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Almost convergence and generalized weighted mean II

Murat Kirişci*

*Correspondence: mkirisci@hotmail.com; murat.kirisci@istanbul.edu.tr
Department of Mathematical Education, Hasan Ali Yücel Education Faculty, İstanbul University, Vefa 34470, Fatih, Istanbul, Turkey

Abstract

In this paper, we investigate some new sequence spaces, which naturally emerge from the concepts of almost convergence and generalized weighted mean. The object of this paper is to introduce the new sequence spaces obtained as the matrix domain of generalized weighted mean in the spaces of almost null and almost convergent sequences. Furthermore, the beta and gamma dual spaces of the new spaces are determined and some classes of matrix transformations are characterized.

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1 Introduction

By a *sequence space*, we understand a linear subspace of the space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences which contains ϕ, the set of all finitely non-zero sequences, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We write ℓ_∞, c, and c_0 for the classical spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1, and ℓ_p, we denote the spaces of all bounded, convergent, absolutely, and p-absolutely convergent series, respectively.

A sequence spaces μ with a linear topology is called a K-space if each of the maps $p_i : \mu \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called an FK-space if μ is a complete linear metric space; a BK-space is a normed FK-space.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a normed space μ is called a *Schauder basis* if for every $x \in \mu$, there is a unique sequence $(\beta_n)_{n=0}^{\infty}$ of scalars such that

$$x = \sum_{n=0}^{\infty} \beta_n b^{(n)}.$$

The sequence $(e^{(n)})$ is the Schauder basis for ℓ_p and c_0, and $\{e, e^{(n)}\}$ is the Schauder basis for the space c, while the space ℓ_∞ has no Schauder basis, where $e^{(n)}$ and e denote the sequences whose only non-zero entry is a 1 in the nth place for each $n \in \mathbb{N}$ and $e = (1, 1, 1, \ldots)$.

A subset M of a metric space (X, d) is said to be *dense* in X if $\overline{M} = X$. A metric space (X, d) is said to be *separable* if it contains a countable subset which is dense in X. Note that a nonseparable space has no Schauder basis.

Let λ and μ be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ into

μ, and we denote it by writing $A : \lambda \to \mu$ if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; here

$$(Ax)_n = \sum_k a_{nk} x_k \quad \text{for each } n \in \mathbb{N}. \tag{1.1}$$

By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and each $x \in \lambda$ and we have $Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in \mu$ for all $x \in \lambda$. A sequence x is said to be A-summable to α if Ax converges to α, which is called the A-limit of x. Also by $(\lambda : \mu; p)$, we denote the subset of $(\lambda : \mu)$ for which limits or sums are preserved whenever there is a limit or sum on the spaces λ and μ. The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\} \tag{1.2}$$

which is a sequence space. If $A = (a_{nk})$ is triangle, that is to say, $a_{nn} \neq 0$ and $a_{nk} = 0$ for all $k > n$, then one can easily observe that the sequence spaces λ_A and λ are linearly isomorphic, *i.e.*, $\lambda_A \cong \lambda$ [1]. We write U for the set of all sequences $u = (u_k)$ such that $(u_k) \neq 0$ for all $k \in \mathbb{N}$. For $u \in U$, let $1/u = (1/u_k)$. Let $u, v \in U$ and define the *generalized weighted mean* or *factorable matrix* $G(u, v) = (g_{nk})$ by

$$g_{nk} = \begin{cases} u_n v_k & (0 \le k \le n), \\ 0 & (k > n), \end{cases}$$

for all $k, n \in \mathbb{N}$; here u_n depends only on n and v_k only on k.

We shall write throughout for brevity

$$a(n,k) = \sum_{j=0}^{n} a_{jk}, \qquad a(n,k,m) = \frac{1}{n+1} \sum_{j=0}^{n} a_{m+j,k}, \qquad \Delta a_{nk} = a_{nk} - a_{n,k+1}$$

for all $k, m, n \in \mathbb{N}$.

The main purpose of present paper is to introduce the sequence spaces $f_0(G)$ and $f(G)$ derived as the domain of the generalized weighted mean in the spaces f_0 and f of almost null and almost convergent sequences, and to determine the β- and γ-duals of these spaces. Furthermore, some classes of matrix mappings on/in the space $f(G)$ are characterized.

2 Spaces of almost null and almost convergent sequences

The shift operator P is defined on ω by $(Px)_m = x_{m+1}$ for all $n \in \mathbb{N}$. A Banach limit L is defined on ℓ_∞, as a non-negative linear functional, such that $L(Px) = L(x)$ and $L(e) = 1$. A sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are α [2] and is denoted by $f\text{-}\lim x_k = \alpha$. Let P^i be the composition of P with itself i times and let us write for a sequence $x = (x_k)$

$$t_{mn}(x) = \frac{1}{n+1} \sum_{i=0}^{n} \left(P^i x\right)_m \quad \text{for all } m, n \in \mathbb{N}. \tag{2.1}$$

Lorentz [2] proved that f-$\lim x_k = \alpha$ if and only if $\lim_{n\to\infty} t_{mn}(x) = \alpha$ uniformly in m. It is well known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal.

The spaces f_0 and f of almost null and almost convergent sequences are defined as follows:

$$f_0 = \Bigl\{x = (x_k) \in \omega : \lim_{n\to\infty} t_{mn}(x) = 0 \text{ uniformly in } m\Bigr\},$$

$$f = \Bigl\{x = (x_k) \in \omega : \exists\alpha \in \mathbb{C} \ni \lim_{n\to\infty} t_{mn}(x) = \alpha \text{ uniformly in } m\Bigr\},$$

where $t_{mn}(x)$ is defined by (2.1). Also, by fs, we denote the space of all almost convergent series.

3 New sequence spaces and their duals

In this section, we introduce the sequence spaces $f_0(G)$ and $f(G)$ and give some results concerning them, and we determine their beta and gamma duals.

Malkowsky and Savaş [3] have defined the sequence space $Z = (u, v; X)$, which consists of all sequences whose $G(u, v)$-transforms are in $X \in \{\ell_\infty, c, c_0, \ell_p\}$, where $u, v \in U$. The space $Z(u, v; X)$ defined by

$$Z = Z(u, v; X) = \left\{x = (x_j) \in \omega : y = \left(\sum_{j=0}^{k} u_k v_j x_j\right) \in X\right\}.$$

Altay and Başar [4] constructed the new paranormed sequence spaces $\lambda(u, v; p)$ defined by

$$\lambda(u, v; p) = \left\{x = (x_j) \in \omega : y = \left(\sum_{j=0}^{k} u_k v_j x_j\right) \in \lambda(p)\right\},$$

where $\lambda \in \{\ell_\infty, c, c_0\}$.

Afterward, Altay and Başar [5] studied the sequence space $\ell(u, v; p)$ as follows:

$$\ell(u, v; p) = \left\{x = (x_j) \in \omega : y = \left(\sum_{j=0}^{k} u_k v_j x_j\right) \in \ell(p)\right\}.$$

Şimşek *et al.* [6] have introduced a modular structure of the sequence spaces defined by Altay and Başar [5] and studied Kadec-Klee and uniform Opial properties of this sequence space on Köthe sequence spaces.

The new sequence spaces $f_0(G)$ and $f(G)$ are the set of all sequences whose $G(u, v)$-transforms are in the spaces f_0 and f, that is,

$$f_0(G) = \Bigl\{x = (x_k) \in \omega : \lim_{n\to\infty} t_{mn}\bigl(G(u, v)x\bigr) = 0 \text{ uniformly in } m\Bigr\},$$

$$f(G) = \Bigl\{x = (x_k) \in \omega : \exists\alpha \in \mathbb{C} \ni \lim_{n\to\infty} t_{mn}\bigl(G(u, v)x\bigr) = \alpha \text{ uniformly in } m\Bigr\}.$$

By the notation of (1.2), the sequence spaces $f_0(G)$ and $f(G)$ are restated as

$$f_0(G) = (f_0)_{G(u,v)} \quad \text{and} \quad f(G) = f_{G(u,v)}.$$

Define the sequence $y = (y_k)$ by the $G(u,v)$-transform of a sequence $x = (x_k)$,

$$y_n = \sum_{k=0}^{n} u_n v_k x_k \quad \text{for all } n \in \mathbb{N}. \tag{3.1}$$

Theorem 3.1 *The sequence spaces $f(G)$ and $f_0(G)$ are BK-spaces with the same norm given by*

$$\|x\|_{f(G)} = \|G(u,v)x\|_f = \sup_{m,n\in\mathbb{N}} \left|t_{mn}\big(G(u,v)x\big)\right|, \tag{3.2}$$

where

$$t_{mn}\big(G(u,v)x\big) = \frac{1}{m+1}\sum_{j=0}^{m}\big\{G(u,v)x\big\}_{n+j} = \frac{1}{m+1}\sum_{j=0}^{m}\sum_{k=0}^{n+j} u_{n+j}v_k x_k \quad \textit{for all } m,n \in \mathbb{N}.$$

Proof f_0 and f endowed with the norm $\|\cdot\|_\infty$ are BK-spaces (Boos [7, Example 7.3.2(b)]) and $G(u,v)$ is a triangle matrix. Theorem 4.3.2 of Wilansky [8, p. 61] gives the fact that $f(G)$ and $f_0(G)$ are BK-spaces with the norm $\|\cdot\|_{f(G)}$. □

Theorem 3.2 *The sequence spaces $f_0(G)$ and $f(G)$ strictly include the spaces f_0 and f, respectively.*

Proof By the definition on the sequence spaces f_0 and f, it is immediate that $f \subset f(G)$ and $f_0 \subset f_0(G)$.

Now, we should show that these inclusions are strict. We consider the sequence $t = (t_k)$ defined by

$$t_k = \begin{cases} \frac{1}{u_n v_k}, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}. \end{cases} \tag{3.3}$$

The sequence is not almost convergent but Gt is almost convergent to 1/2. This step completes the proof. □

Theorem 3.3 *The inclusions $c \subset f(G)$ and $f(G) \subset \ell_\infty$ strictly hold.*

Proof It is clear that $c \subset f(G)$ and $f(G) \subset \ell_\infty$ because of Theorem 3.2 and $c \subset f \subset \ell_\infty$. Further, we show that these inclusions are strict.

Consider the sequence $x = G^{-1}(u,v)y$ with the sequence y in the set $\ell_\infty \setminus f$ given by Miller and Orhan [9] as $y = (0,\dots,0,1,\dots,1,0,\dots,0,1,\dots,1,\dots)$, where the blocks of 0's are increasing by factors of 100 and blocks of 1's are increasing by factors of 10. Then the sequence x is not in $f(G)$ but in the space ℓ_∞. This shows that the inclusion $f(G) \subset \ell_\infty$ is strict.

Since the inclusion $f \subset f(G)$ strictly holds by Theorem 3.2, by combining this fact with the well-known strict inclusion $c \subset f$, one can easily see that the inclusion $c \subset f(G)$ also strictly holds, as was desired. □

It is known from Corollary 3.3 of Başar and Kirişçi [10] that the Banach space f has no Schauder basis. It is also known from Theorem 2.3 of Jarrah and Malkowsky [11] that the domain μ_A of a matrix A in a normed sequence space μ has a basis if and only if μ has a basis whenever $A = (a_{nk})$ is a triangle. Combining these two facts one can immediately conclude that neither the space $f_0(G)$ nor the space $f(G)$ have a Schauder basis.

Lemma 3.4 [12, Theorem 2.1] *Let λ, μ be the BK-spaces and $B^U_\mu = (b_{nk})$ be defined via the sequence $\alpha = (\alpha_k) \in \mu$ and triangle matrix $U = (u_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} \alpha_j u_{nj} v_{jk}$$

for all $k, n \in \mathbb{N}$. Then the inclusion $\mu\lambda_U \subset \lambda_U$ holds if and only if the matrix $B^U_\mu = UD_\alpha U^{-1}$ is in the classes $(\lambda : \lambda)$, where D_α is the diagonal matrix defined by $[D_\alpha]_{nn} = \alpha_n$ for all $n \in \mathbb{N}$.

Lemma 3.5 [12, Theorem 3.1] *$B^U_\mu = (b_{nk})$ be defined via a sequence $a = (a_k) \in \omega$ and inverse of the triangle matrix $U = (u_{nk})$ by*

$$b_{nk} = \sum_{j=k}^{n} a_j v_{jk}$$

for all $k, n \in \mathbb{N}$. Then

$$\lambda^\beta_U = \{a = (a_k) \in \omega : B^U \in (\lambda : c)\}$$

and

$$\lambda^\gamma_U = \{a = (a_k) \in \omega : B^U \in (\lambda : \ell_\infty)\}.$$

From Lemma 3.4 and Lemma 3.5, we may give the theorem determining the β- and γ-duals of the sequence space $f(G)$.

Theorem 3.6 *Let $u, v \in U$ and $z = (z_k) \in \omega$. Define the matrix $E = (e_{nk})$ by*

$$e_{nk} = \begin{cases} \frac{1}{u_k}(\frac{z_k}{v_k} - \frac{z_{k+1}}{v_{k+1}}) & (k < n), \\ \frac{z_k}{u_k v_k} & (k = n), \\ 0 & (k > n), \end{cases} \tag{3.4}$$

for all $k, n \in \mathbb{N}$. Then

$$\{f(G)\}^\beta = \{z = (z_k) \in \omega : E \in (f : c)\}$$

and

$$\{f(G)\}^\gamma = \{z = (z_k) \in \omega : E \in (f : \ell_\infty)\}.$$

Proof Consider the equality

$$\sum_{j=0}^{k} z_j x_j = \sum_{j=0}^{k-1} \frac{1}{u_j}\left(\frac{z_j}{v_j} - \frac{z_{j+1}}{v_{j+1}}\right) y_j + \frac{1}{u_k v_k} y_k z_k = (Ey)_k, \tag{3.5}$$

where $E = (e_{nk})$ is defined by (3.4). We therefore observe by (3.5) that $zx = (z_k x_k) \in cs$ or bs whenever $x = (x_k) \in f(G)$ if and only if $Ey \in c$ or ℓ_∞ whenever $y = (y_k) \in f$. We obtain from Lemma 3.4 and Lemma 3.5 the result that $z = (z_k) \in \{f(G)\}^\beta$ or $z = (z_k) \in \{f(G)\}^\gamma$ if and only if $E \in (f : c)$ or $E \in (f : \ell_\infty)$, which is what we wished to prove. □

As a direct consequence of Theorem 3.6, we have the following.

Corollary 3.7 *Let $u, v \in U$ for all $k \in \mathbb{N}$. Then*

$$\{f(G)\}^\beta = \left\{z = (z_k) \in \omega : \left\{\frac{1}{u_k}\left(\frac{z_k}{v_k} - \frac{z_{k+1}}{v_{k+1}}\right)\right\} \in \ell_1 \textit{ and } \left(\frac{z_k}{u_k v_k}\right) \in c\right\}$$

and

$$\{f(G)\}^\gamma = \left\{z = (z_k) \in \omega : \left\{\frac{1}{u_k}\left(\frac{z_k}{v_k} - \frac{z_{k+1}}{v_{k+1}}\right)\right\} \in \ell_1 \textit{ and } \left(\frac{z_k}{u_k v_k}\right) \in \ell_\infty\right\}.$$

4 Some matrix mappings related to the space $f(G)$

In this section, we give two theorems characterizing the classes of matrix transformations from the sequence space $f(G)$ into any given sequence space μ and from any sequence space μ into the given sequence space $f(G)$.

We write throughout for brevity

$$\widetilde{a}_{nk} = \frac{1}{u_k}\left(\frac{a_{nk}}{v_k} - \frac{a_{n,k+1}}{v_{k+1}}\right) \quad \text{and} \quad b_{nk} = \sum_{j=0}^{n} u_n v_j a_{jk} \tag{4.1}$$

for all $k, n \in \mathbb{N}$.

Lemma 4.1 *Let $A = (a_{nk})$ be an infinite matrix. Then the following statements hold*:

(i) (*cf.* [13]) $A \in (f : \ell_\infty)$ *if and only if*

$$\sup_{n\in\mathbb{N}} \sum_k |a_{nk}| < \infty. \tag{4.2}$$

(ii) (*cf.* [13]) $A \in (f : c)$ *if and only if* (4.2) *holds, and there are* $\alpha_k, \alpha \in \mathbb{C}$ *such that*

$$\lim_{n\to\infty} a_{nk} = \alpha_k \quad \textit{for each } k \in \mathbb{N}, \tag{4.3}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0, \tag{4.4}$$

$$\lim_{n\to\infty} \sum_k \left|\Delta(a_{nk} - \alpha_k)\right| = \alpha. \tag{4.5}$$

(iii) (*cf.* [14]) $A \in (f : f)$ *if and only if* (4.2) *holds and*

$$f\text{-}\lim_{n\to\infty} a_{nk} = \alpha_k \quad \textit{for each fixed } k \in \mathbb{N}, \tag{4.6}$$

$$f\text{-}\lim_{n\to\infty} \sum_k a_{nk} = \alpha, \tag{4.7}$$

$$\lim_{m\to\infty} \sum_k \bigl|\Delta\bigl[a(n,k,m) - \alpha_k\bigr]\bigr| = 0 \quad \textit{uniformly in } n. \tag{4.8}$$

Theorem 4.2 *Suppose that the entries of the infinite matrices* $E = (e_{nk})$ *and* $F = (f_{nk})$ *are connected with the relation*

$$e_{nk} = \sum_{j=k}^{\infty} u_j v_k f_{nj} \quad \textit{or} \quad f_{nk} = \frac{1}{u_k}\left(\frac{e_{nk}}{v_k} - \frac{e_{n,k+1}}{v_{k+1}}\right) \tag{4.9}$$

for all $k, n \in \mathbb{N}$ *and* μ *be any given sequence space. Then* $E \in (f(G) : \mu)$ *if and only if* $\{e_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^\beta$ *for all* $n \in \mathbb{N}$ *and* $F \in (f : \mu)$.

Proof Let μ be any given sequence. Suppose that (4.9) holds between the infinite matrices $E = (e_{nk})$ and $F = (f_{nk})$, and we take into account that the spaces $f(G)$ and f are linearly isomorphic.

Let $E \in (f(G) : \mu)$ and take any $y = (y_k) \in f$. Then $FG(u,v)$ exists and $\{e_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^\beta$, which yields the result that (4.9) is necessary and $\{f_{nk}\}_{k\in\mathbb{N}} \in f^\beta$ for each $n \in \mathbb{N}$. Hence, Fy exists for each $y \in f$ and thus by letting $m \to \infty$ in the equality

$$\sum_{k=0}^{m} e_{nk} x_k = \sum_{k=0}^{m-1} \frac{1}{u_k}\left(\frac{e_{nk}}{v_k} - \frac{e_{n,k+1}}{v_{k+1}}\right) y_k + \frac{e_{nm}}{u_m v_m} y_m \quad \text{for all } m, n \in \mathbb{N}$$

we obtain $Ex = Fy$, which leads to the consequence $F \in (f : \mu)$.

Conversely, let $\{e_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^\beta$ for each $n \in \mathbb{N}$ and $F \in (f : \mu)$, and we take any $x = (x_k) \in f(G)$. Then Ex exists. Therefore, we obtain from the equality

$$\sum_{k=0}^{m} f_{nk} y_k = \sum_{k=0}^{m} \sum_{j=k}^{m} u_j v_k f_{nj} x_k \quad \text{for all } m, n \in \mathbb{N}$$

as $m \to \infty$ the result that $Fy = Ex$ and this shows that $E \in (f(G) : \mu)$. This completes the proof. □

By changing the roles of the spaces $f_0(G)$ and $f(G)$ with μ, we have the following theorem.

Theorem 4.3 *Suppose that the entries of the infinite matrices* $A = (a_{nk})$ *and* $B = (b_{nk})$ *are connected with the relation* (4.1) *and* μ *be any given sequence space. Then* $A \in (\mu : f(G))$ *if and only if* $B \in (\mu : f)$.

Proof Let $z = (z_k) \in \mu$ and consider the following equality:

$$\sum_{j=0}^{n} \sum_{k=0}^{m} v_n u_j a_{jk} z_k = \sum_{k=0}^{m} b_{nk} z_k \quad \text{for all } m, n \in \mathbb{N}. \tag{4.10}$$

Equation (4.10) yields as $m \to \infty$ the result that $(Bz)_n = \{G(u,v)(Az)\}_n$. Therefore, one can immediately observe from this that $Az \in f(G)$ whenever $z \in \mu$ if and only if $Bz \in f$ whenever $z \in \mu$. This completes the proof. □

It is of course so that Theorem 4.2 and Theorem 4.3 have several consequences depending on the choice of sequence space μ and the sequences $u = (u_n)$ and $v = (v_k)$. Therefore by Theorem 4.2 and Theorem 4.3, necessary and sufficient conditions for $(f(G) : \mu)$ and $(\mu : f(G))$ may be derived by replacing the entries of E and $\widetilde{A}$ by those of the entries of $F = EG^{-1}(u,v)$ and $B = G(u,v)\widetilde{A}$, respectively, where the necessary and sufficient conditions on the matrices F and B are read from the concerning results in the existing literature.

If we get the space ℓ_∞ and the spaces e^r_∞, r^t_∞, and X_∞, which are isomorphic to ℓ_∞ instead of μ in Theorem 4.2, we obtain the following corollaries.

Corollary 4.4 *$A \in (f(G) : \ell_\infty)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^\beta$ and*

$$\sup_{n\in\mathbb{N}} \sum_k |\widetilde{a}_{nk}| < \infty. \tag{4.11}$$

Corollary 4.5 *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $C = (c_{nk})$ by*

$$c_{nk} = \sum_{j=0}^{n} \binom{n}{j} (1-r)^{n-j} r^j a_{jk} \quad (k, n \in \mathbb{N}).$$

Then the necessary and sufficient conditions in order for A to belong to the class $(f(G) : e^r_\infty)$ are obtained from Theorem 4.2 by replacing the entries of the matrix A by those of the matrix C; here $e^r_\infty = \{x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} |\sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k x_k| < \infty\}$ as defined Altay et al. [15] *and Altay and Başar* [16].

Corollary 4.6 *Let $A = (a_{nk})$ be an infinite matrix and define the matrix $D = (d_{nk})$ by*

$$d_{nk} = \frac{1}{T_n} \sum_{j=0}^{n} t_j a_{jk} \quad (k, n \in \mathbb{N}),$$

where $T_n = \sum_{k=0}^{n} t_k$ for all $n \in \mathbb{N}$. Then the necessary and sufficient conditions in order for A to belong to the class $(f(G) : r^t_\infty)$ are obtained from in Theorem 4.2 by replacing the entries of the matrix A by those of the matrix D; here r^t_∞ is the space of all sequences whose R^t-transforms are in the space ℓ_∞ [17].

Remark 4.7 In the case $t = e$ in the space r^t_∞, this space reduces to the Cesàro sequence space of non-absolute type X_∞ [18]. Then Corollary 4.6 also includes the characterization of class $(f(G) : X_\infty)$, as a special case.

As in Corollaries 4.4-4.6 and Remark 4.7, the following corollaries are obtained for $\mu = \{f, f(E), \widehat{f}, \widetilde{f}\}$; here the spaces $f(E), \widehat{f}, \widetilde{f}$ are isomorphic to the space f.

Corollary 4.8 *$A \in (f(G):f)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^{\beta}$, (4.11) holds and there are $\alpha_k, \alpha \in \mathbb{C}$ such that*

$$f\text{-}\lim_{n\to\infty} \widetilde{a}_{nk} = \alpha_k \quad \text{for each } k \in \mathbb{N}; \tag{4.12}$$

$$f\text{-}\lim_{n\to\infty} \sum_k \widetilde{a}_{nk} = \alpha; \tag{4.13}$$

$$\lim_{n\to\infty} \sum_k \bigl|\Delta\bigl[\widetilde{a}(n,k,m) - \alpha_k\bigr]\bigr| = 0 \quad \text{uniformly in } m, \tag{4.14}$$

where $\widetilde{a}(n,k,m) = \frac{1}{n+1}\sum_{j=0}^{n} \widetilde{a}_{m+j,k}$.

Corollary 4.9 *Let $A = (a_{nk})$ be an infinite matrix and the matrix $C = (c_{nk})$ be defined by Corollary* 4.5. *Then the necessary and sufficient conditions in order for A to belong to the class $(f(G):f(E))$ are obtained from Corollary* 4.8 *by replacing the entries of the matrix A by those of the matrix C; here*

$$f(E) = \left\{x = (x_k) \in \omega : \exists l \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{j=0}^{m} \sum_{k=0}^{n+j} \frac{\binom{n+j}{k}(1-r)^{n+j-k} r^k x_k}{m+1} = l \text{ uniformly in } n\right\}$$

defined by Kirişci [19].

Corollary 4.10 *Let $A = (a_{nk})$ be an infinite matrix and the matrix $H = (h_{nk})$ be defined by $h_{nk} = sa_{n-1,k} + ra_{nk}$. Then the necessary and sufficient conditions in order for A to belong to the class $(f(G):\widehat{f})$ are obtained from Corollary* 4.8 *by replacing the entries of the matrix A by those of the matrix H; here*

$$\widehat{f} = \left\{x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{j=0}^{m} \frac{sx_{k-1+j} + rx_{k+j}}{m+1} = \alpha \text{ uniformly in } k\right\}$$

defined by Başar and Kirişci [10].

Corollary 4.11 *Let $A = (a_{nk})$ be an infinite matrix and the matrix $M = (m_{nk})$ defined by $m_{nk} = \sum_{j=k}^{\infty} \frac{a_{nj}}{j+1}$. Then the necessary and sufficient conditions in order for A to belong to the class $(f(G):\widetilde{f})$ are obtained from Corollary* 4.8 *by replacing the entries of the matrix A by those of the matrix M; here*

$$\widetilde{f} = \left\{x = (x_k) \in \omega : \exists \alpha \in \mathbb{C} \ni \lim_{m\to\infty} \sum_{k=0}^{n} \frac{1}{n+1} \sum_{j=0}^{k} \frac{x_{j+p}}{k+1} = \alpha \text{ uniformly in } p\right\}$$

defined by Kayaduman and Şengönul [20].

Now, we list the following conditions:

$$\sup_{n\in\mathbb{N}} \sum_k \frac{1}{q+1} \left|\sum_{i=n}^{n+q} a_{ik}\right| < \infty, \tag{4.15}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha, \tag{4.16}$$

$$\sup_{n\in\mathbb{N}}\sum_k |\Delta a_{nk}| < \infty, \tag{4.17}$$

$$\lim_{k\to\infty} a_{nk} = 0 \quad \text{for each fixed } n \in \mathbb{N}, \tag{4.18}$$

$$\lim_{n\to\infty}\sum_k \left|\Delta^2 a_{nk}\right| = \alpha, \tag{4.19}$$

$$\lim_{m\to\infty}\sum_k \left|a(n,k,m) - \alpha_k\right| = 0 \quad \text{uniformly in } n, \tag{4.20}$$

$$\lim_{q\to\infty}\sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q} \Delta\left[\left(a(n+i,k) - \alpha_k\right)\right]\right| = 0 \quad \text{uniformly in } n, \tag{4.21}$$

$$\sup_{n\in\mathbb{N}}\sum_k \left|\Delta a(n,k)\right| < \infty, \tag{4.22}$$

$$f\text{-}\lim a(n,k) = \alpha_k \quad \text{exists for each fixed } k \in \mathbb{N}, \tag{4.23}$$

$$\lim_{q\to\infty}\sum_k \frac{1}{q+1}\left|\sum_{i=0}^{q} \Delta^2\left[a(n+i,k) - \alpha_k\right]\right| = 0 \quad \text{uniformly in } n, \tag{4.24}$$

$$\sup_{n\in\mathbb{N}}\sum_k \left|a(n,k)\right| < \infty, \tag{4.25}$$

$$\sum_k a_{nk} = \alpha_k \quad \text{for each fixed } k \in \mathbb{N}, \tag{4.26}$$

$$\sum_n \sum_k a_{nk} = \alpha, \tag{4.27}$$

$$\lim_{n\to\infty}\sum_k \left|\Delta\left[a(n,k) - \alpha_k\right]\right| = 0. \tag{4.28}$$

Prior to giving some consequences as an application of this idea, we give the following basic lemma, which is the collection of the characterization of matrix transformations related to almost convergence.

Lemma 4.12 *Let $A = (a_{nk})$ be an infinite matrix. Then,*

(i) $A = (a_{nk}) \in (fs : \ell_\infty)$ *if and only if* (4.17) *and* (4.18) *hold*.
(ii) $A = (a_{nk}) \in (fs : c)$ *if and only if* (4.3) *and* (4.17)-(4.19) *hold* [21].
(iii) $A = (a_{nk}) \in (c : f)$ *if and only if* (4.6), (4.7), *and* (4.15) *hold* [22].
(iv) $A = (a_{nk}) \in (\ell_\infty : f)$ *if and only if* (4.6), (4.8), *and* (4.15) *hold* [14].
(v) $A = (a_{nk}) \in (bs : f)$ *if and only if* (4.6), (4.17),(4.18), *and* (4.21) *hold* [23].
(vi) $A = (a_{nk}) \in (fs : f)$ *if and only if* (4.6), (4.8), (4.18), *and* (4.24) *hold* [24].
(vii) $A = (a_{nk}) \in (cs : f)$ *if and only if* (4.6) *and* (4.17) *hold* [25].
(viii) $A = (a_{nk}) \in (bs : fs)$ *if and only if* (4.18) *and* (4.21)-(4.23) *hold* [23].
(ix) $A = (a_{nk}) \in (fs : fs)$ *if and only if* (4.17) *and* (4.22)-(4.24) *hold* [24].
(x) $A = (a_{nk}) \in (cs : fs)$ *if and only if* (4.22) *and* (4.23) *hold* [25].
(xi) $A = (a_{nk}) \in (f : cs)$ *if and only if* (4.25)-(4.28) *hold* [24].

Now, we can give the following results.

Corollary 4.13 *The following statements hold*:

(i) *$A = (a_{nk}) \in (f(G) : c)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^{\beta}$ for all $n \in \mathbb{N}$ and* (4.2)-(4.5) *hold with $\widetilde{a}_{nk}$ instead of a_{nk}.*
(ii) *$A \in (f(G) : c_0)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^{\beta}$ for all $n \in \mathbb{N}$ and* (4.2) *holds,* (4.3) *and* (4.5) *hold with $\alpha_k = 0$ and* (4.4) *holds with $\alpha = 0$ as $\overline{a}_{nk}$ instead of a_{nk}.*
(iii) *$A = (a_{nk}) \in (f(G) : bs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^{\beta}$ for all $n \in \mathbb{N}$ and* (4.25) *holds with $\widetilde{a}_{nk}$ instead of a_{nk}.*
(iv) *$A = (a_{nk}) \in (f(G) : cs)$ if and only if $\{a_{nk}\}_{k\in\mathbb{N}} \in \{f(G)\}^{\beta}$ for all $n \in \mathbb{N}$ and* (4.25)-(4.28) *hold with $\widetilde{a}_{nk}$ instead of a_{nk}.*

Corollary 4.14 *We have*:
(i) *$A = (a_{nk}) \in (\ell_\infty : f(G))$ if and only if* (4.2), (4.6), *and* (4.20) *hold with b_{nk} instead of a_{nk}.*
(ii) *$A = (a_{nk}) \in (f : f(G))$ if and only if* (4.2), (4.6), (4.7), *and* (4.8) *hold with b_{nk} instead of a_{nk}.*
(iii) *$A = (a_{nk}) \in (c : f(G))$ if and only if* (4.2), (4.6), *and* (4.7) *hold with b_{nk} instead of a_{nk}.*
(iv) *$A = (a_{nk}) \in (bs : f(G))$ if and only if* (4.17), (4.18), (4.6), *and* (4.21) *hold with b_{nk} instead of a_{nk}.*
(v) *$A = (a_{nk}) \in (fs : f(G))$ if and only if* (4.18), (4.6), (4.8), *and* (4.21) *hold with b_{nk} instead of a_{nk}.*
(vi) *$A = (a_{nk}) \in (cs : f(G))$ if and only if* (4.17) *and* (4.6) *hold with b_{nk} instead of a_{nk}.*
(vii) *$A = (a_{nk}) \in (bs : fs(G))$ if and only if* (4.18), (4.21), *and* (4.23) *hold with b_{nk} instead of a_{nk}.*
(viii) *$A = (a_{nk}) \in (fs : fs(G))$ if and only if* (4.21)-(4.24) *hold with b_{nk} instead of a_{nk}.*
(ix) *$A = (a_{nk}) \in (cs : fs(G))$ if and only if* (4.22) *and* (4.23) *hold with b_{nk} instead of a_{nk}.*

Here $fs(G)$ denotes the domain of the $G(u,v)$-generalized weighted mean in the sequence space fs.

5 Conclusion

As an essential work on the algebraic and topological properties of the spaces f_0 and f, Başar and Kirişçi [10] have recently introduced the sequence spaces $\widehat{f}_0$ and $\widehat{f}$ derived by the domain of the generalized difference matrix $B(r,s)$ in the sequence spaces f_0 and f, respectively. Following Başar and Kirişçi [10], Kayaduman and Şengönül have studied the domain $\widetilde{f}_0$ and $\widetilde{f}$ of the Cesàro mean of order one in the spaces f_0 and f, in [20]. They have determined the β- and γ-duals of the new spaces $\widetilde{f}_0$ and $\widetilde{f}$, and they characterize some classes of matrix transformations on/in the new sequence spaces. They complete the paper by a nice section including some core theorems related to the matrix classes on/in the new sequence space $\widetilde{f}$. Quite recently, in [26], Sönmez has introduced the domain $f(B)$ of the triple band matrix $B(r,s,t)$ in the sequence space f. In this paper, the β- and γ-duals of the space $f(B)$ are determined. Furthermore, the classes $(f(B) : \mu)$ and $(\mu : f(B))$ of infinite matrices are characterized together with some other classes, where μ is any given sequence space. Finally, in [27] Candan has studied the sequence spaces $f_0(\widetilde{B})$ and $f(\widetilde{B})$ as the domain of the double sequential band matrix $\widetilde{B}(\widetilde{r},\widetilde{s})$ in the sequence spaces f_0 and f.

Since Kirişçi and Başar [28], Başar and Kirişçi [10], Kayaduman and Şengönül [20], Sönmez [26, 29], and Candan [27, 30] are recent works on the domain of certain triangle matrices in the spaces f_0, f, and in the classical sequence spaces, the present paper is their natural continuation. Also these spaces are special cases of the notion of A-almost convergence and $\mathcal{F}_B$-convergence ([31, 32]) as well as analogous to the definition introduced in [33].

Finally, we should note that the investigation of the domain of some particular limitation matrices, namely Cesàro means of order m, Nörlund means, *etc.*, in the spaces f_0 and f will lead to new results which are not comparable with the present results.

Competing interests
The author declares that they have no competing interests.

Author's contributions
MK defined the new almost sequence spaces derived by generalized weighted mean and studied some properties. MK computed the duals of new spaces and characterized the matrix classes. In last section, it was summarized to studies in manuscripts and given some open problems by MK. The author read and approved the final manuscript.

Article's information
Some of the results of this study presented in First International Conference on Analysis and Applied Mathematics (ICAAM 2012, Gumushane University, Turkey) [34].

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