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Strongly singular Calderón-Zygmund operators and commutators on weighted Morrey spaces

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Abstract
In this paper, the authors establish the boundedness of the strongly singular Calderón-Zygmund operator on weighted Morrey spaces. Moreover, the boundedness of the commutator generated by the strongly singular Calderón-Zygmund operator and the weighted *BMO* function on weighted Morrey spaces is also obtained.
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1 Introduction

The strongly singular non-convolution operator was introduced by Alvarez and Milman in [1], whose properties are similar to those of the Calderón-Zygmund operator, but the kernel is more singular near the diagonal than that of the standard case. Furthermore, following a suggestion of Stein, the authors in [1] showed that the pseudo-differential operators with symbols in the class $\mathcal{S}^{-\beta}_{\alpha,\delta}$, where $0 < \delta \le \alpha < 1$ and $n(1-\alpha)/2 \le \beta < n/2$, are included in the strongly singular Calderón-Zygmund operator. Thus, the strongly singular Calderón-Zygmund operator correlates closely with both the theory of Calderón-Zygmund singular integrals in harmonic analysis and the theory of pseudo-differential operators in *PDE*.

Definition 1.1 Let $T : \mathcal{S} \to \mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are satisfied.

(1) T can be extended into a continuous operator from $L^2(\mathbf{R}^n)$ into itself.

(2) There exists a function $K(x,y)$ continuous away from the diagonal $\{(x,y) : x = y\}$ such that

$$\bigl|K(x,y) - K(x,z)\bigr| + \bigl|K(y,x) - K(z,x)\bigr| \le C\frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}},$$

if $2|y-z|^\delta \le |x-z|$ for some $0 < \delta \le 1$ and $0 < \alpha < 1$. And $\langle Tf, g\rangle = \iint K(x,y)f(y)g(x)\,dy\,dx$, for $f, g \in \mathcal{S}$ with disjoint supports.

(3) For some $n(1-\alpha)/2 \le \beta < n/2$, both T and its conjugate operator T^* can be extended to continuous operators from L^q to L^2, where $1/q = 1/2 + \beta/n$.

Alvarez and Milman [1, 2] discussed the boundedness of the strongly singular Calderón-Zygmund operator on Lebesgue spaces. Lin [3] proved the boundedness of the strongly singular Calderón-Zygmund operator on Morrey spaces. Furthermore, Lin and Lu [4] showed the boundedness of the strongly singular Calderón-Zygmund operator on Herz-type Hardy spaces.

Suppose that T is a strongly singular Calderón-Zygmund operator and b is a locally integrable function on $\mathbf{R}^n$. The commutator $[b, T]$ generated by b and T is defined as follows:

$$[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

The authors in [5] obtained the boundedness of the commutators generated by strongly singular Calderón-Zygmund operators and Lipschitz functions on Lebesgue spaces. Lin and Lu [4] proved the boundedness of the commutators of strongly singular Calderón-Zygmund operators on Hardy-type spaces. Moreover, Lin and Lu [3, 6] discussed the boundedness of the commutator $[b, T]$ on Morrey spaces when b is a BMO function or a Lipschitz function, respectively.

The classical Morrey space was originally introduced by Morrey in [7] to study the local behavior of solutions of second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, one can refer to [7, 8]. In [9], Chiarenza and Frasca showed the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator and the Calderón-Zygmund singular integral operator on Morrey spaces. In 2010, Fu and Lu [10] established the boundedness of weighted Hardy operators and their commutators on Morrey spaces.

In 2009, Komori and Shirai [11] defined the weighted Morrey spaces and studied the boundedness of the Hardy-Littlewood maximal operator, the fractional integral operator, and the classical Calderón-Zygmund singular integral operator on these weighted spaces. In 2012, Wang [12] showed the boundedness of commutators generated by classical Calderón-Zygmund operators and weighted BMO functions on weighted Morrey spaces. In 2013, the authors in [13] proved the boundedness of some sublinear operators and their commutators on weighted Morrey spaces.

Inspired by the above results, the main purpose of this paper is to overcome the stronger singularity near the diagonal and establish the boundedness properties of the strongly singular Calderón-Zygmund operators and their commutators on weighted Morrey spaces.

Let us first recall some necessary definitions and notations.

Definition 1.2 ([14]) A non-negative measurable function ω is said to be in the Muckenhoupt class A_p with $1 < p < \infty$ if for every cube Q in $\mathbf{R}^n$, there exists a positive constant C independent of Q such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{1-p'}\,dx\right)^{p-1} \le C,$$

where Q denotes a cube in $\mathbf{R}^n$ with the side parallel to the coordinate axes and $1/p + 1/p' = 1$. When $p = 1$, a non-negative measurable function ω is said to belong to A_1, if there exists a constant $C > 0$ such that for any cube Q,

$$\frac{1}{|Q|}\int_Q \omega(y)\,dy \le C\omega(x), \quad \text{a.e. } x \in Q.$$

It is well known that if $\omega \in A_p$ with $1 < p < \infty$, then $\omega \in A_r$ for all $r > p$, and $\omega \in A_q$ for some $1 < q < p$.

Definition 1.3 ([15]) A weighted function ω belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^r\,dx\right)^{\frac{1}{r}} \le C\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)$$

holds for every cube Q in $\mathbf{R}^n$.

It is well known that if $\omega \in A_p$ with $1 \le p < \infty$, then there exists a $r > 1$ such that $\omega \in RH_r$. It follows directly from Hölder's inequality that $\omega \in RH_r$ implies $\omega \in RH_s$ for all $1 < s < r$.

Definition 1.4 Let $1 \le p < \infty$ and ω be a weighted function. A locally integrable function b is said to be in the weighted BMO space $BMO_p(\omega)$ if

$$\|b\|_{BMO_p(\omega)} = \sup_Q\left(\frac{1}{\omega(Q)}\int_Q \left|b(x) - b_Q\right|^p \omega(x)^{1-p}\,dx\right)^{1/p} < \infty,$$

where $b_Q = \frac{1}{|Q|}\int_Q b(y)\,dy$ and the supremum is taken over all cubes $Q \subset \mathbf{R}^n$.

Moreover, we denote simply $BMO(\omega)$ when $p = 1$.

Definition 1.5 The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \left|f(y)\right|\,dy.$$

We set $M_s(f) = M(|f|^s)^{1/s}$, where $0 < s < \infty$.

The sharp maximal operator $M^\sharp$ is defined by

$$M^\sharp(f)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \left|f(y) - f_Q\right|\,dy \sim \sup_{Q\ni x}\inf_{a\in\mathbf{C}}\frac{1}{|Q|}\int_Q \left|f(y) - a\right|\,dy,$$

where $f_Q = \frac{1}{|Q|}\int_Q f(x)\,dx$. We define the t-sharp maximal operator $M_t^\sharp(f) = M^\sharp(|f|^t)^{1/t}$, where $0 < t < 1$.

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega(f)(x) = \sup_{Q\ni x}\frac{1}{\omega(Q)}\int_Q \left|f(y)\right|\omega(y)\,dy.$$

We also set $M_{s,\omega}(f) = M_\omega(|f|^s)^{1/s}$, where $0 < s < \infty$.

Definition 1.6 ([11]) Let $1 \le p < \infty$, $0 < k < 1$, and ω be a weighted function. Then the weighted Morrey space $L^{p,k}(\omega)$ is defined by

$$L^{p,k}(\omega) = \left\{f \in L^p_{\mathrm{loc}}(\omega) : \|f\|_{L^{p,k}(\omega)} < \infty\right\},$$

where

$$\|f\|_{L^{p,k}(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)^k} \int_Q |f(x)|^p \omega(x)\, dx \right)^{1/p},$$

and the supremum is taken over all cubes Q in $\mathbf{R}^n$.

Definition 1.7 ([11]) Let $1 \le p < \infty$ and $0 < k < 1$. Then for two weighted functions u and v, the weighted Morrey space $L^{p,k}(u,v)$ is defined by

$$L^{p,k}(u,v) = \left\{ f \in L^p_{\mathrm{loc}}(u) : \|f\|_{L^{p,k}(u,v)} < \infty \right\},$$

where

$$\|f\|_{L^{p,k}(u,v)} = \sup_Q \left(\frac{1}{v(Q)^k} \int_Q |f(x)|^p u(x)\, dx \right)^{1/p}.$$

2 Main results

Now we state our main results as follows.

Theorem 2.1 *Let T be a strongly singular Calderón-Zygmund operator, and α, β, δ be given in Definition* 1.1. *If* $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$, $0 < k < 1$, *and* $\omega \in A_{2\beta p/[n(1-\alpha)+2\beta]}$, *then T is bounded on* $L^{p,k}(\omega)$.

Theorem 2.2 *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given in Definition* 1.1 *and* $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$. *Suppose* $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$, $0 < k < 1$, *and* $\omega \in A_1 \cap RH_r$ *with* $r > \frac{(n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$. *If* $b \in BMO(\omega)$, *then* $[b,T]$ *is bounded from* $L^{p,k}(\omega)$ *to* $L^{p,k}(\omega^{1-p},\omega)$.

If we consider the extreme cases $\alpha \to 1$ and $\beta \to 0$ in Definition 1.1, then the strongly singular Calderón-Zygmund operator comes back to the classical Calderón-Zygmund operator. Thus, we get the boundedness of the classical Calderón-Zygmund operator and its commutator on weighted Morrey spaces as corollaries of Theorem 2.1 and Theorem 2.2.

Corollary 2.1 *Let T be a classical Calderón-Zygmund operator. If* $1 < p < \infty$, $0 < k < 1$, *and* $\omega \in A_p$, *then T is bounded on* $L^{p,k}(\omega)$.

Corollary 2.2 *Let T be a classical Calderón-Zygmund operator,* $1 < p < \infty$, $0 < k < 1$ *and* $\omega \in A_1$. *If* $b \in BMO(\omega)$, *then* $[b,T]$ *is bounded from* $L^{p,k}(\omega)$ *to* $L^{p,k}(\omega^{1-p},\omega)$.

Remark 2.1 Actually, Corollary 2.1 and Corollary 2.2 have been exactly obtained in [11] and [12] in the special case $\delta = 1$. Thus, from this perspective, Theorem 2.1 and Theorem 2.2 generalized the corresponding results in [11, 12], and the range of the index in Theorem 2.1 and Theorem 2.2 is reasonable.

3 Preliminaries

Before we give the proofs of our main results, we need some lemmas.

Lemma 3.1 ([1]) *If T is a strongly singular Calderón-Zygmund operator, then T can be defined to be a continuous operator from L^∞ to BMO.*

Lemma 3.2 ([2]) *If T is a strongly singular Calderón-Zygmund operator, then T is of weak (L^1, L^1) type.*

By Lemma 3.1, Lemma 3.2, Definition 1.1, and interpolation theory, we find that T is bounded on L^p, $1 < p < \infty$. Besides the (L^p, L^p)-boundedness, the strongly singular Calderón-Zygmund operator T still has other kinds of boundedness properties on Lebesgue spaces. By interpolating between $(L^2, L^{q'})$ and (L^∞, BMO), where q is given in Definition 1.1 and $1/q + 1/q' = 1$, T is bounded from L^u to L^v with $2 \leq u < \infty$ and $v = \frac{uq'}{2}$. It is easy to see that $0 < \frac{u}{v} \leq \alpha$ in this situation. Then we interpolate between $(L^2, L^{q'})$ and weak (L^1, L^1) to obtain the boundedness of T from L^u to L^v, where $1 < u \leq 2$ and $v = \frac{uq'}{2q' - uq' + 2u - 2}$. In this situation, $0 < \frac{u}{v} \leq \alpha$ if and only if $\frac{n(1-\alpha)+2\beta}{2\beta} \leq u \leq 2$. In a word, the boundedness properties of the strongly singular Calderón-Zygmund operator on Lebesgue spaces can be summarized as follows.

Remark 3.1 The strongly singular Calderón-Zygmund operator T is bounded on L^p for $1 < p < \infty$. And T is bounded from L^u to L^v, $\frac{n(1-\alpha)+2\beta}{2\beta} \leq u < \infty$ and $0 < \frac{u}{v} \leq \alpha$. In particular, if we restrict $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$ in (3) of Definition 1.1, then T is bounded from L^u to L^v, $\frac{n(1-\alpha)+2\beta}{2\beta} < u < \infty$, and $0 < \frac{u}{v} < \alpha$.

Lemma 3.3 ([16, 17]) *Let $\omega \in A_1$. Then for any $1 \leq p < \infty$, there exists an absolute constant $C > 0$ such that $\|b\|_{BMO_p(\omega)} \leq C\|b\|_{BMO(\omega)}$.*

Lemma 3.4 ([11]) *If $1 < p < \infty$, $0 < k < 1$, and $\omega \in A_p$, then M is bounded on $L^{p,k}(\omega)$.*

Lemma 3.5 ([12]) *Let $1 < p < \infty$, $0 < k < 1$, and $\omega \in A_\infty$, then for any $1 < s < p$, we have*

$$\left\|M_{s,\omega}(f)\right\|_{L^{p,k}(\omega)} \leq C\|f\|_{L^{p,k}(\omega)}.$$

Lemma 3.6 ([12]) *Let $0 < t < 1$, $1 < p < \infty$, and $0 < k < 1$. If $u, v \in A_\infty$, then we have*

$$\left\|M_t(f)\right\|_{L^{p,k}(u,v)} \leq C\left\|M_t^\sharp(f)\right\|_{L^{p,k}(u,v)}$$

for all functions f such that the left-hand side is finite. In particular, when $u = v = \omega$ and $\omega \in A_\infty$, we have

$$\left\|M_t(f)\right\|_{L^{p,k}(\omega)} \leq C\left\|M_t^\sharp(f)\right\|_{L^{p,k}(\omega)}$$

for all functions f such that the left-hand side is finite.

Lemma 3.7 *Given $\varepsilon > 0$, we have $\ln x \leq \frac{1}{\varepsilon}x^\varepsilon$, for all $x \geq 1$.*

Let $\varphi(x) = \ln x - \frac{1}{\varepsilon}x^\varepsilon$, $x \geq 1$. The above result comes from the monotone property of the function φ.

Lemma 3.8 *If T is a strongly singular Calderón-Zygmund operator, α, β, δ are given in Definition 1.1, and $0 < t < 1$, then for all $\frac{n(1-\alpha)+2\beta}{2\beta} \leq s < \infty$, there exists a positive constant C such that*

$$M_t^\sharp(Tf)(x) \leq CM_s(f)(x), \quad x \in \mathbf{R}^n$$

for every bounded and compactly supported function f.

Proof For any ball $B = B(x_0, r_B) \subset \mathbf{R}^n$ which contains x, there are two cases.

Case 1: $r_B > 1$.

We have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \big||T(f)(y)|^t - |T(f\chi_{(2B)^c})(x_0)|^t\big|\,dy\right)^{1/t} \\
&\quad\le \left(\frac{1}{|B|}\int_B |T(f)(y) - T(f\chi_{(2B)^c})(x_0)|^t\,dy\right)^{1/t} \\
&\quad\le C\left(\frac{1}{|B|}\int_B |T(f\chi_{2B})(y)|^t\,dy\right)^{1/t} \\
&\qquad + C\left(\frac{1}{|B|}\int_B |T(f\chi_{(2B)^c})(y) - T(f\chi_{(2B)^c})(x_0)|^t\,dy\right)^{1/t} \\
&\quad:= I_1 + I_2.
\end{aligned}$$

For I_1, by Hölder's inequality and the L^s-boundedness of T, we get

$$\begin{aligned}
I_1 &\le C\frac{1}{|B|}\int_B |T(f\chi_{2B})(y)|\,dy \\
&\le C\left(\frac{1}{|B|}\int_B |T(f\chi_{2B})(y)|^s\,dy\right)^{1/s} \\
&\le C\left(\frac{1}{|B|}\int_{2B} |f(y)|^s\,dy\right)^{1/s} \\
&\le CM_s(f)(x).
\end{aligned}$$

Since $r_B > 1$ and $2|y - x_0|^\alpha \le |z - x_0|$ for any $y \in B$, $z \in (2B)^c$, by Hölder's inequality and (2) of Definition 1.1, we have

$$\begin{aligned}
I_2 &\le \frac{C}{|B|}\int_B |T(f\chi_{(2B)^c})(y) - T(f\chi_{(2B)^c})(x_0)|\,dy \\
&\le \frac{C}{|B|}\int_B\int_{(2B)^c} |K(y,z) - K(x_0,z)||f(z)|\,dz\,dy \\
&\le \frac{C}{|B|}\int_B\int_{(2B)^c} \frac{|y - x_0|^\delta}{|z - x_0|^{n+\frac{\delta}{\alpha}}}|f(z)|\,dz\,dy \\
&\le Cr_B^\delta \sum_{j=1}^\infty \int_{2^{j+1}B\setminus 2^jB} \frac{|f(z)|}{|z - x_0|^{n+\frac{\delta}{\alpha}}}\,dz \\
&\le Cr_B^\delta \sum_{j=1}^\infty (2^j r_B)^{-\frac{\delta}{\alpha}} \frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |f(z)|\,dz \\
&\le Cr_B^{\delta-\frac{\delta}{\alpha}} \sum_{j=1}^\infty (2^j)^{-\frac{\delta}{\alpha}} \left(\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B} |f(z)|^s\,dz\right)^{1/s} \\
&\le CM_s(f)(x) r_B^{\delta-\frac{\delta}{\alpha}} \sum_{j=1}^\infty (2^j)^{-\frac{\delta}{\alpha}} \\
&\le CM_s(f)(x).
\end{aligned}$$

Case 2: $0 < r_B \leq 1$.

Denote $\tilde{B} = B(x_0, r_B^\alpha)$. There is

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \left| \left|T(f)(y)\right|^t - \left|T(f\chi_{(2\tilde{B})^c})(x_0)\right|^t \right| dy\right)^{1/t} \\
&\leq \left(\frac{1}{|B|}\int_B \left|T(f)(y) - T(f\chi_{(2\tilde{B})^c})(x_0)\right|^t dy\right)^{1/t} \\
&\leq C\left(\frac{1}{|B|}\int_B \left|T(f\chi_{2\tilde{B}})(y)\right|^t dy\right)^{1/t} \\
&\quad + C\left(\frac{1}{|B|}\int_B \left|T(f\chi_{(2\tilde{B})^c})(y) - T(f\chi_{(2\tilde{B})^c})(x_0)\right|^t dy\right)^{1/t} \\
&:= II_1 + II_2.
\end{aligned}$$

Since $\frac{n(1-\alpha)+2\beta}{2\beta} \leq s < \infty$, by Remark 3.1, there exists an l such that T is bounded from L^s into L^l and $0 < \frac{s}{l} \leq \alpha$. It follows from Hölder's inequality that

$$\begin{aligned}
II_1 &\leq \frac{C}{|B|}\int_B \left|T(f\chi_{2\tilde{B}})(y)\right| dy \\
&\leq C\left(\frac{1}{B}\int_B \left|T(f\chi_{2\tilde{B}})(y)\right|^l dy\right)^{1/l} \\
&\leq C|B|^{-1/l}\left(\int_{2\tilde{B}} \left|f(y)\right|^s dy\right)^{1/s} \\
&= Cr^{-n/l+\alpha n/s}\left(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}} \left|f(y)\right|^s dy\right)^{1/s} \\
&\leq CM_s(f)(x).
\end{aligned}$$

Since $0 < r_B \leq 1$ and $2|y - x_0|^\alpha \leq |z - x_0|$ for any $y \in B$, $z \in (2\tilde{B})^c$, similarly to I_2, we have

$$\begin{aligned}
II_2 &\leq \frac{C}{|B|}\int_B \left|T(f\chi_{(2\tilde{B})^c})(y) - T(f\chi_{(2\tilde{B})^c})(x_0)\right| dy \\
&\leq \frac{C}{|B|}\int_B \int_{(2\tilde{B})^c} \left|K(y,z) - K(x_0,z)\right|\left|f(z)\right| dz\, dy \\
&\leq \frac{C}{|B|}\int_B \int_{(2\tilde{B})^c} \frac{|y - x_0|^\delta}{|z - x_0|^{n+\frac{\delta}{\alpha}}}\left|f(z)\right| dz\, dy \\
&\leq Cr_B^\delta \sum_{j=1}^\infty \int_{2^{j+1}\tilde{B}\setminus 2^j\tilde{B}} \frac{|f(z)|}{|z - x_0|^{n+\frac{\delta}{\alpha}}}\, dz \\
&\leq Cr_B^\delta \sum_{j=1}^\infty \left(2^j r_B^\alpha\right)^{-\frac{\delta}{\alpha}} \frac{1}{|2^{j+1}\tilde{B}|}\int_{2^{j+1}\tilde{B}} \left|f(z)\right| dz \\
&\leq C\sum_{j=1}^\infty \left(2^j\right)^{-\frac{\delta}{\alpha}}\left(\frac{1}{|2^{j+1}\tilde{B}|}\int_{2^{j+1}\tilde{B}} \left|f(z)\right|^s dz\right)^{1/s} \\
&\leq CM_s(f)(x).
\end{aligned}$$

Therefore, combining the estimates in both cases, there is

$$M_t^\sharp(Tf)(x) \sim \sup_{B\ni x}\inf_{a\in\mathbf{C}}\left(\frac{1}{|B|}\int_B\left|\left|T(f)(y)\right|^t - a\right|dy\right)^{\frac{1}{t}} \le CM_s(f)(x). \qquad \square$$

Lemma 3.9 *Let $\omega \in A_1$ and f be a function in $BMO(\omega)$. Suppose $1 \le p < \infty$, $x \in \mathbf{R}^n$, and $r_1, r_2 > 0$. Then*

$$\begin{aligned}&\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\left|f(y) - f_{B(x,r_2)}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}\\ &\quad\le C\|f\|_{BMO(\omega)}\omega(x)\left(1 + \left|\ln\frac{r_2}{r_1}\right|\right)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}}.\end{aligned}$$

Proof Without loss of generality, we may assume that $0 < r_1 \le r_2$ and omit the case $0 < r_2 < r_1$ since their similarity. For $0 < r_1 \le r_2$, there are $k_1, k_2 \in \mathbf{Z}$ such that $2^{k_1-1} < r_1 \le 2^{k_1}$ and $2^{k_2-1} < r_2 \le 2^{k_2}$. Then $k_1 \le k_2$ and

$$(k_2 - k_1 - 1)\ln 2 < \ln\frac{r_2}{r_1} < (k_2 - k_1 + 1)\ln 2.$$

Thus, we have

$$\begin{aligned}&\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\left|f(y) - f_{B(x,r_2)}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}\\
&\quad\le\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\left|f(y) - f_{B(x,2^{k_1})}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}\\
&\qquad+\left(\left|f_{B(x,r_2)} - f_{B(x,2^{k_2})}\right| + \left|f_{B(x,2^{k_2})} - f_{B(x,2^{k_1})}\right|\right)\\
&\qquad\times\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}\\
&\quad\le\left(\frac{2^n}{|B(x,2^{k_1})|}\int_{B(x,2^{k_1})}\left|f(y) - f_{B(x,2^{k_1})}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}\\
&\qquad+\left(\left|f_{B(x,r_2)} - f_{B(x,2^{k_2})}\right| + \sum_{j=k_1}^{k_2-1}\left|f_{B(x,2^{j+1})} - f_{B(x,2^j)}\right|\right)\\
&\qquad\times\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}\\
&\quad\le C\|f\|_{BMO(\omega)}\omega(x)^{\frac{1}{p}} + \left(\frac{1}{|B(x,r_2)|}\int_{B(x,r_2)}\left|f(y) - f_{B(x,2^{k_2})}\right|dy\right.\\
&\qquad\left.+\sum_{j=k_1}^{k_2-1}\frac{1}{|B(x,2^j)|}\int_{B(x,2^j)}\left|f(y) - f_{B(x,2^{j+1})}\right|dy\right)\\
&\qquad\times\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}}.\end{aligned}$$

Write

$$\begin{aligned}
&\frac{1}{|B(x,r_2)|}\int_{B(x,r_2)}\left|f(y)-f_{B(x,2^{k_2})}\right|dy \\
&\quad\le \frac{2^n}{|B(x,2^{k_2})|}\left(\int_{B(x,2^{k_2})}\left|f(y)-f_{B(x,2^{k_2})}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_{B(x,2^{k_2})}\omega(y)\,dy\right)^{\frac{1}{p'}} \\
&\quad\le 2^n\frac{\omega(B(x,2^{k_2}))}{|B(x,2^{k_2})|} \\
&\qquad\times\left(\frac{1}{\omega(B(x,2^{k_2}))}\int_{B(x,2^{k_2})}\left|f(y)-f_{B(x,2^{k_2})}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}} \\
&\quad\le 2^n\|f\|_{BMO(\omega)}\omega(x)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{|B(x,2^j)|}\int_{B(x,2^j)}\left|f(y)-f_{B(x,2^{j+1})}\right|dy \\
&\quad\le \frac{2^n}{|B(x,2^{j+1})|}\left(\int_{B(x,2^{j+1})}\left|f(y)-f_{B(x,2^{j+1})}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}} \\
&\qquad\times\left(\int_{B(x,2^{j+1})}\omega(y)\,dy\right)^{\frac{1}{p'}} \\
&\quad\le 2^n\|f\|_{BMO(\omega)}\omega(x).
\end{aligned}$$

If $1 < p < \infty$, then by the fact $\omega \in A_1 \subset A_{p'}$, we have

$$\begin{aligned}
&\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}} \\
&\quad=\left[\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\omega(y)^{1-p}\,dy\right)^{p'-1}\right]^{\frac{1}{p(p'-1)}} \\
&\quad\le C\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\omega(y)\,dy\right)^{-\frac{1}{p(p'-1)}} \\
&\quad= C\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}}.
\end{aligned}$$

If $p = 1$, then the above estimate holds obviously.

Thus,

$$\begin{aligned}
&\left(\frac{1}{|B(x,r_1)|}\int_{B(x,r_1)}\left|f(y)-f_{B(x,r_2)}\right|^p\omega(y)^{1-p}\,dy\right)^{\frac{1}{p}} \\
&\quad\le C\|f\|_{BMO(\omega)}\omega(x)^{\frac{1}{p}}
\end{aligned}$$

$$+ C\left(2^n\|f\|_{BMO(\omega)}\omega(x) + \sum_{j=k_1}^{k_2-1} 2^n\|f\|_{BMO(\omega)}\omega(x)\right)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}}$$
$$\leq C\|f\|_{BMO(\omega)}\omega(x)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}} + C\|f\|_{BMO(\omega)}\omega(x)(k_2-k_1+1)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}}$$
$$\leq C\|f\|_{BMO(\omega)}\omega(x)\left(1+\left|\ln\frac{r_2}{r_1}\right|\right)\left(\frac{\omega(B(x,r_1))}{|B(x,r_1)|}\right)^{-\frac{1}{p'}}.$$

This completes the proof of Lemma 3.9. □

Lemma 3.10 *Let T be a strongly singular Calderón-Zygmund operator, α, β, δ be given in Definition 1.1 and $\frac{n(1-\alpha)}{2} < \beta < \frac{n}{2}$. Let $0 < t < 1$, $\frac{n(1-\alpha)+2\beta}{2\beta} < s < \infty$, $\omega \in A_1 \cap RH_r$ with $r > \frac{(n(1-\alpha)+2\beta)(s-1)}{2\beta s - n(1-\alpha) - 2\beta}$, and $b \in BMO(\omega)$, then we have*

$$M_t^\sharp([b,T]f)(x) \leq C\|b\|_{BMO(\omega)}\big(\omega(x)M_{s,\omega}(Tf)(x) + \omega(x)M_{s,\omega}(f)(x)\big), \quad a.e.\ x \in \mathbf{R}^n.$$

Proof For any ball $B = B(x, r_B)$ with the center x and radius r_B, there are two cases.

Case 1: $r_B > 1$.

We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2B}$ and χ_{2B} denotes the characteristic function of $2B$. Observe that

$$\begin{aligned}
&[b,T](f)(y) \\
&\quad = \big(b(y) - b_{2B}\big)T(f)(y) - T\big((b-b_{2B})f\big)(y) \\
&\quad = \big(b(y) - b_{2B}\big)T(f)(y) - T\big((b-b_{2B})f_1\big)(y) - T\big((b-b_{2B})f_2\big)(y).
\end{aligned}$$

Since $0 < t < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B \Big|\big|[b,T](f)(y)\big|^t - \big|T\big((b-b_{2B})f_2\big)(x)\big|^t\Big|\,dy\right)^{1/t} \\
&\quad \leq \left(\frac{1}{|B|}\int_B \big|[b,T](f)(y) + T\big((b-b_{2B})f_2\big)(x)\big|^t\,dy\right)^{1/t} \\
&\quad \leq C\left(\frac{1}{|B|}\int_B \big|\big(b(y)-b_{2B}\big)T(f)(y)\big|^t\,dy\right)^{1/t} \\
&\qquad + C\left(\frac{1}{|B|}\int_B \big|T\big((b-b_{2B})f_1\big)(y)\big|^t\,dy\right)^{1/t} \\
&\qquad + C\left(\frac{1}{|B|}\int_B \big|T\big((b-b_{2B})f_2\big)(y) - T\big((b-b_{2B})f_2\big)(x)\big|^t\,dy\right)^{1/t} \\
&\quad := I_1 + I_2 + I_3.
\end{aligned}$$

We are now going to estimate each term, respectively. Since $\omega \in A_1$, it follows from Hölder's inequality and Lemma 3.3 that

$$\begin{aligned}
I_1 &\leq \frac{C}{|B|}\int_B \big|\big(b(y)-b_{2B}\big)T(f)(y)\big|\,dy \\
&\leq \frac{C}{|B|}\left(\int_B \big|b(y)-b_{2B}\big|^{s'}\omega(y)^{1-s'}\,dy\right)^{1/s'}\left(\int_B \big|Tf(y)\big|^s\omega(y)\,dy\right)^{1/s}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|b\|_{BMO(\omega)} \frac{\omega(B)}{|B|} \left(\frac{1}{\omega(B)} \int_B \left| Tf(y) \right|^s \omega(y)\, dy \right)^{1/s} \\
&\le C\|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(Tf)(x).
\end{aligned}
$$

Applying Kolmogorov's inequality [15], Lemma 3.2, Hölder's inequality, and Lemma 3.3, we get

$$
\begin{aligned}
I_2 &\le \frac{C}{|B|^{1/t}} \left(|B|^{1-t} \left\| (b - b_{2B}) f_1 \right\|_1^t \right)^{1/t} \\
&= \frac{C}{|B|} \int_{2B} \left| \left(b(y) - b_{2B} \right) f(y) \right| dy \\
&\le \frac{C}{|B|} \left(\int_{2B} \left| b(y) - b_{2B} \right|^{s'} \omega(y)^{1-s'}\, dy \right)^{1/s'} \left(\int_{2B} \left| f(y) \right|^s \omega(y)\, dy \right)^{1/s} \\
&\le C\|b\|_{BMO(\omega)} \frac{\omega(2B)}{|2B|} \left(\frac{1}{\omega(2B)} \int_{2B} \left| f(y) \right|^s \omega(y)\, dy \right)^{1/s} \\
&\le C\|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(f)(x).
\end{aligned}
$$

Since $r_B > 1$ and $2|y - x|^\alpha \le |z - x|$ for any $y \in B$, $z \in (2B)^c$, by (2) of Definition 1.1, we have

$$
\begin{aligned}
I_3 &\le \frac{C}{|B|} \int_B \left| T\left((b - b_{2B}) f_2 \right)(y) - T\left((b - b_{2B}) f_2 \right)(x) \right| dy \\
&\le \frac{C}{|B|} \int_B \int_{(2B)^c} \left| K(y,z) - K(x,z) \right| \left| b(z) - b_{2B} \right| \left| f(z) \right| dz\, dy \\
&\le C \sum_{j=1}^{\infty} \frac{1}{|B|} \int_B \int_{2^{j+1}B \setminus 2^j B} \frac{|y - x|^\delta}{|z - x|^{n + \frac{\delta}{\alpha}}} \left| b(z) - b_{2B} \right| \left| f(z) \right| dz\, dy \\
&\le C r_B^\delta \sum_{j=1}^{\infty} \left(2^j r_B \right)^{-\frac{\delta}{\alpha}} \frac{1}{|B|} \int_B \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B \setminus 2^j B} \left| b(z) - b_{2B} \right| \left| f(z) \right| dz\, dy \\
&\le C r_B^{\delta - \frac{\delta}{\alpha}} \sum_{j=1}^{\infty} \left(2^j \right)^{-\frac{\delta}{\alpha}} \frac{1}{|2^{j+1}B|} \int_{2^{j+1}B} \left| b(z) - b_{2B} \right| \left| f(z) \right| dz.
\end{aligned}
$$

Applying Hölder's inequality and Lemma 3.9, we get

$$
\begin{aligned}
I_3 &\le C \sum_{j=1}^{\infty} \left(2^j \right)^{-\frac{\delta}{\alpha}} \frac{1}{|2^{j+1}B|} \left(\int_{2^{j+1}B} \left| b(z) - b_{2B} \right|^{s'} \omega(z)^{1-s'}\, dz \right)^{1/s'} \\
&\quad \times \left(\int_{2^{j+1}B} \left| f(z) \right|^s \omega(z)\, dz \right)^{1/s} \\
&\le C \sum_{j=1}^{\infty} \left(2^j \right)^{-\frac{\delta}{\alpha}} \frac{1}{|2^{j+1}B|} \left| 2^{j+1}B \right|^{\frac{1}{s'}} \|b\|_{BMO(\omega)} j \omega(x) \left(\frac{\omega(2^{j+1}B)}{|2^{j+1}B|} \right)^{-\frac{1}{s}} \\
&\quad \times \left(\int_{2^{j+1}B} \left| f(z) \right|^s \omega(z)\, dz \right)^{1/s} \\
&\le C\|b\|_{BMO(\omega)} \omega(x) \sum_{j=1}^{\infty} j \left(2^j \right)^{-\frac{\delta}{\alpha}} \left(\frac{1}{\omega(2^{j+1}B)} \int_{2^{j+1}B} \left| f(z) \right|^s \omega(z)\, dz \right)^{1/s}
\end{aligned}
$$

$$\le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(f)(x)\sum_{j=1}^{\infty} j\big(2^j\big)^{-\frac{\delta}{\alpha}}$$
$$\le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(f)(x).$$

Case 2: $0 < r_B \le 1$.

Since $r > \frac{(n(1-\alpha)+2\beta)(s-1)}{2\beta s-n(1-\alpha)-2\beta}$, that is, $\frac{n(1-\alpha)+2\beta}{2\beta} < \frac{rs}{s+r-1}$, there exists an s_0 such that $\frac{n(1-\alpha)+2\beta}{2\beta} < s_0 < \frac{rs}{s+r-1}$. For the index s_0 which we chose, by Remark 3.1, there exists an l_0 such that T is bounded from L^{s_0} to L^{l_0} and $0 < \frac{s_0}{l_0} < \alpha$. Then we can take a θ satisfying $0 < \frac{s_0}{l_0} < \theta < \alpha$.

Let $\tilde{B} = B(x, r_B^\theta)$. We decompose $f = f_3 + f_4$, where $f_3 = f\chi_{2\tilde{B}}$ and $\chi_{2\tilde{B}}$ denotes the characteristic function of $2\tilde{B}$. Write

$$\begin{aligned}
&[b,T](f)(y)\\
&\quad= \big(b(y)-b_{2B}\big)T(f)(y) - T\big((b-b_{2B})f\big)(y)\\
&\quad= \big(b(y)-b_{2B}\big)T(f)(y) - T\big((b-b_{2B})f_3\big)(y) - T\big((b-b_{2B})f_4\big)(y).
\end{aligned}$$

Since $0 < t < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B\Big|\big|[b,T](f)(y)\big|^t - \big|T\big((b-b_{2B})f_4\big)(x)\big|^t\Big|\,dy\right)^{1/t}\\
&\quad\le \left(\frac{1}{|B|}\int_B\big|[b,T](f)(y) + T\big((b-b_{2B})f_4\big)(x)\big|^t\,dy\right)^{1/t}\\
&\quad\le C\left(\frac{1}{|B|}\int_B\big|\big(b(y)-b_{2B}\big)T(f)(y)\big|^t\,dy\right)^{1/t} + C\left(\frac{1}{|B|}\int_B\big|T\big((b-b_{2B})f_3\big)(y)\big|^t\,dy\right)^{1/t}\\
&\qquad + C\left(\frac{1}{|B|}\int_B\big|T\big((b-b_{2B})f_4\big)(y) - T\big((b-b_{2B})f_4\big)(x)\big|^t\,dy\right)^{1/t}\\
&\quad:= II_1 + II_2 + II_3.
\end{aligned}$$

Similarly to estimate I_1, we have

$$II_1 \le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(Tf)(x).$$

Since $1 < s_0 < s < \infty$, there exists an l ($1 < l < \infty$) such that $\frac{1}{s_0} = \frac{1}{s} + \frac{1}{l}$. By Hölder's inequality and the (L^{s_0}, L^{l_0})-boundedness of T, we have

$$\begin{aligned}
II_2 &\le \frac{C}{|B|}\int_B\big|T\big((b-b_{2B})f_3\big)(y)\big|\,dy\\
&\le C\left(\frac{1}{|B|}\int_B\big|T\big((b-b_{2B})f_3\big)(y)\big|^{l_0}\,dy\right)^{\frac{1}{l_0}}\\
&\le C|B|^{-\frac{1}{l_0}}\left(\int_{2\tilde{B}}\big|b(y)-b_{2B}\big|^{s_0}\big|f(y)\big|^{s_0}\,dy\right)^{\frac{1}{s_0}}\\
&\le C|B|^{-\frac{1}{l_0}}\left(\int_{2\tilde{B}}\big|b(y)-b_{2B}\big|^{l}\omega(y)^{-\frac{l}{s}}\,dy\right)^{\frac{1}{l}}\left(\int_{2\tilde{B}}\big|f(y)\big|^{s}\omega(y)\,dy\right)^{\frac{1}{s}}\\
&\le CM_{s,\omega}(f)(x)\omega(2\tilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}\left(\int_{2\tilde{B}}\big|b(y)-b_{2B}\big|^{l}\omega(y)^{-\frac{l}{s}}\,dy\right)^{\frac{1}{l}}.
\end{aligned}$$

Let $p_0 = \frac{(r-1)(s-s_0)}{s(s_0-1)}$. Since $s_0 < \frac{rs}{s+r-1}$, we get $1 < p_0 < \infty$. So $p_0' = \frac{(r-1)(s-s_0)}{rs-(s+r-1)s_0}$ and $1 < p_0' < \infty$. Applying Hölder's inequality for p_0 and p_0', Lemma 3.9, and noticing that $r = \frac{lp_0}{s'} - \frac{p_0}{p_0'}$, we get

$$\begin{aligned}
II_2 &\le CM_{s,\omega}(f)(x)\omega(2\tilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}\left(\int_{2\tilde{B}}\left|b(y)-b_{2B}\right|^{lp_0'}\omega(y)^{1-lp_0'}\,dy\right)^{\frac{1}{lp_0'}} \\
&\quad\times\left(\int_{2\tilde{B}}\omega(y)^{\frac{lp_0}{s'}-\frac{p_0}{p_0'}}\,dy\right)^{\frac{1}{lp_0}} \\
&\le CM_{s,\omega}(f)(x)\omega(2\tilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}|2\tilde{B}|^{\frac{1}{lp_0'}}\|b\|_{BMO(\omega)}\omega(x) \\
&\quad\times\left(1+\left|\ln\frac{r_B^\theta}{r_B}\right|\right)\left(\frac{\omega(2\tilde{B})}{|2\tilde{B}|}\right)^{-\frac{1}{(lp_0')'}}\left[\left(\frac{1}{|2\tilde{B}|}\int_{2\tilde{B}}\omega(y)^r\,dy\right)^{\frac{1}{r}}\right]^{\frac{r}{lp_0}}|2\tilde{B}|^{\frac{1}{lp_0}} \\
&\le CM_{s,\omega}(f)(x)\omega(2\tilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}|2\tilde{B}|^{\frac{1}{l}}\|b\|_{BMO(\omega)}\omega(x) \\
&\quad\times\left(1+\left|\ln\frac{r_B^\theta}{r_B}\right|\right)\left(\frac{\omega(2\tilde{B})}{|2\tilde{B}|}\right)^{-\frac{1}{(lp_0')'}}\left(\frac{\omega(2\tilde{B})}{|2\tilde{B}|}\right)^{\frac{r}{lp_0}} \\
&\le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(f)(x)\omega(2\tilde{B})^{\frac{1}{s}}|B|^{-\frac{1}{l_0}}|2\tilde{B}|^{\frac{1}{l}} \\
&\quad\times\left(1+(1-\theta)\ln\frac{1}{r_B}\right)\left(\frac{\omega(2\tilde{B})}{|2\tilde{B}|}\right)^{-\frac{1}{s}}.
\end{aligned}$$

The inequality $0 < \frac{s_0}{l_0} < \theta$ implies that $\varepsilon_1 := n(\frac{\theta}{s_0} - \frac{1}{l_0}) > 0$. By Lemma 3.7, we have

$$\begin{aligned}
II_2 &\le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(f)(x)|B|^{-\frac{1}{l_0}}|2\tilde{B}|^{\frac{1}{l}+\frac{1}{s}}\left(1+\frac{1}{\varepsilon_1}r_B^{-\varepsilon_1}\right) \\
&\le C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(f)(x)r_B^{n(\frac{\theta}{s_0}-\frac{1}{l_0})-\varepsilon_1} \\
&= C\|b\|_{BMO(\omega)}\omega(x)M_{s,\omega}(f)(x).
\end{aligned}$$

The fact $\theta < \alpha$ implies that $\varepsilon_2 := \frac{\delta}{\alpha}(\alpha - \theta) > 0$. For any $y \in B$ and $z \in (2\tilde{B})^c$, we have $2|y-x|^\alpha \le 2r_B^\alpha \le 2r_B^\theta \le |z-x|$ since $0 < r_B \le 1$. It follows from (2) of Definition 1.1, Hölder's inequality, Lemma 3.9, and Lemma 3.7 that

$$\begin{aligned}
II_3 &\le \frac{C}{|B|}\int_B\left|T\big((b-b_{2B})f_4\big)(y)-T\big((b-b_{2B})f_4\big)(x)\right|dy \\
&\le \frac{C}{|B|}\int_B\int_{(2\tilde{B})^c}\left|K(y,z)-K(x,z)\right|\left|b(z)-b_{2B}\right|\left|f(z)\right|dz\,dy \\
&\le C\sum_{j=1}^\infty\frac{1}{|B|}\int_B\int_{2^{j+1}\tilde{B}\setminus 2^j\tilde{B}}\frac{|y-x|^\delta}{|z-x|^{n+\frac{\delta}{\alpha}}}\left|b(z)-b_{2B}\right|\left|f(z)\right|dz\,dy \\
&\le Cr_B^\delta\sum_{j=1}^\infty\left(2^jr_B^\theta\right)^{-\frac{\delta}{\alpha}}\frac{1}{|2^{j+1}\tilde{B}|}\int_{2^{j+1}\tilde{B}\setminus 2^j\tilde{B}}\left|b(z)-b_{2B}\right|\left|f(z)\right|dz \\
&\le Cr_B^{\delta-\frac{\theta\delta}{\alpha}}\sum_{j=1}^\infty\left(2^j\right)^{-\frac{\delta}{\alpha}}\frac{1}{|2^{j+1}\tilde{B}|}\left(\int_{2^{j+1}\tilde{B}}\left|b(z)-b_{2B}\right|^{s'}\omega(y)^{1-s'}\,dy\right)^{\frac{1}{s'}}
\end{aligned}$$

$$\begin{aligned}
&\times \left(\int_{2^{j+1}\tilde{B}} |f(z)|^s \omega(y)\,dy\right)^{\frac{1}{s}} \\
&\leq C r_B^{\delta-\frac{\theta\delta}{\alpha}} \sum_{j=1}^{\infty} (2^j)^{-\frac{\delta}{\alpha}} \frac{1}{|2^{j+1}\tilde{B}|} |2^{j+1}\tilde{B}|^{\frac{1}{s'}} \|b\|_{BMO(\omega)} \omega(x) \\
&\quad \times \left(j + (1-\theta)\ln\frac{1}{r_B}\right) \left(\frac{\omega(2^{j+1}\tilde{B})}{|2^{j+1}\tilde{B}|}\right)^{-\frac{1}{s}} M_{s,\omega}(f)(x)\,\omega(2^{j+1}\tilde{B})^{\frac{1}{s}} \\
&\leq C r_B^{\delta-\frac{\theta\delta}{\alpha}} \sum_{j=1}^{\infty} (2^j)^{-\frac{\delta}{\alpha}} \|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(f)(x) \left(j + (1-\theta)\ln\frac{1}{r_B}\right) \\
&\leq C\|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(f)(x) r_B^{\delta-\frac{\theta\delta}{\alpha}} \sum_{j=1}^{\infty} (2^j)^{-\frac{\delta}{\alpha}} \left(j + \frac{1}{\varepsilon_2} r_B^{-\varepsilon_2}\right) \\
&\leq C\|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(f)(x) r_B^{\frac{\delta}{\alpha}(\alpha-\theta)-\varepsilon_2} \sum_{j=1}^{\infty} j (2^j)^{-\frac{\delta}{\alpha}} \\
&\leq C\|b\|_{BMO(\omega)} \omega(x) M_{s,\omega}(f)(x).
\end{aligned}$$

Combining the estimates in both cases, we have

$$\begin{aligned}
M_t^{\sharp}([b,T]f)(x) &\sim \sup_{r_B>0} \inf_{a\in\mathbf{C}} \left(\frac{1}{|B(x,r_B)|} \int_{B(x,r_B)} \left||[b,T](f)(y)|^t - a\right| dy\right)^{\frac{1}{t}} \\
&\leq C\|b\|_{BMO(\omega)} \left(\omega(x) M_{s,\omega}(Tf)(x) + \omega(x) M_{s,\omega}(f)(x)\right). \qquad \square
\end{aligned}$$

4 Proof of the main results

Now we are able to prove our main results.

Proof of Theorem 2.1 Since $\frac{n(1-\alpha)+2\beta}{2\beta} < p < \infty$ and $\omega \in A_{2\beta p/[n(1-\alpha)+2\beta]}$, there exists an l such that $1 \leq l < \frac{2p\beta}{n(1-\alpha)+2\beta}$ and $\omega \in A_l$. Since $\frac{n(1-\alpha)+2\beta}{2\beta} < \frac{p}{l} \leq p$, there exists an s such that $\frac{n(1-\alpha)+2\beta}{2\beta} < s < \frac{p}{l} \leq p$. It follows from $\frac{p}{s} > l$ that $\omega \in A_{p/s}$. Applying Lemma 3.6, Lemma 3.8, and Lemma 3.4, we have

$$\begin{aligned}
\|T(f)\|_{L^{p,k}(\omega)} &\leq \|M_t(Tf)\|_{L^{p,k}(\omega)} \leq C\|M_t^{\sharp}(Tf)\|_{L^{p,k}(\omega)} \\
&\leq C\|M_s(f)\|_{L^{p,k}(\omega)} = C\|M(|f|^s)\|_{L^{p/s,k}(\omega)}^{1/s} \\
&\leq C\||f|^s\|_{L^{p/s,k}(\omega)}^{1/s} = C\|f\|_{L^{p,k}(\omega)}.
\end{aligned}$$

This completes the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2 Since $r > \frac{(n(1-\alpha)+2\beta)(p-1)}{2\beta p-n(1-\alpha)-2\beta}$, that is $p > \frac{(n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}$, there exists an s such that $p > s > \frac{(n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta} > \frac{n(1-\alpha)+2\beta}{2\beta}$. Since $s > \frac{(n(1-\alpha)+2\beta)(r-1)}{2\beta r-n(1-\alpha)-2\beta}$, we have $r > \frac{(n(1-\alpha)+2\beta)(s-1)}{2\beta s-n(1-\alpha)-2\beta}$. Applying Lemma 3.6 and Lemma 3.10, we thus have

$$\begin{aligned}
&\|[b,T](f)\|_{L^{p,k}(\omega^{1-p},\omega)} \\
&\quad \leq \|M_t([b,T]f)\|_{L^{p,k}(\omega^{1-p},\omega)} \\
&\quad \leq C\|M_t^{\sharp}([b,T]f)\|_{L^{p,k}(\omega^{1-p},\omega)}
\end{aligned}$$

$$\begin{aligned}&\leq C\|b\|_{BMO(\omega)}\big(\big\|\omega(\cdot)M_{s,\omega}(Tf)\big\|_{L^{p,k}(\omega^{1-p},\omega)}+\big\|\omega(\cdot)M_{s,\omega}(f)\big\|_{L^{p,k}(\omega^{1-p},\omega)}\big)\\&=C\|b\|_{BMO(\omega)}\big(\big\|M_{s,\omega}(Tf)\big\|_{L^{p,k}(\omega)}+\big\|M_{s,\omega}(f)\big\|_{L^{p,k}(\omega)}\big).\end{aligned}$$

Therefore, by using Lemma 3.5 and Theorem 2.1, we obtain

$$\begin{aligned}\big\|[b,T](f)\big\|_{L^{p,k}(\omega^{1-p},\omega)}&\leq C\|b\|_{BMO(\omega)}\big(\|Tf\|_{L^{p,k}(\omega)}+\|f\|_{L^{p,k}(\omega)}\big)\\&\leq C\|b\|_{BMO(\omega)}\|f\|_{L^{p,k}(\omega)}.\end{aligned}$$

This completes the proof of Theorem 2.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL put forward the ideas of the paper, and the authors completed the paper together. They also read and approved the final manuscript.

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