\,dy+\int_{\eta+\frac{1}{x-\gamma}}^1\frac{(x-\gamma)^{-\frac{\alpha+\beta}{2}}}{(y-\eta)^{\frac{\alpha+\beta}{2}+1}}\,dy \\
&= \frac{4}{\alpha+\beta}-\frac{2}{\alpha+\beta}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&\ge \frac{4(1-\eta)^{-\frac{\alpha+\beta}{2}}}{\alpha+\beta}(x-\gamma)^{-\frac{\alpha+\beta}{2}}-\frac{2(1-\eta)^{-\frac{\alpha+\beta}{2}}}{\alpha+\beta}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&= \frac{2}{\alpha+\beta}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}}.
\end{aligned}
$$

Since for $y \ge 1$, $y-\eta \ge \frac{1}{x-\gamma}$, by the improved Euler-Maclaurin summation formula (*cf.* [6]), it follows that

$$
\begin{aligned}
-\int_1^\infty \rho(y)h_y'(x,y)\,dy &= \left(\frac{\alpha+\beta}{2}+1\right)(x-\gamma)^{-\frac{\alpha+\beta}{2}}\int_1^\infty \rho(y)(y-\eta)^{-\frac{\alpha+\beta}{2}-2}\,dy \\
&> -\frac{1}{8}\left(\frac{\alpha+\beta}{2}+1\right)(x-\gamma)^{-\frac{\alpha+\beta}{2}}(1-\eta)^{-\frac{\alpha+\beta}{2}-2}.
\end{aligned}
$$

In view of (13) and the above results, for $1-\eta \ge \frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, we find

$$
\begin{aligned}
R(x) &> \frac{2}{\alpha+\beta}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}}-\frac{1}{2}(1-\eta)^{-\frac{\alpha+\beta}{2}-1}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&\quad -\frac{1}{8}\left(\frac{\alpha+\beta}{2}+1\right)(1-\eta)^{-\frac{\alpha+\beta}{2}-2}(x-\gamma)^{-\frac{\alpha+\beta}{2}} \\
&= \left[\frac{2(1-\eta)^2}{\alpha+\beta}-\frac{1-\eta}{2}-\frac{2+\alpha+\beta}{16}\right]\frac{(x-\gamma)^{-\frac{\alpha+\beta}{2}}}{(1-\eta)^{2+\frac{\alpha+\beta}{2}}} \ge 0.
\end{aligned}
$$

Hence for $x > \gamma$, we have $R(x) > 0$, and then $\varpi(x) < \omega(n) = \frac{4}{\alpha+\beta}$.

On the other-hand, since $h(x,y)$ is decreasing with respect to $y > \eta$, we find

$$\begin{aligned}\varpi(x) &> \int_1^\infty \frac{(\min\{1,(x-\gamma)(y-\eta)\})^\beta}{(\max\{1,(x-\gamma)(y-\eta)\})^\alpha} \frac{(x-\gamma)^{\frac{\alpha-\beta}{2}}}{(y-\eta)^{1-\frac{\alpha-\beta}{2}}}\,dy \\ &\overset{t=(x-\gamma)(y-\eta)}{=} \int_{(1-\eta)(x-\gamma)}^\infty \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\frac{\alpha-\beta}{2}-1}\,dt = \frac{4}{\alpha+\beta}\bigl(1-\theta(x)\bigr),\end{aligned}$$

where $\theta(x) := \frac{\alpha+\beta}{4}\int_0^{(1-\eta)(x-\gamma)} \frac{(\min\{1,t\})^\beta}{(\max\{1,t\})^\alpha} t^{\frac{\alpha-\beta}{2}-1}\,dt \in (0,1)$.

(i) For $0 < x-\gamma \le \frac{1}{1-\eta}$, we obtain

$$\theta(x) = \frac{\alpha+\beta}{4}\int_0^{(1-\eta)(x-\gamma)} t^{\beta+\frac{\alpha-\beta}{2}-1}\,dt = \frac{1}{2}(1-\eta)^{\frac{\alpha+\beta}{2}}(x-\gamma)^{\frac{\alpha+\beta}{2}}.$$

(ii) For $x-\gamma > \frac{1}{1-\eta}$, it follows that

$$\begin{aligned}\theta(x) &= \frac{\alpha+\beta}{4}\left[\int_0^1 t^{\beta+\frac{\alpha-\beta}{2}-1}\,dt + \int_1^{(1-\eta)(x-\gamma)} t^{-\alpha+\frac{\alpha-\beta}{2}-1}\,dt\right] \\ &= 1-\frac{1}{2}(1-\eta)^{-\frac{\alpha+\beta}{2}}(x-\gamma)^{-\frac{\alpha+\beta}{2}}.\end{aligned}$$

Hence we have (11) and (12). □

Lemma 3 *Let the assumptions of Lemma* 2 *be fulfilled and additionally, let* $0 < p < 1$ *or* $p < 0$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_n \ge 0$, $n \in \mathbf{N}$, $f(x)$ *be a non-negative measurable function in* (γ,∞). *Then we have the following inequalities*:

$$\begin{aligned}J &:= \left\{\sum_{n=1}^\infty (n-\beta)^{\frac{p(\alpha-\beta)}{2}-1}\left[\int_\gamma^\infty \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha} f(x)\,dx\right]^p\right\}^{\frac{1}{p}} \\ &\ge \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\int_\gamma^\infty \varpi(x)(x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}}, \qquad (14)\end{aligned}$$

$$\begin{aligned}L_1 &:= \left\{\int_\gamma^\infty \frac{(x-\alpha)^{\frac{q(\alpha-\beta)}{2}-1}}{[\varpi(x)]^{q-1}}\left[\sum_{n=1}^\infty \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}\right]^q dx\right\}^{\frac{1}{q}} \\ &\ge \left\{\frac{4}{\alpha+\beta}\sum_{n=1}^\infty (n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1} a_n^q\right\}^{\frac{1}{q}}. \qquad (15)\end{aligned}$$

Proof For $0 < p < 1$, setting $k(x,n) := \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}$, by the reverse Hölder inequality (*cf.* [19]) and (11), it follows that

$$\begin{aligned}&\left[\int_\gamma^\infty \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha} f(x)\,dx\right]^p \\ &= \left\{\int_\gamma^\infty k(x,n)\left[\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}{(n-\eta)^{(1-\frac{\alpha-\beta}{2})/p}} f(x)\right]\left[\frac{(n-\gamma)^{(1-\frac{\alpha-\beta}{2})/p}}{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}\right]dx\right\}^p\end{aligned}$$

$$\begin{aligned}
&\geq \int_{\gamma}^{\infty} k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^{p}(x)\,dx \\
&\quad\times\left\{\int_{\gamma}^{\infty} k(x,n)\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}\,dx\right\}^{p-1} \\
&= \left\{\omega(n)(n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1}\right\}^{p-1}\int_{\gamma}^{\infty} k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^{p}(x)\,dx \\
&= \left(\frac{4}{\alpha+\beta}\right)^{p-1}(n-\eta)^{1-\frac{p(\alpha-\beta)}{2}}\int_{\gamma}^{\infty} k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^{p}(x)\,dx.
\end{aligned}$$

Then by the Lebesgue term by term integration theorem (*cf.* [20]), we have

$$\begin{aligned}
J &\geq \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\sum_{n=1}^{\infty}\int_{\gamma}^{\infty} k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^{p}(x)\,dx\right\}^{\frac{1}{p}} \\
&= \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\int_{\gamma}^{\infty}\sum_{n=1}^{\infty} k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}f^{p}(x)\,dx\right\}^{\frac{1}{p}} \\
&= \left(\frac{4}{\alpha+\beta}\right)^{\frac{1}{q}}\left\{\int_{\gamma}^{\infty}\varpi(x)(x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1}f^{p}(x)\,dx\right\}^{\frac{1}{p}},
\end{aligned}$$

and then (14) follows. By the reverse Hölder inequality, for $q < 0$, we have

$$\begin{aligned}
\left[\sum_{n=1}^{\infty} k(x,n)a_n\right]^{q} &= \left\{\sum_{n=1}^{\infty} k(x,n)\left[\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}{(n-\eta)^{(1-\frac{\alpha-\beta}{2})/p}}\right]\left[\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})/p}a_n}{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})/q}}\right]\right\}^{q} \\
&\leq \left\{\sum_{n=1}^{\infty} k(x,n)\frac{(x-\gamma)^{(1-\frac{\alpha-\beta}{2})(p-1)}}{(n-\eta)^{1-\frac{\alpha-\beta}{2}}}\right\}^{q-1}\sum_{n=1}^{\infty} k(x,n)\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}a_n^{q} \\
&= \frac{[\varpi(x)]^{q-1}}{(x-\gamma)^{\frac{q(\alpha-\beta)}{2}-1}}\sum_{n=1}^{\infty} k(x,n)\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}a_n^{q}.
\end{aligned}$$

By the Lebesgue term by term integration theorem, we have

$$\begin{aligned}
L_1 &\geq \left\{\int_{\gamma}^{\infty}\sum_{n=1}^{\infty} k(x,n)\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}a_n^{q}\,dx\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{n=1}^{\infty}\int_{\gamma}^{\infty} k(x,n)\frac{(n-\eta)^{(1-\frac{\alpha-\beta}{2})(q-1)}}{(x-\gamma)^{1-\frac{\alpha-\beta}{2}}}a_n^{q}\,dx\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{n=1}^{\infty}\omega(n)(n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1}a_n^{q}\right\}^{\frac{1}{q}},
\end{aligned}$$

and in view of (11), inequality (15) follows. For $p < 0$, by the same way we still have (14) and (15). □

Lemma 4 *Let the assumptions of Lemma* 2 *be fulfilled and additionally, let* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \varepsilon < \frac{p}{2}(\alpha+\beta)$. *Setting* $\widetilde{f}(x) = (x-\gamma)^{\frac{\alpha-\beta}{2}+\frac{\varepsilon}{p}-1}$, $x \in (\gamma, \gamma+1)$; $\widetilde{f}(x) = 0$, $x \in [\gamma+1, \infty)$, *and* $\widetilde{a}_n = (n-\eta)^{\frac{\alpha-\beta}{2}-\frac{\varepsilon}{q}-1}$, $n \in \mathbf{N}$, *then we have*

$$\widetilde{I} := \sum_{n=1}^{\infty} \widetilde{a}_n \int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} \widetilde{f}(x)\,dx$$
$$< \frac{1}{\varepsilon} \frac{(\alpha+\beta)}{(\frac{\alpha+\beta}{2})^2 - (\frac{\varepsilon}{p})^2} \left[\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right], \tag{16}$$

$$\widetilde{H} := \left\{\int_{\gamma}^{\infty} (x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1} \widetilde{f}^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1} \widetilde{a}_n^q\right\}^{\frac{1}{q}}$$
$$> \frac{1}{\varepsilon}\bigl(1-\varepsilon O(1)\bigr)^{\frac{1}{p}} \left\{\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right\}^{\frac{1}{q}}. \tag{17}$$

Proof We find

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty} (n-\eta)^{\frac{\alpha-\beta}{2}-\frac{\varepsilon}{q}-1} \int_{\gamma}^{\gamma+1} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} (x-\gamma)^{\frac{\alpha-\beta}{2}+\frac{\varepsilon}{p}-1}\,dx \\
&< \sum_{n=1}^{\infty} (n-\eta)^{\frac{\alpha-\beta}{2}-\frac{\varepsilon}{q}-1} \int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} (x-\gamma)^{\frac{\alpha-\beta}{2}+\frac{\varepsilon}{p}-1}\,dx \\
&= \frac{\alpha+\beta}{(\frac{\alpha+\beta}{2})^2 - (\frac{\varepsilon}{p})^2} \left[\frac{1}{(1-\eta)^{\varepsilon+1}} + \sum_{n=1}^{\infty} \frac{1}{(n-\eta)^{\varepsilon+1}}\right] \\
&< \frac{\alpha+\beta}{(\frac{\alpha+\beta}{2})^2 - (\frac{\varepsilon}{p})^2} \left[\frac{1}{(1-\eta)^{\varepsilon+1}} + \int_{1}^{\infty} \frac{dy}{(y-\eta)^{\varepsilon+1}}\right] \\
&= \frac{1}{\varepsilon} \frac{(\alpha+\beta)}{(\frac{\alpha+\beta}{2})^2 - (\frac{\varepsilon}{p})^2} \left[\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right],
\end{aligned}$$

and then (16) is valid. We obtain

$$\begin{aligned}
\widetilde{H} &= \left\{\int_{\gamma}^{\gamma+1} \left[1 - \frac{1}{2}(1-\eta)^{\frac{\alpha+\beta}{2}}(x-\gamma)^{\frac{\alpha+\beta}{2}}\right](x-\gamma)^{\varepsilon-1}\,dx\right\}^{\frac{1}{p}} \\
&\quad \times \left\{\frac{1}{(1-\eta)^{\varepsilon+1}} + \sum_{n=2}^{\infty} \frac{1}{(n-\eta)^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&> \left(\frac{1}{\varepsilon} - O(1)\right)^{\frac{1}{p}} \left\{\frac{1}{(1-\eta)^{\varepsilon+1}} + \int_{1}^{\infty} \frac{dy}{(y-\eta)^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\bigl(1-\varepsilon O(1)\bigr)^{\frac{1}{p}} \left\{\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right\}^{\frac{1}{q}},
\end{aligned}$$

and so (17) is valid. □

3 Main results

We introduce the functions

$$\Phi(x) := (x-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1}, \qquad \widetilde{\Phi}(x) = \bigl(1-\theta(x)\bigr)\Phi(x) \quad \bigl(x \in (\gamma,\infty)\bigr),$$
$$\Psi(n) := (n-\eta)^{q(1-\frac{\alpha-\beta}{2})-1} \quad (n \in \mathbf{N}),$$

wherefrom $[\Phi(x)]^{1-q} = (x-\gamma)^{q\frac{\alpha-\beta}{2}-1}$, $[\widetilde{\Phi}(x)]^{1-q} = (1-\theta(x))^{1-q}(x-\gamma)^{q\frac{\alpha-\beta}{2}-1}$ and $[\Psi(n)]^{1-p} = (n-\eta)^{p\frac{\alpha-\beta}{2}-1}$.

Theorem 1 *If* $0 < \alpha+\beta \le 2$, $\gamma \in \mathbf{R}$, $\eta \le 1-\frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, $0 < p < 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x), a_n \ge 0$, $f \in L_{p,\widetilde{\Phi}}(\gamma,\infty)$, $a = \{a_n\}_{n=1}^{\infty} \in l_{q,\Psi}$, $\|f\|_{p,\widetilde{\Phi}} > 0$ *and* $\|a\|_{q,\Psi} > 0$, *then we have the following equivalent inequalities*:

$$\begin{aligned} I &:= \sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta}}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} f(x)\,dx \\ &= \int_{\gamma}^{\infty} f(x) \sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx > \frac{4}{\alpha+\beta}\|f\|_{p,\widetilde{\Phi}}\|a\|_{q,\Psi}, \end{aligned} \tag{18}$$

$$J = \left\{ \sum_{n=1}^{\infty} [\Psi(n)]^{1-p} \left[\int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx \right]^p \right\}^{\frac{1}{p}} > \frac{4}{\alpha+\beta}\|f\|_{p,\widetilde{\Phi}}, \tag{19}$$

$$L := \left\{ \int_{\gamma}^{\infty} [\widetilde{\Phi}(x)]^{1-q} \left[\sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} \right]^q dx \right\}^{\frac{1}{q}} > \frac{4}{\alpha+\beta}\|a\|_{q,\Psi}, \tag{20}$$

where the constant $\frac{4}{\alpha+\beta}$ *is the best possible in the above inequalities.*

Proof The two expressions for I in (18) follow from Lebesgue's term by term integration theorem. By (14) and (11), we have (19). By the reverse Hölder inequality, we have

$$I = \sum_{n=1}^{\infty} \left[\Psi^{\frac{-1}{q}}(n) \int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx \right] \left[\Psi^{\frac{1}{q}}(n) a_n \right] \ge J\|a\|_{q,\Psi}.$$

Then by (19), we have (18). On the other-hand, assume that (18) is valid. Setting

$$a_n := [\Psi(n)]^{1-p} \left[\int_{\gamma}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}}\,dx \right]^{p-1}, \quad n \in \mathbf{N},$$

it follows that $J^{p-1} = \|a\|_{q,\Psi}$. By (14), we find $J > 0$. If $J = \infty$, then (19) is trivially valid; if $J < \infty$, then by (18), we have

$$\|a\|_{q,\Psi}^q = J^{q(p-1)} = J^p = I > \frac{4}{\alpha+\beta}\|f\|_{p,\widetilde{\Phi}}\|a\|_{q,\Psi},$$

therefore $\|a\|_{q,\Psi}^{q-1} = J > \frac{4}{\alpha+\beta}\|f\|_{p,\widetilde{\Phi}}$, that is, (19) is equivalent to (18). On the other-hand, by (11) we have $[\varpi(x)]^{1-q} > (1-\theta(x))^{1-q}(\frac{4}{\alpha+\beta})^{1-q}$. Then in view of (15), we have (20). By the Hölder inequality, we find

$$I = \int_{\gamma}^{\infty} \left[\widetilde{\Phi}^{\frac{1}{p}}(x) f(x) \right] \left[\widetilde{\Phi}^{\frac{-1}{p}}(x) \sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} \right] dx \ge \|f\|_{p,\widetilde{\Phi}} L.$$

Then by (20), we have (18). On the other-hand, assume that (18) is valid. Setting

$$f(x) := [\widetilde{\Phi}(x)]^{1-q} \left[\sum_{n=1}^{\infty} \frac{(\min\{1,(x-\gamma)(n-\eta)\})^{\beta} a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} \right]^{q-1}, \quad x \in (\gamma,\infty),$$

then $L^{q-1} = \|f\|_{p,\widetilde{\Phi}}$. By (15), we find $L > 0$. If $L = \infty$, then (20) is trivially valid; if $L < \infty$, then by (18), we have

$$\|f\|^p_{p,\widetilde{\Phi}} = L^{p(q-1)} = I > \frac{4}{\alpha+\beta}\|f\|_{p,\widetilde{\Phi}}\|a\|_{q,\Psi},$$

therefore $\|f\|^{p-1}_{p,\widetilde{\Phi}} = L > \frac{4}{\alpha+\beta}\|a\|_{q,\Psi}$, that is, (20) is equivalent to (18). Hence, (18), (19), and (20) are equivalent.

If there exists a positive number k ($\geq \frac{4}{\alpha+\beta}$), such that (18) is valid as we replace $\frac{4}{\alpha+\beta}$ with k, then in particular, it follows that $\widetilde{I} > k\widetilde{H}$. In view of (16) and (17), we have

$$\frac{(\alpha+\beta)}{(\frac{\alpha+\beta}{2})^2 - (\frac{\varepsilon}{p})^2}\left[\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right] > k\bigl(1-\varepsilon O(1)\bigr)^{\frac{1}{p}}\left\{\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right\}^{\frac{1}{q}},$$

and $\frac{4}{\alpha+\beta} \geq k$ ($\varepsilon \to 0^+$). Hence $k = \frac{4}{\alpha+\beta}$ is the best value of (18).

By the equivalence of the inequalities, the constant factor $\frac{4}{\alpha+\beta}$ in (19) and (20) is the best possible. □

For $p < 0$, we have the dual forms of (18), (19), and (20) as follows:

Theorem 2 *If* $0 < \alpha+\beta \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\alpha+\beta}{8}(1+\sqrt{3+\frac{4}{\alpha+\beta}})$, $p < 0$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x), a_n \geq 0$, $f \in L_{p,\Phi}(\gamma,\infty)$, $a = \{a_n\}_{n=1}^\infty \in l_{q,\Psi}$, $\|f\|_{p,\Phi} > 0$ *and* $\|a\|_{q,\Psi} > 0$, *then we have the following equivalent inequalities*:

$$\sum_{n=1}^\infty a_n \int_\gamma^\infty \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha} f(x)\,dx > \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}\|a\|_{q,\Psi}, \tag{21}$$

$$\left\{\sum_{n=1}^\infty \bigl[\Psi(n)\bigr]^{1-p}\left[\int_\gamma^\infty \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta f(x)}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}\,dx\right]^p\right\}^{\frac{1}{p}} > \frac{4}{\alpha+\beta}\|f\|_{p,\Phi}, \tag{22}$$

$$\left\{\int_\gamma^\infty \bigl[\Phi(x)\bigr]^{1-q}\left[\sum_{n=1}^\infty \frac{(\min\{1,(x-\gamma)(n-\eta)\})^\beta a_n}{(\max\{1,(x-\gamma)(n-\eta)\})^\alpha}\right]^q dx\right\}^{\frac{1}{q}} > \frac{4}{\alpha+\beta}\|a\|_{q,\Psi}, \tag{23}$$

where the constant $\frac{4}{\alpha+\beta}$ *is the best possible in the above inequalities.*

Proof By means of Lemma 3 and the same way, we can prove that (21), (22), and (23) are valid and equivalent. For $0 < \varepsilon \frac{|p|}{2}(\alpha+\beta)$, setting $\widetilde{f}(x)$ and $\widetilde{a}_n$ as Lemma 4, if there exists a positive number k ($\geq \frac{4}{\alpha+\beta}$), such that (21) is valid as we replace $\frac{4}{\alpha+\beta}$ with k, then in particular, by (16), it follows that

$$\begin{aligned}
&\frac{\alpha+\beta}{(\frac{\alpha+\beta}{2})^2 - (\frac{\varepsilon}{p})^2}\left[\frac{\varepsilon}{(1-\eta)^{\varepsilon+1}} + \frac{1}{(1-\eta)^{\varepsilon}}\right] \\
&\quad > \varepsilon\widetilde{I} > \varepsilon k\left\{\int_\gamma^{\gamma+1}(x-\gamma)^{\varepsilon-1}\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^\infty \frac{1}{(n-\eta)^{\varepsilon+1}}\right\}^{\frac{1}{q}} \\
&\quad > \varepsilon k\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p}}\left\{\int_1^\infty \frac{dy}{(y-\eta)^{\varepsilon+1}}\right\}^{\frac{1}{q}} = k\left\{\frac{1}{(1-\eta)^{\varepsilon}}\right\}^{\frac{1}{q}},
\end{aligned}$$

and $\frac{4}{\alpha+\beta} \geq k$ ($\varepsilon \to 0^+$). Hence $k = \frac{4}{\alpha+\beta}$ is the best value of (21). By the equivalence of the inequalities, the constant factor $\frac{4}{\alpha+\beta}$ in (21) and (22) is the best possible. □

Remark (i) Since we find

$$\min_{0<\alpha+\beta\leq 2}\left\{1-\frac{\alpha+\beta}{8}\left(1+\sqrt{3+\frac{4}{\alpha+\beta}}\right)\right\} = \frac{3-\sqrt{5}}{4} = 0.19^+ > 0,$$

then for $\eta = \gamma = 0$ in (18), we have

$$\theta_0(x) = \begin{cases} \frac{1}{2}x^{\frac{\alpha+\beta}{2}}, & 0 < x \leq 1, \\ 1-\frac{1}{2}x^{-\frac{\alpha+\beta}{2}}, & x > 1, \end{cases}$$

and the following inequality:

$$\begin{aligned}&\sum_{n=1}^{\infty} a_n \int_0^{\infty} \frac{(\min\{1,xn\})^{\beta}}{(\max\{1,xn\})^{\alpha}} f(x)\,dx \\ &\quad > \frac{4}{\alpha+\beta}\left\{\int_0^{\infty}\bigl(1-\theta_0(x)\bigr)x^{p(1-\frac{\alpha-\beta}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\frac{\alpha-\beta}{2})-1} a_n^q\right\}^{\frac{1}{q}}. \end{aligned} \tag{24}$$

Hence (18) is a more accurate inequality of (24).

(ii) For $\beta = 0$ in (18), we have $0 < \alpha \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\alpha}{8}(1+\sqrt{3+\frac{4}{\alpha}})$,

$$\theta_1(x) = \begin{cases} \frac{1}{2}(x-\gamma)^{\frac{\alpha}{2}}, & 0 < x-\gamma \leq \frac{1}{1-\eta}, \\ 1-\frac{1}{2}(x-\gamma)^{-\frac{\alpha}{2}}, & x-\gamma > \frac{1}{1-\eta} \end{cases}$$

and the following inequality:

$$\begin{aligned}&\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{f(x)\,dx}{(\max\{1,(x-\gamma)(n-\eta)\})^{\alpha}} \\ &\quad > \frac{4}{\alpha}\left\{\int_{\gamma}^{\infty}\bigl(1-\theta_1(x)\bigr)(x-\gamma)^{p(1-\frac{\alpha}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} (n-\eta)^{q(1-\frac{\alpha}{2})-1} a_n^q\right\}^{\frac{1}{q}}; \end{aligned} \tag{25}$$

for $\alpha = 0$ in (18), we have $0 < \beta \leq 2$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\beta}{8}(1+\sqrt{3+\frac{4}{\beta}})$,

$$\theta_2(x) = \begin{cases} \frac{1}{2}(x-\gamma)^{\frac{\beta}{2}}, & 0 < x-\gamma \leq \frac{1}{1-\eta}, \\ 1-\frac{1}{2}(x-\gamma)^{-\frac{\beta}{2}}, & x-\gamma > \frac{1}{1-\eta} \end{cases}$$

and the following inequality:

$$\begin{aligned}&\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \bigl(\min\{1,(x-\gamma)(n-\eta)\}\bigr)^{\beta} f(x)\,dx \\ &\quad > \frac{4}{\beta}\left\{\int_{\gamma}^{\infty}\bigl(1-\theta_2(x)\bigr)(x-\gamma)^{p(1+\frac{\beta}{2})-1} f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} (n-\eta)^{q(1+\frac{\beta}{2})-1} a_n^q\right\}^{\frac{1}{q}}; \end{aligned} \tag{26}$$

for $\beta = \alpha = \lambda$ in (18), we have $0 < \lambda \leq 1$, $\gamma \in \mathbf{R}$, $\eta \leq 1 - \frac{\lambda}{4}(1 + \sqrt{3 + \frac{2}{\lambda}})$,

$$\theta_3(x) = \begin{cases} \frac{1}{2}(x-\gamma)^{\lambda}, & 0 < x - \gamma \leq \frac{1}{1-\eta}, \\ 1 - \frac{1}{2}(x-\gamma)^{-\lambda}, & x - \gamma > \frac{1}{1-\eta} \end{cases}$$

and the following inequality:

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \left[\frac{\min\{1, (x-\gamma)(n-\eta)\}}{\max\{1, (x-\gamma)(n-\eta)\}}\right]^{\lambda} f(x)\,dx$$
$$> \frac{2}{\lambda}\left\{\int_{\gamma}^{\infty} \bigl(1 - \theta_3(x)\bigr)(x-\gamma)^{p-1} f^p(x)\,dx\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} (n-\eta)^{q-1} a_n^q\right\}^{\frac{1}{q}}. \tag{27}$$

(iii) Setting $y = \frac{1}{x-\gamma} + \gamma$, $g(y) = (y-\gamma)^{\alpha-\beta-2} f(\frac{1}{y-\gamma} + \gamma)$, $\varphi(y) = (y-\gamma)^{p(1-\frac{\alpha-\beta}{2})-1}$ and $\widetilde{\varphi}(y) = (1 - \theta(\frac{1}{y-\gamma} + \gamma))\varphi(y)$ in (18), by simplification, we obtain the following inequality with the homogeneous kernel:

$$\sum_{n=1}^{\infty} a_n \int_{\gamma}^{\infty} \frac{(\min\{y-\gamma, n-\eta\})^{\beta}}{(\max\{y-\gamma, n-\eta\})^{\alpha}} g(y)\,dy > \frac{4}{\alpha+\beta} \|g\|_{p,\widetilde{\varphi}} \|a\|_{q,\Psi}. \tag{28}$$

It is evident that (28) is equivalent to (18), and then the same constant factor $\frac{4}{\alpha+\beta}$ in (28) is still the best possible. In the same way, we can find the following inequalities equivalent to (28) with the same best possible constant factor $\frac{4}{\alpha+\beta}$:

$$\left\{\sum_{n=1}^{\infty} \bigl[\Psi(n)\bigr]^{1-p} \left[\int_{\gamma}^{\infty} \frac{(\min\{y-\gamma, n-\eta\})^{\beta} g(y)}{(\max\{y-\gamma, n-\eta\})^{\alpha}}\,dy\right]^p\right\}^{\frac{1}{p}} > \frac{4}{\alpha+\beta} \|f\|_{p,\widetilde{\varphi}}, \tag{29}$$

$$\left\{\int_{\gamma}^{\infty} \bigl[\widetilde{\varphi}(y)\bigr]^{1-q} \left[\sum_{n=1}^{\infty} \frac{(\min\{y-\gamma, n-\eta\})^{\beta} a_n}{(\max\{y-\gamma, n-\eta\})^{\alpha}}\right]^q dy\right\}^{\frac{1}{q}} > \frac{4}{\alpha+\beta} \|a\|_{q,\Psi}. \tag{30}$$

(iv) Applying the same way in Theorem 2, we still can obtain some particular dual forms as (i) and (ii) and some equivalent inequalities similar to (28), (29), and (30).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by BY and QC. BY prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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