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The general traveling wave solutions of the Fisher type equations and some related problems

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Dedicated to Professor Seiki Mori on the occasion of his 70th birthday.

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Abstract

In this article, we introduce two recent results with respect to the integrality and exact solutions of the Fisher type equations and their applications. We obtain the sufficient and necessary conditions of integrable and general meromorphic solutions of these equations by the complex method. Our results are of the corresponding improvements obtained by many authors. All traveling wave exact solutions of many nonlinear partial differential equations are obtained by making use of our results. Our results show that the complex method provides a powerful mathematical tool for solving a great number of nonlinear partial differential equations in mathematical physics. We will propose four analogue problems and expect that the answer is positive, at last.

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1 Introduction

Nonlinear partial differential equations (NLPDEs) are widely used as models to describe many important dynamical systems in various fields of science, particularly in fluid mechanics, solid state physics, plasma physics and nonlinear optics. Exact solutions of NLPDEs of mathematical physics have attracted significant interest in the literature. Over the last years, much work has been done on the construction of exact solitary wave solutions and periodic wave solutions of nonlinear physical equations. Many methods have been developed by mathematicians and physicists to find special solutions of NLPDEs, such as the inverse scattering method [1], the Darboux transformation method [2], the Hirota bilinear method [3], the Lie group method [4], the bifurcation method of dynamic systems [5–7], the sine-cosine method [8], the tanh-function method [9, 10], the Fan-expansion method [11], and the homogenous balance method [12]. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations. Recently, Kudryashov *et al.* [13–16] found exact meromorphic solutions for some nonlinear ordinary differential equations by using Laurent series and gave some basic results. Following their work, the complex method was introduced by Yuan *et al.*

[17–19]. In this article, we survey two recent results with respect to the integrality and exact solutions of the Fisher type equations and their applications. We obtain the sufficient and necessary conditions of integrable and general meromorphic solutions of these equations by the complex method. Our results are of the corresponding improvements obtained by many authors. All traveling wave exact solutions of many nonlinear partial differential equations are obtained by making use of our results. Our results show that the complex method provides a powerful mathematical tool for solving a great number of nonlinear partial differential equations in mathematical physics. We will propose four analogue problems and expect that the answer is positive, at last.

2 Fisher type equations with degree two

In 2013, Yuan *et al.* [17] derived all traveling wave exact solutions by using the complex method for a type of ordinary differential equations (ODEs)

$$Aw'' + Bw + Cw^2 + D = 0, \tag{1}$$

where A, B, C and D are arbitrary constants.

In order to state these results, we need some concepts and notations.

A meromorphic function $w(z)$ means that $w(z)$ is holomorphic in the complex plane $\mathbb{C}$ except for poles. α, b, c, c_i and c_{ij} are constants which may be different from each other in different place. We say that a meromorphic function f belongs to the class W if f is an elliptic function, or a rational function of $e^{\alpha z}$, $\alpha \in \mathbb{C}$, or a rational function of z.

Theorem 2.1 *Suppose that $AC \neq 0$, then all meromorphic solutions w of Eq.* (1) *belong to the class W. Furthermore, Eq.* (1) *has the following three forms of solutions*:

(I) *The elliptic general solutions*

$$w_{1d}(z) = -6\frac{A}{C}\left\{-\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + F}{\wp(z) - E}\right]^2\right\} + 6\frac{AE}{C} - \frac{B}{2C}.$$

Here, $4DC = -12A^2 g_2 + B^2$, $F^2 = 4E^3 - g_2 E - g_3$, g_3 *and* E *are arbitrary.*

(II) *The simply periodic solutions*

$$w_{1s}(z) = -6\frac{A}{C}\alpha^2 \coth^2\frac{\alpha}{2}(z - z_0) - \frac{A}{2C}\alpha^2 - \frac{B}{2C},$$

where $4DC = -A^2\alpha^4 + B^2$, $z_0 \in \mathbb{C}$.

(III) *The rational function solutions*

$$w_{1r}(z) = -\frac{6\frac{A}{C}}{(z - z_0)^2} - \frac{B}{2C},$$

where $4CD = B^2$, $z_0 \in \mathbb{C}$.

Equation (1) is an important auxiliary equation, because many nonlinear evolution equations can be converted to Eq. (1) using the traveling wave reduction. For instance, the classical KdV equation, the Boussinesq equation, the (3 + 1)-dimensional Jimbo-Miwa equation and the Benjamin-Bona-Mahony equation can be converted to Eq. (1) [17].

In 2013, Yuan *et al.* [20] employed the complex method to obtain first all meromorphic solutions of the equation

$$Aw'' + Bw' + Cw + Dw^2 + E = 0, \tag{2}$$

where A, B, C, D, E are arbitrary constants.

Theorem 2.2 *Suppose that $AD \neq 0$, then Eq.* (2) *is integrable if and only if $B = 0, \pm\frac{5}{\sqrt{6}} \times \sqrt{-2AD\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}}, \pm\frac{5i}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}}$. Furthermore, the general solutions of Eq.* (1) *are of the following form*:

(i) *If $B = 0$, then we have the elliptic general solutions of Eq.* (2)

$$w_{2d}(z) = -6\frac{A}{D}\left\{-\wp(z) + \frac{1}{4}\left[\frac{\wp'(z) + M}{\wp(z) - N}\right]^2\right\} + 6\frac{AN}{D} - \frac{C}{2D}.$$

Here, $12A^2g_2 = C^2$, $M^2 = 4N^3 - g_2N - g_3$, g_3 and N are arbitrary.

In particular, which degenerates to the simply periodic solutions

$$w_{2s}(z) = -6\frac{A}{D}\alpha^2\coth^2\frac{\alpha}{2}(z - z_0) - \frac{A}{2D}\alpha^2 - \frac{C}{2D},$$

where $A^2\alpha^4 = C^2$, $z_0 \in \mathbb{C}$.

And the rational function solutions

$$w_{2r}(z) = -\frac{6\frac{A}{D}}{(z - z_0)^2} - \sqrt{\frac{C^2}{4D^2} - \frac{E}{D}} - \frac{C}{2D},$$

where $C^2 = 4DE$, $z_0 \in \mathbb{C}$.

(ii) *If $B = \pm\frac{5}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}}$, then the general solutions of Eq.* (2)

$$\begin{aligned} w_{g2}(z) = {} & \exp\left\{\mp\frac{2}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}z}\right\} \\ & \times \wp\left(\sqrt{-\frac{D}{A}}\exp\left\{\mp\frac{1}{\sqrt{6}}\sqrt{-\frac{D}{A}\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}z}\right\} - s_0; 0, g_3\right) \\ & - \sqrt{\frac{C^2}{4D^2} - \frac{E}{D}} - \frac{C}{2D}, \end{aligned}$$

where $\sqrt{\frac{C^2}{4D^2}} = -\frac{C}{2D}$, both s_0 and g_3 are arbitrary constants.

In particular, which degenerates to the one parameter family of solutions

$$w_{f2}(z) = 2\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}\frac{1}{\{1 - \exp\{\pm\frac{(z-z_0)}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2} - \frac{E}{D}}}\}\}^2} - \sqrt{\frac{C^2}{4D^2} - \frac{E}{D}} - \frac{C}{2D},$$

where $\sqrt{\frac{C^2}{4D^2}} = -\frac{C}{2D}$, $z_0 \in \mathbb{C}$.

(iii) *If* $B = \pm\frac{5i}{\sqrt{6}}\sqrt{-2AD\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}$, *then the general solutions of Eq.* (2)

$$\begin{aligned} w_{g2,i}(z) &= \exp\left\{\mp\frac{2i}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}z\right\} \\ &\quad\times\wp\left(\sqrt{\frac{D}{A}}\exp\left\{\mp\frac{i}{\sqrt{6}}\sqrt{-\frac{D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}z\right\}-s_0;0,g_3\right) \\ &\quad-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{3C}{2D}, \end{aligned}$$

where $\sqrt{\frac{C^2}{4D^2}} = -\frac{C}{2D}$, *and both* s_0 *and* g_3 *are arbitrary constants.*

In particular, which degenerates to the one parameter family of solutions

$$\begin{aligned} w_{f2,i}(z) &= -2\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}\frac{1}{\{1-\exp\{\pm\frac{i(z-z_0)}{\sqrt{6}}\sqrt{-\frac{2D}{A}\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}}\}\}^2} \\ &\quad-\sqrt{\frac{C^2}{4D^2}-\frac{E}{D}}-\frac{3C}{2D}, \end{aligned}$$

where $\sqrt{\frac{C^2}{4D^2}} = -\frac{C}{2D}$, $z_0 \in \mathbb{C}$.

The Fisher equation with degree two

Consider the Fisher equation

$$u_t = \nu u_{xx} + su(1-u),$$

which is a nonlinear diffusion equation as a model for the propagation of a mutant gene with an advantageous selection intensity s. It was suggested by Fisher as a deterministic version of the stochastic model for the spatial spread of a favored gene in a population in 1936.

Set $t' = st$ and $x' = (\frac{s}{\nu})^{\frac{1}{2}}x$ and drop the primes, then the above equation becomes

$$u_t = u_{xx} + u(1-u). \qquad \text{(Fisher)}$$

By substituting

$$u(x,t) = w(z), \qquad z = x - ct,$$

into Eq. (Fisher) and integrating it, we obtain

$$w'' + cw' + w(1-w) = 0.$$

It is converted to Eq. (2), where

$$A = 1, \qquad B = c, \qquad C = 1, \qquad D = -1, \qquad E = 0.$$

Three nonlinear pseudoparabolic physical models

The one-dimensional Oskolkov equation, the Benjamin-Bona-Mahony-Peregrine-Burgers equation and the Oskolkov-Benjamin-Bona-Mahony-Burgers equation are the special cases of our Eq. (2).

The one-dimensional Oskolkov equation has the form

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + uu_x = 0, \tag{Oskolkov}$$

where $\lambda \neq 0$, $\alpha \in \mathbb{R}$.

Substituting

$$u(x,t) = w(z), \quad z = x - ct,$$

into Eq. (Oskolkov) and integrating the equation, we have

$$\lambda w'' - \alpha w' - cw + \frac{1}{2}w^2 = 0.$$

It is converted to Eq. (2), where

$$A = \lambda, \qquad B = -\alpha, \qquad C = -c, \qquad D = \frac{1}{2}, \qquad E = 0.$$

The Benjamin-Bona-Mahony-Peregrine-Burgers equation is of the form

$$u_t - u_{xxt} - \alpha u_{xx} + +\gamma u_x + \theta uu_x + \beta u_{xxx} = 0, \tag{BBMPB}$$

where α is a positive constant, θ and β are nonzero real numbers.

Substituting

$$u(x,t) = w(z), \quad z = x - ct,$$

into Eq. (BBMPB), we get

$$(c+\beta)w'' - \alpha w' + (\gamma - c)w + \frac{\theta}{2}w^2 = 0.$$

It is converted to Eq. (2), where

$$A = c + \beta, \qquad B = -\alpha, \qquad C = \gamma - c, \qquad D = \frac{\theta}{2}, \qquad E = 0.$$

The Oskolkov-Benjamin-Bona-Mahony-Burgers equation is of the form

$$u_t - u_{xxt} - \alpha u_{xx} + +\gamma u_x + \theta uu_x = 0, \tag{OBBMB}$$

where α is a positive constant, θ is a nonzero real number.

Substituting

$$u(x,t) = w(z), \quad z = x - ct,$$

into Eq. (OBBMB), we deduce

$$cw'' - \alpha w' + (\gamma - c)w + \frac{\theta}{2}w^2 = 0.$$

It is converted to Eq. (2), where

$$A = c, \qquad B = -\alpha, \qquad C = \gamma - c, \qquad D = \frac{\theta}{2}, \qquad E = 0.$$

The KdV-Burgers equation

The KdV-Burgers equation is of the form

$$u_t + uu_x + u_{xxx} - \alpha u_{xx} = 0, \tag{KdV-B}$$

where α is a constant.

Substituting the traveling wave transformation

$$u(x,t) = w(z), \quad z = x + Ct,$$

into Eq. (KdV-B) and integrating it yields the auxiliary ordinary differential equation

$$w'' - \alpha w' + \frac{1}{2}w^2 + Cw + E = 0,$$

where E is an integral constant. It is converted to Eq. (2), where

$$A = 1, \qquad B = -\alpha, \qquad C = C, \qquad D = \frac{1}{2}, \qquad E = E.$$

3 Fisher type equations with degree three

In 2012, Yuan *et al.* [21] employed the complex method to find all meromorphic solutions of the auxiliary ordinary differential equations

$$Aw'' + Bw + Cw^3 + D = 0, \tag{3}$$

where A, B, C and D are arbitrary constants.

Theorem 3.1 [21] *Suppose that $AC \neq 0$, then all meromorphic solutions w of Eq.* (3) *belong to the class W. Furthermore, Eq.* (3) *has the following three forms of solutions*:

(I) *The elliptic function solutions*

$$w_{3d}(z) = \pm\frac{1}{2}\sqrt{-\frac{2A}{C}} \times \frac{(-\wp + c)(4\wp c^2 + 4\wp^2 c + 2\wp' d - \wp g_2 - cg_2)}{((12c^2 - g_2)\wp + 4c^3 - 3cg_2)\wp' + (4\wp^3 + 12c\wp^2 - 3g_2\wp - cg_2)d}.$$

Here, $g_3 = 0$, $d^2 = 4c^3 - g_2 c$, g_2 and c are arbitrary.

(II) *The simply periodic solutions*

$$w_{3s,1}(z) = \alpha\sqrt{-\frac{A}{2C}}\left(\coth\frac{\alpha}{2}(z-z_0) - \coth\frac{\alpha}{2}(z-z_0-z_1) - \coth\frac{\alpha}{2}z_1\right)$$

and

$$w_{3s,2}(z) = \alpha\sqrt{-\frac{A}{2C}}\tanh\frac{\alpha}{2}(z-z_0),$$

where $z_0 \in \mathbb{C}$, $B = A\alpha^2(\frac{1}{2} + \frac{3}{2\sinh^2\frac{\alpha}{2}z_1})$, $D = \sqrt{-\frac{A}{2C}}\frac{\tanh\frac{\alpha}{2}z_1}{\sinh^2\frac{\alpha}{2}z_1}$, $z_1 \neq 0$ *in the former formula, or* $B = \frac{A\alpha^2}{2}$, $D = 0$.

(III) *The rational function solutions*

$$w_{3r,1}(z) = \pm\sqrt{-\frac{2A}{C}}\frac{1}{z-z_0}$$

and

$$w_{3r,2}(z) = \pm\sqrt{-\frac{2A}{Cz_1^2}}\left(\frac{z_1}{z-z_0} - \frac{z_1}{z-z_0-z_1} - 1\right),$$

where $z_0 \in \mathbb{C}$. $B = 0$, $D = 0$ *in the former case, or given* $z_1 \neq 0$, $B = \frac{6A}{z_1^2}$, $D = \mp 2C(\frac{-2A}{Cz_1^2})^{3/2}$.

In 2013, Yuan *et al.* [22] considered the following equation:

$$Aw'' + Bw + Cw^2 + w^3 + D = 0, \tag{4}$$

where A, B, C and D are arbitrary constants. They obtained the following result and gave its two applications.

Theorem 3.2 *Suppose that* $A \neq 0$, *then all meromorphic solutions* w *of Eq.* (4) *belong to the class* W. *Furthermore, Eq.* (4) *has the following three forms of solutions*:

(I) *All elliptic function solutions*

$$w_{4d}(z) = -\frac{C}{3} \pm \sqrt{-\frac{A}{2}} \times \frac{(-\wp + E)(4\wp E^2 + 4\wp^2 E + 2\wp' F - \wp g_2 - E g_2)}{((12E^2 - g_2)\wp + 4E^3 - 3Eg_2)\wp' + 4F\wp^3 + 12FE\wp^2 - 3Fg_2\wp - FEg_2},$$

where $A(C^2 - 9B) = 12C\sqrt{-\frac{A}{2}}$, $27D = C^3$, $g_3 = 0$, $F^2 = 4E^3 - g_2 E$, g_2 *and* E *are arbitrary constants.*

(II) *All simply periodic solutions*

$$w_{4s,1}(z) = \pm\sqrt{-\frac{A}{2}}\alpha\coth\frac{\alpha}{2}(z-z_0) - \frac{C}{3}$$

and

$$w_{4s,2}(z) = \pm\sqrt{-\frac{A}{2}}\alpha\left(\coth\frac{\alpha}{2}(z-z_0) - \coth\frac{\alpha}{2}(z-z_0-z_1)\right) - \frac{C}{3} \mp \sqrt{-\frac{A}{2}}\alpha\coth\frac{\alpha}{2}z_1,$$

where $z_0 \in \mathbb{C}$. $A(2C^2 + 9A\alpha^2 - 18B) = 24C\sqrt{-\frac{A}{2}}$, $27D - C^3 = 27\alpha^2\sqrt{-\frac{A}{2}}$ *in the former case, or* $z_1 \neq 0$, $8C\sqrt{-\frac{A}{2}} + 6AB = 3A^2\alpha^2(\frac{3}{\sinh^2\frac{\alpha}{2}z_1} + 1)$,

$$162D\sqrt{-\frac{A}{2}} = \left(2C\sqrt{-\frac{A}{2}} \mp 3A\alpha\coth\frac{\alpha}{2}z_1\right) \times \left(\frac{108A\alpha^2}{\sinh^2\frac{\alpha}{2}z_1} + 3C^2 \mp 9C\alpha\sqrt{-\frac{A}{2}}\coth\frac{\alpha}{2}z_1\right).$$

(III) *All rational function solutions*

$$w_{4r,1}(z) = \pm\frac{2\sqrt{-\frac{A}{2}}}{z-z_0} - \frac{C}{3}$$

and

$$w_{4r,2}(z) = \pm\frac{2\sqrt{-\frac{A}{2}}}{z-z_0} \mp \frac{2\sqrt{-\frac{A}{2}}}{z-z_0-z_1} \mp \frac{2\sqrt{-\frac{A}{2}}}{z_1} - \frac{C}{3},$$

where $z_0 \in \mathbb{C}$. $A(C^2 - 9B) = 12C\sqrt{-\frac{A}{2}}$, $27D = C^3$ *in the former case, or* $A(\frac{54A}{z_1^2} + C^2 - 9B) = 12C\sqrt{-\frac{A}{2}}$, $\frac{4A^2}{z_1^3} = (\frac{C^3}{27} + \frac{2C}{z_1^2} - D)\sqrt{-\frac{A}{2}}$, $z_1 \neq 0$.

Very recently, Yuan *et al.* [23] studied the differential equation

$$Aw'' + Bw' + Cw + Dw^3 = 0, \tag{5}$$

where A, B, C, D are arbitrary constants.

They got the following theorem.

Theorem 3.3 *Suppose that* $AD \neq 0$, *then Eq.* (5) *is integrable if and only if* $B = 0, \pm\frac{3}{\sqrt{2}}\sqrt{AC}$. *Furthermore, the general solutions of Eq.* (5) *are of the following form*:

(I) [21] *When* $B = 0$, *the elliptic general solutions of Eq.* (5)

$$w_{5d,1}(z) = \pm\sqrt{-\frac{2A}{D}}\frac{\wp'(z-z_0; g_2, 0)}{\wp(z-z_0; g_2, 0)},$$

where z_0 *and* g_2 *are arbitrary. In particular, it degenerates to the simply periodic solutions and rational solutions*

$$w_{5s,1}(z) = \alpha\sqrt{-\frac{A}{2D}}\tanh\frac{\alpha}{2}(z-z_0)$$

and

$$w_{5r}(z) = \pm\sqrt{-\frac{2A}{D}}\frac{1}{z-z_0},$$

where $C = \frac{A\alpha^2}{2}$ *and* $z_0 \in \mathbb{C}$.

(II) *When* $B = \pm\frac{3}{\sqrt{2}}\sqrt{AC}$, *the general solutions of Eq.* (5)

$$w_{5g,1}(z) = \pm\frac{1}{2}\exp\left\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}z\right\}\frac{\wp'(\sqrt{-\frac{D}{C}}\exp\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}z\} - s_0; g_2, 0)}{\wp(\sqrt{-\frac{D}{C}}\exp\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}z\} - s_0; g_2, 0)},$$

where $\wp(s : g_2, 0)$ *is the Weierstrass elliptic function, both* s_0 *and* g_2 *are arbitrary constants. In particular,* $w_{5g,1}(z)$ *degenerates to the one parameter family of solutions*

$$w_{5f,1}(z) = \pm\sqrt{-\frac{C}{D}}\frac{1}{1-\exp\{\mp\frac{1}{\sqrt{2}}\sqrt{\frac{C}{A}}(z-z_0)\}},$$

where $z_0 \in \mathbb{C}$.

All exact solutions of Eq. (Newell-Whitehead), the nonlinear Schrödinger Eq. (NLS) and Eq. (Fisher 3) can be converted to Eq. (5) making use of the traveling wave reduction.

The Newell-Whitehead equation

The Newell-Whitehead equation is of the form

$$u_{xx} - u_t - ru^3 + su = 0, \tag{Newell-Whitehead}$$

where r, s are constants.

Substituting

$$u(x,t) = w(z), \quad z = x + \omega t,$$

into Eq. (Newell-Whitehead) gives

$$w'' - \omega w' + sw - rw^3 = 0.$$

It is converted to Eq. (5), where

$$A = 1, \qquad B = -\omega, \qquad C = 1, \qquad D = -1.$$

The NLS equation

The NLS equation is of the form

$$iu_t + \alpha u_{xx} + \beta|u|^2u = 0, \tag{NLS}$$

where α, β are nonzero constants.

Substituting

$$u(x,t) = w(z)e^{kx-\omega t}, \quad z = x + ct,$$

into Eq. (NLS) gives

$$\alpha w'' + i(2\alpha k - c)w' + (\omega - \alpha k^2)w + \beta w^3 = 0.$$

It is converted to Eq. (5), where

$$A = \alpha, \qquad B = i(2\alpha k - c), \qquad C = \omega - \alpha k^2, \qquad D = \beta.$$

The Fisher equation with degree three

The Fisher equation with degree three is of the form

$$u_t = u_{xx} + u(1 - u^2). \tag{Fisher 3}$$

Substituting

$$u(x,t) = w(z), \quad z = x - ct,$$

into Eq. (Fisher 3) gives

$$w'' + cw' + w(1 - w^2) = 0.$$

It is converted to Eq. (5), where

$$A = 1, \qquad B = c, \qquad C = 1, \qquad D = -1.$$

4 The complex method and some problems

In order to state our complex method, we need some notations and results.

Set $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$, $r_j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $r = (r_0, r_1, \ldots, r_m)$, $j = 0, 1, \ldots, m$. We define a differential monomial denoted by

$$M_r[w](z) := [w(z)]^{r_0} [w'(z)]^{r_1} [w''(z)]^{r_2} \cdots [w^{(m)}(z)]^{r_m}.$$

$p(r) := r_0 + r_1 + \cdots + r_m$ is called the degree of $M_r[w]$. A differential polynomial is defined by

$$P(w, w', \ldots, w^{(m)}) := \sum_{r \in I} a_r M_r[w],$$

where a_r are constants, and I is a finite index set. The total degree is defined by $\deg P(w, w', \ldots, w^{(m)}) := \max_{r \in I}\{p(r)\}$.

We will consider the following complex ordinary differential equations:

$$P(w, w', \ldots, w^{(m)}) = bw^n + c, \tag{6}$$

where $b \neq 0$, c are constants, $n \in \mathbb{N}$.

Let $p, q \in \mathbb{N}$. Suppose that Eq. (6) has a meromorphic solution w with a pole at $z = 0$. We say that Eq. (6) satisfies the weak $\langle p, q\rangle$ condition if substituting Laurent series

$$w(z) = \sum_{k=-q}^{\infty} c_k z^k, \quad q > 0, c_{-q} \neq 0 \tag{7}$$

into Eq. (6), we can determine p distinct Laurent singular parts as follows:

$$\sum_{k=-q}^{-1} c_k z^k.$$

In order to give the representations of elliptic solutions, we need some notations and results concerning elliptic functions [24].

Let ω_1, ω_2 be two given complex numbers such that $\operatorname{Im} \frac{\omega_1}{\omega_2} > 0$, $L = L[2\omega_1, 2\omega_2]$ be a discrete subset $L[2\omega_1, 2\omega_2] = \{\omega \mid \omega = 2n\omega_1 + 2m\omega_2, n, m \in \mathbb{Z}\}$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. The discriminant is $\Delta = \Delta(c_1, c_2) := c_1^3 - 27c_2^2$, and we have

$$s_n = s_n(L) := \sum_{\omega \in L \setminus \{0\}} \frac{1}{\omega^n}.$$

The Weierstrass elliptic function $\wp(z) := \wp(z, g_2, g_3)$ is a meromorphic function with double periods $2\omega_1$, $2\omega_2$, satisfying the equation

$$\left(\wp'(z)\right)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3, \tag{8}$$

where $g_2 = 60s_4$, $g_3 = 140s_6$, and $\Delta(g_2, g_3) \neq 0$.

Theorem 4.1 [24, 25] *The Weierstrass elliptic functions $\wp(z) := \wp(z, g_2, g_3)$ have two successive degeneracies, and we have the addition formula*:

(i) *Degeneracy to simply periodic functions (i.e., rational functions of one exponential e^{kz}) according to*

$$\wp\left(z, 3d^2, -d^3\right) = 2d - \frac{3d}{2}\coth^2\sqrt{\frac{3d}{2}}z \tag{9}$$

if one root e_j is double ($\Delta(g_2, g_3) = 0$).

(ii) *Degeneracy to rational functions of z according to*

$$\wp(z, 0, 0) = \frac{1}{z^2}$$

if one root e_j is triple ($g_2 = g_3 = 0$).

(iii) *We have the addition formula*

$$\wp(z - z_0) = -\wp(z) - \wp(z_0) + \frac{1}{4}\left[\frac{\wp'(z) + \wp'(z_0)}{\wp(z) - \wp(z_0)}\right]^2. \tag{10}$$

By the above notations and results, we can give the following method, let us call it the complex method, to find exact solutions of some PDEs.

Step 1. Substituting the transform $T: u(x,t) \to w(z)$, $(x,t) \to z$ into a given PDE gives nonlinear ordinary differential equations (6).

Step 2. Substitute Eq. (7) into Eq. (6) to determine that the weak $\langle p,q\rangle$ condition holds, and pass the Painlevé test for Eq. (6).

Step 3. Find the meromorphic solutions $w(z)$ of Eq. (6) with a pole at $z = 0$, which have $m-1$ integral constants.

Step 4. By the addition formula of Theorem 4.1 we obtain all meromorphic solutions $w(z-z_0)$.

Step 5. Substituting the inverse transform T^{-1} into these meromorphic solutions $w(z-z_0)$, we get all exact solutions $u(x,t)$ of the original given PDE.

Proof of Theorem 2.2 *in case* $E = 0$ By substituting Eq. (7) into Eq. (2) we have $q = 2$, $p = 1$, $c_{-2} = \frac{6A}{D}$, $c_{-1} = -\frac{6B}{5D}$, $c_0 = \frac{1}{50}\frac{25AC-B^2}{AD}$, $c_1 = -\frac{B^3}{250A^2D}$, $c_2 = \frac{C^2}{40AD} - \frac{7B^4}{5{,}000A^3D}$, $c_3 = \frac{11BC^2}{600A^2D} - \frac{79B^5}{75{,}000A^4D}$ and

$$0 \times c_4 + B^2\left(B - \frac{5\sqrt{AC}}{\sqrt{6}}\right)\left(B + \frac{5\sqrt{AC}}{\sqrt{6}}\right)\left(B - \frac{5i\sqrt{AC}}{\sqrt{6}}\right)\left(B + \frac{5i\sqrt{AC}}{\sqrt{6}}\right) = 0.$$

For the Laurent expansion (7) to be valid, B satisfies this equation and c_4 is an arbitrary constant. Therefore, $B = 0$ or $B = \pm\frac{5\sqrt{AC}}{\sqrt{6}}$ or $B = \pm\frac{5i\sqrt{AC}}{\sqrt{6}}$, where $i^2 = -1$. For other B it would be necessary to add logarithmic terms to the expansion, thus giving a branch point rather than a pole.

(i) For $B = 0$, Eq. (2) is completely integrable by standard techniques and the solutions are expressible in terms of elliptic functions (*cf.* [17]); *i.e.*, the elliptic general solutions of Eq. (2)

$$w_{1d}(z) = -6\frac{A}{D}\left\{-\wp(z) + \frac{1}{4}\left[\frac{\wp'(z)+M}{\wp(z)-N}\right]^2\right\} + 6\frac{AN}{D} - \frac{C}{2D}.$$

Here, $12A^2g_2 = C^2$, $M^2 = 4N^3 - g_2N - g_3$, g_3 and N are arbitrary.

In particular, which degenerates to the simply periodic solutions

$$w_{1s}(z) = -6\frac{A}{D}\alpha^2\coth^2\frac{\alpha}{2}(z-z_0) - \frac{A}{2D}\alpha^2 - \frac{C}{2D},$$

where $A^2\alpha^4 = C^2$, $z_0 \in \mathbb{C}$.

And the rational function solutions

$$w_{1r}(z) = -\frac{6\frac{A}{D}}{(z-z_0)^2},$$

where $C = 0$, $z_0 \in \mathbb{C}$.

For $B = \pm\frac{5\sqrt{AC}}{6}, \pm\frac{5i\sqrt{AC}}{6}$, we transform Eq. (2) into the autonomous part of the first Painlevé equation. In this way we find the general solutions.

(ii) For $B = \pm\frac{5\sqrt{AC}}{\sqrt{6}}$, setting $w(z) = f(z)u(s)$, $s = g(z)$, and substituting in Eq. (2), we find that the equation for $u(s)$ is

$$-\frac{A}{D}(g')^2u'' = \frac{u'g'}{D}\left\{2A\frac{f'}{f} + A\frac{g''}{g'} + B\right\} + \frac{u}{D}\left\{A\frac{f''}{f} + B\frac{f'}{f} + C\right\} + fu^2. \tag{11}$$

If we take f and g such that

$$\begin{aligned} &A\frac{f''}{f} + B\frac{f'}{f} + C = 0, \\ &2A\frac{f'}{f} + A\frac{g''}{g'} + B = 0, \end{aligned} \tag{12}$$

then Eq. (11) for u is integrable. By (12), one takes $f(z) = \exp\{\alpha z\}$ and

$$g(z) = \beta \exp\left\{-\left(\frac{B}{A} + 2\alpha\right)z\right\},$$

where $\alpha = \mp\frac{2}{\sqrt{6}}\sqrt{\frac{C}{A}}$, $\beta^2 = -\frac{D}{C}$.

Thus Eq. (11) reduces to

$$u'' = 6u^2. \tag{13}$$

The general solutions of Eq. (13) are the Weierstrass elliptic functions $u(s) = \wp(s - s_0; 0, g_3)$, where s_0 and g_3 are two arbitrary constants.

Therefore, when $B = \pm\frac{5\sqrt{AC}}{\sqrt{6}}$, the general solutions of Eq. (2)

$$w_{g1}(z) = \exp\left\{\mp\frac{2}{\sqrt{6}}\sqrt{\frac{C}{A}}z\right\}\wp\left(\sqrt{-\frac{D}{A}}\exp\left\{\mp\frac{1}{\sqrt{6}}\sqrt{\frac{C}{A}}z\right\} - s_0; 0, g_3\right),$$

where both s_0 and g_3 are arbitrary constants. In particular, by Theorem 4.1 and $g_3 = 0$, $w_{g,i}(z)$ degenerates to the one parameter family of solutions

$$w_{f1}(z) = -\frac{C}{D}\frac{1}{\{1 - \exp\{\pm\frac{(z-z_0)}{\sqrt{6}}\sqrt{\frac{C}{A}}\}\}^2},$$

where $z_0 \in \mathbb{C}$.

(iii) For $B = \pm\frac{5i\sqrt{AC}}{\sqrt{6}}$, setting $w(z) = f(z)u(s) - \frac{C}{D}$, $s = g(z)$, and substituting in Eq. (2), we obtain that the equation for $u(s)$ is

$$u'' = 6u^2, \tag{14}$$

where

$$f(z) = \exp\{\alpha z\}, \qquad g(z) = \beta \exp\left\{-\left(\frac{B}{A} + 2\alpha\right)z\right\},$$

where $\alpha = \mp\frac{2i}{\sqrt{6}}\sqrt{\frac{C}{A}}$, $\beta^2 = \frac{D}{C}$. The general solutions of Eq. (14) are the Weierstrass elliptic functions $u(s) = \wp(s - s_0; 0, g_3)$, where s_0 and g_3 are two arbitrary constants.

Therefore, when $B = \pm\frac{5i\sqrt{AC}}{\sqrt{6}}$, we know the general solutions of Eq. (2),

$$w_{g1,i}(z) = \exp\left\{\mp\frac{2i}{\sqrt{6}}\sqrt{\frac{C}{A}}z\right\}\wp\left(\sqrt{\frac{D}{A}}\exp\left\{\mp\frac{i}{\sqrt{6}}\sqrt{\frac{C}{A}}z\right\} - s_0; 0, g_3\right) - \frac{C}{D},$$

where both s_0 and g_3 are arbitrary constants. In particular, by Theorem 4.1 and $g_3 = 0$, $w_{g,i}(z)$ degenerates to the one parameter family of solutions

$$w_{f1,i}(z) = \frac{C}{D}\frac{1}{\{1-\exp\{\pm\frac{i(z-z_0)}{\sqrt{6}}\sqrt{\frac{C}{A}}\}\}^2} - \frac{C}{D},$$

where $z_0 \in \mathbb{C}$. □

Similarly, in the proof of Theorem 3.3, we transform Eq. (5) into the autonomous part of the second Painlevé equation

$$u'' = 2u^3. \tag{15}$$

Obviously, Eqs. (14) and (15) are also special cases of Eqs. (1) and (3), respectively. We also know that there are six classes of Painlevé equations. Therefore, we ask naturally whether or not there exist other four classes autonomous parts of Painlevé equations could be transformed by $w(z) = f(z)u(s)$, $s = g(z)$ from the related equations; *i.e*, we propose the following open questions.

Question 4.1 Find all meromorphic solutions of the other four classes autonomous parts of Painlevé equations:

(AP$_3$) $u'' = \frac{(u')^2}{u} + \gamma u^3 + \frac{\delta}{u}$;
(AP$_4$) $u'' = \frac{(u')^2}{2u} + \frac{3}{2}u^3 - 2\alpha u + \frac{\beta}{u}$;
(AP$_5$) $u'' = (\frac{1}{2u} + \frac{1}{u-1})(u')^2 + \frac{\delta u(u+1)}{u-1}$;
(AP$_6$) $u'' = \frac{1}{2}(\frac{1}{u} + \frac{1}{u-1})(u')^2$;

where α, β, γ and δ are arbitrary constants.

Question 4.2 Determine the related equations and find their meromorphic general solutions for each of the above equations (AP$_i$), $i = 3, 4, 5, 6$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WY and YW carried out the design of the study and performed the analysis. BX and JQ participated in its design and coordination. All authors read and approved the final manuscript.

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References

1. Ablowitz, MJ, Clarkson, PA: Solitons, Nonlinear Evolution Equations and Inverse Scattering. London Mathematical Society Lecture Note Series, vol. 149. Cambridge University Press, Cambridge (1991)
2. Matveev, VB, Salle, MA: Darboux Transformations and Solitons. Springer Series in Nonlinear Dynamics. Springer, Berlin (1991)
3. Hirota, R, Satsuma, J: Soliton solutions of a coupled KdV equation. Phys. Lett. A **85**(8-9), 407-408 (1981)
4. Olver, PJ: Applications of Lie Groups to Differential Equations, 2nd edn. Graduate Texts in Mathematics, vol. 107. Springer, New York (1993)
5. Li, JB, Liu, Z: Travelling wave solutions for a class of nonlinear dispersive equations. Chin. Ann. Math., Ser. B **3**(3), 397-418 (2002)
6. Tang, S, Huang, W: Bifurcations of travelling wave solutions for the generalized double sinh-Gordon equation. Appl. Math. Comput. **189**(2), 1774-1781 (2007)
7. Feng, D, He, T, Lü, J: Bifurcations of travelling wave solutions for $(2+1)$-dimensional Boussinesq type equation. Appl. Math. Comput. **185**(1), 402-414 (2007)
8. Tang, S, Xiao, Y, Wang, Z: Travelling wave solutions for a class of nonlinear fourth order variant of a generalized Camassa-Holm equation. Appl. Math. Comput. **210**(1), 39-47 (2009)
9. Tang, S, Zheng, J, Huang, W: Travelling wave solutions for a class of generalized KdV equation. Appl. Math. Comput. **215**(7), 2768-2774 (2009)
10. Malfliet, W, Hereman, W: The tanh method: I. Exact solutions of nonlinear evolution and wave equations. Phys. Scr. **54**(6), 563-568 (1996)
11. Fan, E: Uniformly constructing a series of explicit exact solutions to nonlinear equations in mathematical physics. Chaos Solitons Fractals **16**(5), 819-839 (2003)
12. Wang, ML: Solitary wave solutions for variant Boussinesq equations. Phys. Lett. A **199**, 169-172 (1995)
13. Kudryashov, NA: Meromorphic solutions of nonlinear ordinary differential equations. Commun. Nonlinear Sci. Numer. Simul. **15**, 2778-2790 (2010)
14. Demina, MV, Kudryashov, NA: From Laurent series to exact meromorphic solutions: the Kawahara equation. Phys. Lett. A **374**(39), 4023-4029 (2010)
15. Demina, MV, Kudryashov, NA: Explicit expressions for meromorphic solutions of autonomous nonlinear ordinary differential equations. Commun. Nonlinear Sci. Numer. Simul. **16**(3), 1127-1134 (2011)
16. Kudryashov, NA, Sinelshchikov, DI, Demina, MV: Exact solutions of the generalized Bretherton equation. Phys. Lett. A **375**(7), 1074-1079 (2011)
17. Yuan, WJ, Li, YZ, Lin, JM: Meromorphic solutions of an auxiliary ordinary differential equation using complex method. Math. Methods Appl. Sci. **36**(13), 1776-1782 (2013). doi:10.1002/mma.2723
18. Yuan, WJ, Huang, Y, Shang, YD: All travelling wave exact solutions of two nonlinear physical models. Appl. Math. Comput. **219**(11), 6212-6223 (2013). doi:10.1016/j.amc.2012.12.023
19. Yuan, WJ, Shang, YD, Huang, Y, Wang, H: The representation of meromorphic solutions of certain ordinary differential equations and its applications. Sci. Sin., Math. **43**(6), 563-575 (2013). doi:10.1360/012012-159
20. Yuan, WJ, Huang, ZF, Lai, JC, Qi, JM: The general meromorphic solutions of an auxiliary ordinary differential equation using complex method and its applications. Stud. Math. (2013, to appear)
21. Yuan, WJ, Xiong, WL, Lin, JM, Wu, YH: All meromorphic solutions of an auxiliary ordinary differential equation and its applications. Acta Math. Sci. (2012, to appear)
22. Yuan, WJ, Xiong, WL, Lin, JM, Wu, YH: All meromorphic solutions of a kind of second order ordinary differential equation and its applications. Complex Var. Elliptic Equ. (2013, to appear)
23. Yuan, WJ, Huang, ZF, Fu, MZ, Lai, JC: The general solutions of an auxiliary ordinary differential equation using complex method and its applications. Adv. Differ. Equ. **2014**, 147 (2014). doi:10.1186/1687-1847-2014-147
24. Lang, S: Elliptic Functions, 2nd edn. Springer, New York (1987)
25. Conte, R, Musette, M: Elliptic general analytic solutions. Stud. Appl. Math. **123**(1), 63-81 (2009)

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