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Properties of convergence of a class of iterative processes generated by sequences of self-mappings with applications to switched dynamic systems

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Abstract

This article investigates the convergence properties of iterative processes involving sequences of self-mappings of metric or Banach spaces. Such sequences are built from a set of primary self-mappings which are either expansive or non-expansive self-mappings and some of the non-expansive ones can be contractive including the case of strict contractions. The sequences are built subject to switching laws which select each active self-mapping on a certain activation interval in such a way that essential properties of boundedness and convergence of distances and iterated sequences are guaranteed. Applications to the important problem of stability of dynamic switched systems are also given.

Keywords: expansive; non-expansive; contractive and strictly contractive self-mappings; switched dynamic systems; convergence; fixed point; stability

1 Introduction

The problems of boundedness and convergence of sequences of iterative schemes are very important in numerical analysis and the numerical implementation of discrete schemes. See [1–5] and references therein. In particular, [1] describes in detail and with rigor the associated problems linked to the theory of fixed points in various types of spaces like metric spaces, complete and compact metric spaces and Banach spaces. This book also contains, discusses and compares results of a number of relevant background references on the subject. In other papers, related problems are focused from a computational point of view including the acceleration of convergence using modified numerical methods like Aitken's delta-squared methods or Steffensen's method, [2–5]. On the other hand, there is also a rich background in theory and applications of fixed point theory related to non-expansive, contractive, weakly contractive and strictly contractive mappings as well as related to their counterparts in the framework of common fixed points and coincidence points for several mappings and in the framework of multivalued functions. A (non-exhaustive) list of recent related references is given including new results and a discussion of previous background ones. See, for instance, [1, 6–25], and references therein. Many efforts are also devoted to the formulation of extensions of the above problems to the study of existence and uniqueness of best proximity points in cyclic self-mappings, [8, 13, 14, 16–21], to that of proximal

contractions, [13, 14] and to the characterization of approximate fixed and coincidence points, [22, 23]. Direct applications of fixed point theory to the study of the stability of dynamic systems including the property of ultimate boundedness for the trajectory solutions having mixed non-expansive and expansive properties through time or being subject to impulsive controls have been given in [21] and [24, 25]. Some recent studies of best proximity points of weak ϕ-contractions in ordered metric spaces have been performed in [26]. On the other hand, the existence of best proximity points for 2-cyclic operators in uniformly convex Banach spaces is investigated in [27]. Finally, in [28], it has been proved that some previous fixed point results and some recently announced best proximity results are equivalent.

This paper is focused on the study of boundedness and convergence of sequences of distances and iterated points and the characterization of fixed points of a class of composite self-maps in metric spaces. Such maps are built with combinations of sets of elementary self-maps which can be expansive or non-expansive and the last ones can be contractive (including the case of strict contractions). The composite maps are defined by switching rules which select some self-map (the 'active' self-map) on a certain interval of definition of the running index of the sequence of iterates being built. The above mentioned properties concerning the sequences of iterates being generated from the given initial points are investigated under particular constraints for the switching rule. Note, on the other hand, that the properties of controllability and stability of differential-difference and the various kinds of dynamic systems are of a wide interest in theory and applications [25, 29–52]. See, for instance, related problems associated with continuous-time, discrete-time, digital, and hybrid systems and those involving delayed dynamics [38–42] and [31, 34–37, 44–46], sampled-data systems under constant, non-uniform, and/or multirate sampling and switched systems [34, 36, 45, 46]. In this context, this paper also includes an application of the developed theoretical framework to the stability of (in general) nonlinear switched dynamic systems. The composite self-map generating the trajectory solution sequence from initial conditions is defined with sets of elementary self-maps being associated with distinct active parameterizations which are switched through time by switching rules which guarantee the fulfilment of the suitable properties.

2 Problem statement and main results

Consider a sequence $\{T_{\sigma(n)}\}$ of self-mappings from X to X where (X, d) is a metric space endowed with a metric $d : X \times X \to \mathbf{R}_{0+}$ and an iterative process

$$x_{n+1} = T_{\sigma(n)} x_n, \quad x_0 \in X, \forall n \in \mathbf{Z}_{0+}. \tag{2.1}$$

The subsequent nomenclature is used:

(1) The function $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$ is the so-called switching law where $\bar{q} = \{1, 2, \ldots, q\}$ if $q \in \mathbf{Z}_+$ is finite and $\bar{q} \equiv \mathbf{Z}_+$, otherwise, *i.e.* $q = \mathrm{card}(\bar{q}) = \chi_0$ (*i.e.* the infinity cardinal of a countable set).

(2) q is the set of distinct parameterizations of the sequence $ST = \{T_{\sigma(n)}\}$ of self-mappings on X in the sense that such a sequence contains a finite or infinite set $ST^* = \{T_1, T_2, \ldots, T_q\}$ of distinct self-mappings.

(3) q is the disjoint union of the sets $\bar{q} = q_{sc} \cup q_{0c} \cup q_{ne} \cup q_e$, where q_{sc}, q_{0c}, q_{ne}, and q_e are, respectively, the indexing sets of strict contractions, contractive self-mappings which

are not strict contractions, non-expansive self-mappings which are not contractive, and a class of expansive self-mappings which fulfil a specific expanding condition according to

$$d(T_i x, T_i y) \leq K_i d(x, y), \quad \forall x, y \in X \text{ with } K_i \in (0, 1) \text{ if } i \in q_{sc},$$

$$d(T_i x, T_i y) < d(x, y) = K_i d(x, y), \quad \forall x, y(\neq x) \in X \text{ with } K_i = 1 \text{ if } i \in q_{0c},$$

$$d(T_i x, T_i y) = K_i d(x, y) = d(x, y), \quad \forall x, y \in X \text{ with } K_i = 1 \text{ if } i \in q_{ne}, \tag{2.2}$$

$$K_{0i} d(x, y) \leq d(T_i x, T_i y) \leq K_i d(x, y),$$

$$\forall x, y \in X \text{ with } K_{0i}, K_i(> K_{0i}) \in (1, \infty) \text{ if } i \in q_e.$$

Note that $q_c = q_{sc} \cup q_{0c}$ indexes the whole set of contractive self-mappings in the set ST^*.

(4) The composite mapping $ST : \mathbf{Z}_{0+} \times \mathbf{Z}_+ \times X \to X, \forall n \in \mathbf{Z}_{0+}$ defined by

$$\hat{T}(n, n+m) = T_{\sigma(n+m-1)} \hat{T}(n, n+m-1)$$

$$= T_{\sigma(n+m-1)} T_{\sigma(n+m-2)} \cdots T_{\sigma(n)}, \quad \forall n \in \mathbf{Z}_{0+}, \forall m \in \mathbf{Z}_+; \tag{2.3}$$

in particular,

$$\hat{T}_n = \hat{T}(0, n) = T_{\sigma(n)} \hat{T}_{n-1} = T_{\sigma(n)} T_{\sigma(n-1)} \cdots T_{\sigma(0)} \tag{2.4}$$

with $\sigma(n) \in \bar{q}$, defines the sequence of iterations $\{x_n\}$ defined by

$$x_{n+1} = \hat{T}(m, n+1) x_m = \hat{T}_{\hat{\sigma}(n, m+1)} = T_{\sigma(n)} x_n = \hat{T}_n x_0, \quad \forall x_0 \in X, \forall m, n(\geq m) \in \mathbf{Z}_{0+}, \tag{2.5}$$

where $\hat{\sigma}(n, m+1) = \{\sigma(n), \sigma(n+1), \ldots, \sigma(m)\}$ is the sequence of configurations from n to $m+1$ defined by the switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$. One gets from recursion with (2.1)

$$d(\hat{T}_n x_0, \hat{T}_{n-1} x_0) \leq K(0, n) d(x_0, T_{\sigma(0)} x_0)$$

$$= K(0, n) d(x_0, T_{\sigma(0)} x_0), \quad \forall x_0 \in X, \forall n \in \mathbf{Z}_+ \tag{2.6}$$

if $\sigma(j) \notin q_{0c}, \forall j \in \bar{n} \cup \{0\}$, and

$$d(\hat{T}_{n+1} x_0, \hat{T}_n x_0) \leq K(0, n) K_{0c}(0, n) d(x_0, T_{\sigma(0)} x_0)$$

$$< K(0, n) d(x_0, T_{\sigma(0)} x_0), \quad \forall x_0 \in X, \tag{2.7}$$

if there is some $j \in \bar{n} \cup \{0\}$ such that $\sigma(j) \in q_{0c}$, where

$$K(0, n) = \prod_{j=0}^{n-1} [K_{\sigma(j)}],$$

$$K_{0c}(0, n) = \prod_{j(\in \sigma_{0c})=0}^{n-1} \left[\frac{d(T_{\sigma(j+1)} x_{j+1}, T_{\sigma(j+2)} x_j)}{d(T_{\sigma(j)} x_j, T_{\sigma(j+1)} x_{j+1})} \right] (< 1), \quad \forall n \in \mathbf{Z}_{0+}. \tag{2.8}$$

The following simple example illustrates how the iterative process can work in a real situation.

Example 2.1 Consider the simple scalar discrete equation (2.1) with $x_{n+1} = T_{\sigma(n)}x_n = a_n x_n$ with $x_0 \in \mathbf{R}$, $\forall n \in \mathbf{Z}_{0+}$ under the Euclidean metric. The sequence $\{T_{\sigma(n)} = a_n\}$ is such that $T_{\sigma(n)} \in \{T_1, T_2, T_3\}$, $\forall n \in \mathbf{Z}_{0+}$, where $T_1 = \{z \in \mathbf{R} : z \leq \alpha\}$, $T_2 = \{1\}$, and $T_3 = \{z \in \mathbf{R} : z \geq \beta\}$ for some given real constants $-1 < \alpha < 1$ and $\beta > 1$. It is clear that the self-mapping T_1 on $\mathbf{R}$ is a strict contraction with a contractive constant $K_1 = |\alpha| < 1$, the self-mapping T_2 on $\mathbf{R}$ is non-expansive with constant $K_2 = 1$, and the self-mapping T_3 on $\mathbf{R}$ is expansive with constants $K_{03} = 1$ and $K_3 = \beta > 1$. Note that the set of fixed points of T_i reduces to $\{0\}$, that is, $\text{Fix}(T_i) = \{0\}$ for $i \in \bar{q}\{0\}$ is the unique fixed, and equilibrium, point of T_1 and T_3 while $\text{Fix}(T_2) = \mathbf{R}$ with the whole $\mathbf{R}$ being an equilibrium set of T_2. Note also that $\{0\}$ is a stable equilibrium point of T_1, while it is an unstable equilibrium point of T_3.

The switching law is $\sigma : \mathbf{Z}_{0+} \to \bar{q} = \{1, 2, 3\} \subseteq \mathbf{Z}_+$. If $\sigma(n) = i (\in \bar{q})$, $\forall n \in [n_1, n_2] \cap \mathbf{Z}_{0+}$, it is said that the ith configuration of (2.1) is active in the interval $[n_1, n_2]$. If $\sigma(n_2) = j(\in \bar{q}) \neq i$, then it is said that n_2 is a switching sample of (2.1) since the active configuration becomes modified with such a sample.

Lemma 2.2 *Consider the iterative process* (2.1) *built with a sequence* $\{T_{\sigma(n)}\}$ *of self-mappings from X to X such that (X, d) is a metric space. Assume a given switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$ such that there are a strictly increasing testing sequence of integers $S = S(\sigma) = \{n_k\}$, which may be non-unique, with $n_0 = 0$, and an associate sequence of positive real numbers $S_\rho = S_\rho(S) = \{\rho_k\}$ verifying $n_{k+1} - n_k \leq \mu < +\infty$ and $K(n_k, n_{k+1}) = \prod_{j=n_k}^{n_{k+1}-1}[K_{\sigma(j)}] \leq \rho_k$, $\forall k \in \mathbf{Z}_{0+}$, where $\mu = \mu(S)$. Then the metric has the following properties:*

(i) If $\rho_k < 1$, $\forall k \in \mathbf{Z}_{0+}$ then $d(x_{n+1}, x_n) \to 0$ as $n \to \infty$, $\forall x_0 \in X$, and $d(x_{n_{k+1}}, x_{n_k}) \leq (\sum_{i=0}^{\mu-1} M_0^i)\rho_k d(x_{n_k}, x_{n_{k-1}})$ $\forall k \in \mathbf{Z}_+$, $\forall x_0 \in X$, $d(x_{n_k}, x_{n_{k+1}}) \to 0$ as $k \to \infty$ and $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, $\forall x_0 \in X$, where $M_0 = \max_{i \in \bar{q}}(K_i)$.

(ii) If the switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$ is such that $\rho_k \leq \rho/(\sum_{i=0}^{\mu} M_0^i)$, $\forall k \in \mathbf{Z}_{0+}$, for some given real constant $\rho \in (0, 1)$ then all the sequence of composite self-mappings defined by $\hat{T}(n_k, n_{k+1}) = T_{\sigma(n_{k+1}-1)} \cdots \hat{T}_{\sigma(n_k)}$ on X are strict contractions of contractive constant ρ.

(iii) If $\rho_k > 1$, $\forall k \in \mathbf{Z}_{0+}$ then $d(x_{n+1}, x_n) \to +\infty$ as $n \to \infty$ for any given $x_0 \notin \text{Fix}(T_{\sigma(0)})$ in X.

(iv) Assume that $\rho_k \leq \bar{\rho} < 1$, $\forall k \in \mathbf{Z}_{0+}$ for $n_k \in S$ and $\rho_k \in S_\rho$, $\forall k \in \mathbf{Z}_{0+}$ and the existence of the limits $\lim_{k \to \infty}(n_{k+1} - n_k) = \mu^* \leq \mu < +\infty$ and $\hat{T}(n_k, n_{k+1}) \to \hat{T}^*(0, \mu^*)$ uniformly in X as $k \to \infty$, that is, $\lim_{k \to \infty} \sup \|\hat{T}(n_k, n_{k+1})x - \hat{T}^*(0, \mu^*)x\| = 0$, $\forall x \in X$, where $\mu^* = \mu^*(S)$. Then the self-mapping $\hat{T}^*(0, \mu^*)$ on X is a strict contraction and, for any given $N \in \mathbf{Z}_{0+}$, there is a real constant $\delta = \delta(N)$ such that $\{\delta(N)\} \to 0$ such that one has for all $k, N \in \mathbf{Z}_{0+}$

$$d(x_{k+N+1}, x_{k+N}) \leq \bar{\rho}^k d(x_{n_0}, x_{n_1}) + \frac{1 - \bar{\rho}^k}{1 - \bar{\rho}} \max_{i \in \mathbf{Z}_{0+}}(\delta_{n_i}), \tag{2.9}$$

$$\lim_{k, N \to \infty} d(x_{n_{k+N+1}}, x_{n_{k+N}}) = 0. \tag{2.10}$$

Proof Note that $n_0 = 0$ so that for any $n_k \in S$ and $k \in \mathbf{Z}_{0+}$

$$d(x_{n_{k+1}+1}, x_{n_{k+1}}) \leq K(n_k, n_{k+1})d(x_{n_k}, x_{n_{k+1}}) \leq \rho_k d(x_{n_k}, x_{n_{k+1}}), \quad \forall k \in \mathbf{Z}_{0+}$$

$$\leq \cdots \leq \left(\prod_{i=0}^{k}[\rho_i]\right) d(T_{\sigma(0)}x_0, x_0) < d(T_{\sigma(0)}x_0, x_0), \tag{2.11}$$

$$d(x_{n_{k+1}+j+1}, x_{n_{k+1}+j}) \le \left(\prod_{i=n_k+1}^{n_{k+1}+j} [K_{\sigma(i)}] \right) \left(\prod_{i=0}^{k} [\rho_i] \right) d(T_{\sigma(0)}x_0, x_0)$$

$$\le M(n_k, j) \left(\prod_{i=0}^{k} [\rho_i] \right) d(T_{\sigma(0)}x_0, x_0)$$

$$\le M \left(\prod_{i=0}^{k} [\rho_i] \right) d(T_{\sigma(0)}x_0, x_0) < M \left(\prod_{i=0}^{k} [\rho_i] \right) d(T_{\sigma(0)}x_0, x_0),$$

$$\forall k \in \mathbf{Z}_{0+}, \forall j \in \overline{n_k - n_{k+1} - 1} \cup \{0\} \tag{2.12}$$

(the last inequalities of (2.11)-(2.12) standing if and only if the respective left-hand-sides are nonzero) for some set of real bounded sequences $\{M(n_k, j)\}, j \in \overline{n_{k+2} - n_{k+1}}$ which satisfy $M(n_k, j) \le M = M_0 \max(1, M_0^{\mu-1}) < +\infty$ for some $M \in \mathbf{R}_+$ and $j \in \overline{n_{k+2} - n_{k+1}}$, since $\max_{k \in \mathbf{Z}_{0+}}(n_{k+1} - n_k) \le \mu < +\infty$, $\forall k \in \mathbf{Z}_{0+}$ and $K_i \in (1, \infty)$, $\forall i \in \bar{q}$. There is $k_0 \in \mathbf{Z}_{0+}$ such that $(\prod_{i=0}^{k}[\rho_i]) < 1/M$, since $\rho_i < 1$, $\forall i \in \mathbf{Z}_{0+}$, then one gets from (2.8)

$$d(x_{n_{k+1}+j+1}, x_{n_{k+1}+j}) < d(T_{\sigma(0)}x_0, x_0), \quad \forall k(\ge k_0) \in \mathbf{Z}_{0+}. \tag{2.13}$$

Also, since $\rho_k < 1$, $\forall k \in \mathbf{Z}_{0+}$, one gets from (2.11)-(2.12)

$$\lim_{k \to \infty} d(x_{n_k+j+1}, x_{n_k+j}) = 0, \quad \forall j \in \overline{n_{k+1} - n_k - 1} \cup \{0\}. \tag{2.14}$$

Since (2.14) involves any limit distance in-between any two adjacent elements of the solution sequence $\{x_n\}$ of (2.1) for any given $x_0 \in X$, one gets $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$. On the other hand, one gets from (2.11) and the triangle inequality

$$d(x_{n_{k+1}}, x_{n_k}) \le \sum_{i=1}^{n_{k+1}-n_k-1} d(x_{n_k+i}, x_{n_k+i+1}) \le \rho_k \left(\sum_{i=0}^{\mu-1} M_0^i \right) d(x_{n_k}, x_{n_{k+1}}), \quad \forall k \in \mathbf{Z}_{0+} \tag{2.15}$$

so that $d(x_{n_{k+1}}, x_{n_k}) \le (\sum_{i=0}^{\mu-1} M_0^i) \rho_k d(x_{n_k}, x_{n_{k-1}})$, $\forall k \in \mathbf{Z}_+$ and $d(x_{n_k}, x_{n_{k+1}}) \to 0$ as $k \to \infty$ from the triangle inequality, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ and $\max_{k \in \mathbf{Z}_{0+}}(n_{k+1} - n_k) \le \mu < +\infty$. Hence, property (i) follows. Property (ii) is proven by noting that if $\rho_k \le \rho/(\sum_{i=0}^{\mu-1} M_0^i)$, $\forall k \in \mathbf{Z}_{0+}$, for some real $\rho \in (0, 1)$ then

$$d(x_{n_{k+1}}, x_{n_k}) \le \rho d(x_{n_k}, x_{n_{k-1}}) \le \rho^2 d(x_{n_{k-1}}, x_{n_{k-2}}) \le \cdots \le \rho^k d(x_{n_0}, x_{m_1}). \tag{2.16}$$

Property (iii) is proven from the fact that expansive mappings of the switching law fulfil

$$d(x, y) < K_{0i}d(x, y) \le d(T_ix, T_iy), \quad \forall x, y \in X \text{ with } K_{0i} > 1 \text{ if } i \in q_e$$

and it follows from the assumptions that there is an infinite sequence $S = \{n_k\}$ such that $T_{\sigma(n_k)}$ on X are non-expansive self-mappings. Then $d(x_{n_{k+1}}, x_{n_{k+1}+1}) \ge (\prod_{i=0}^{k}[\rho_i])d(T_{\sigma(0)}x_0, x_0)$, $\forall k \in \mathbf{Z}_{0+}$ so that $\lim_{k \to \infty} d(x_{n_{k+1}}, x_{n_{k+1}+1}) = +\infty$ if $x_0 \notin \text{Fix}(T_{\sigma(0)})$. Thus, $\lim_{k \to \infty} d(x_{n_{k+1}+j}, x_{n_{k+1}+j+1}) = +\infty$ for all $j = 0, 1, \dots, n_{k+2} - n_{k+1} - 1$ since $n_{k+2} - n_{k+1} \le \mu$. Otherwise, it would be impossible with the triangle inequality of the metric to be bounded over a sum of a finite number of distances as $k \to \infty$ of which one is unbounded as $k \to \infty$. Hence,

property (iii) follows. To prove property (iv), note that $\delta_{n_k} = d(\hat{T}(n_{k-1}, n_k)x, \hat{T}(n_k, n_{k+1})x)$ leads to $\delta_{n_k} \to 0$ as $k \to \infty$, so that $\{\delta_{n_k}\}$ is bounded since $(n_{k+1} - n_k) \to \mu^*$ as $k \to \infty$ and $\hat{T}(n_k, n_{k+1}) \to \hat{T}^*(0, \mu^*)$ uniformly in X as $k \to \infty$. Then $\rho_k \leq \bar{\rho} < 1$, $\forall k \in \mathbf{Z}_{0+}$ for $n_k \in S$ and $\rho_k \in S_\rho$, $\forall k \in \mathbf{Z}_{0+}$ and the existence of the limits $\lim_{k \to \infty}(n_{k+1} - n_k) = \mu^* \leq \mu < +\infty$, $\lim_{k \to \infty} \rho_k = \rho^* \leq \bar{\rho} < 1$ and $\hat{T}(n_k, n_{k+1}) \to \hat{T}^*(0, \mu^*)$ uniformly in X as $k \to \infty$ imply that $\hat{T}^*(0, \mu^*)$ is a strict contraction on X and, furthermore, since $[d(\hat{T}(n_{k-1}, n_k)x_{n_k}, \hat{T}(n_k, n_{k+1})x_{n_k}) - d(\hat{T}(0, \mu^*)x_{n_k}, \hat{T}(0, \mu^*)x_{n_k})] \to 0$ as $k \to \infty$, one gets

$$
\begin{aligned}
d(x_{n_{k+1}}, x_{n_k}) &= d\big(\hat{T}(n_k, n_{k+1})x_{n_k}, \hat{T}(n_{k-1}, n_k)x_{n_{k-1}}\big) \\
&\leq d\big(\hat{T}(n_{k-1}, n_k)x_{n_{k-1}}, \hat{T}(n_{k-1}, n_k)x_{n_k}\big) + d\big(\hat{T}(n_{k-1}, n_k)x_{n_k}, \hat{T}(n_k, n_{k+1})x_{n_k}\big) \\
&\leq K(n_{k-1}, n_k)d(x_{n_{k-1}}, x_{n_k}) + \delta_{n_k} \\
&\leq \left(\prod_{i=1}^{k}[\rho_i]\right)d(x_{n_0}, x_{n_1}) + \sum_{i=0}^{k}\prod_{j=i+1}^{k}[\rho_i]\delta_{n_i} \\
&\leq \bar{\rho}^k d(x_{n_0}, x_{n_1}) + \frac{1 - \bar{\rho}^k}{1 - \bar{\rho}}\max_{i \in \mathbf{Z}_{0+}}(\delta_{n_i})
\end{aligned}
\tag{2.17}
$$

and then (2.9) holds. Also, (2.10) follows from (2.9), since $\lim_{N \to \infty} \max_{i(\geq N) \in \mathbf{Z}_{0+}}(\delta_{n_i}) = 0$ for any given $x_0 \in X$ and $\bar{\rho}^{n_{k+N} - n_N} \to 0$ as $k, N \to \infty$, and then

$$
\begin{aligned}
0 &\leq \limsup_{k \to \infty} d(x_{n_{k+1}}, x_{n_{k+2}}) \\
&\leq \limsup_{k \to \infty}\left(d(x_{n_{k+1}}, x_{n_{k+2}}) - \bar{\rho}^{\mu^*}d(x_{n_k}, x_{n_{k+1}}) - \frac{1 - \bar{\rho}^{\mu^*}}{1 - \bar{\rho}}\max_{i \in \mathbf{Z}_{0+}}(\delta_{n_i})\right) \\
&\leq \limsup_{k \to \infty}\left(d(x_{n_{k+1}}, x_{n_{k+2}}) - K(0, \mu^*)d(x_{n_k}, x_{n_{k+1}}) - \frac{1 - \bar{\rho}^{\mu^*}}{1 - \bar{\rho}}\max_{i \in \mathbf{Z}_{0+}}(\delta_{n_i})\right) \leq 0
\end{aligned}
\tag{2.18}
$$

so that $\lim_{k \to \infty} d(x_{n_{k+1}}, x_{n_{k+2}})$ exists and then, since $\lim_{k \to \infty}(n_{k+1} - n_k) = \mu^* \leq \mu < +\infty$ and $\hat{T}(n_k, n_{k+1}) \to \hat{T}^*(0, \mu^*)$ uniformly in X as $k \to \infty$ and (2.14)-(2.15) hold, one gets for any nonnegative integer $j \leq \mu^*$

$$
\begin{aligned}
\lim_{k \to \infty} d\big(x_{n_k}, \hat{T}(n_k, n_{k+1})x_{n_k}\big) &= \lim_{k \to \infty} d\big(x_{n_k}, \hat{T}^*(0, \mu^*)x_{n_k}\big) \\
&= \lim_{k \to \infty} d\big(x_{n_k+j}, \hat{T}(n_{k+j}, n_{k+j+1})x_{n_k}\big) \\
&= \lim_{k \to \infty} d\big(x_{n_k+j}, \hat{T}^*(0, \mu^*)x_{n_k+j}\big) \\
&= \lim_{k \to \infty} d\big(x_{n_k+\mu^*+j}, \hat{T}^*(0, \mu^*)x_{n_k+j}\big) = 0
\end{aligned}
\tag{2.19}
$$

and property (iv) has been proven. □

Remark 2.3 Note that the testing sequence $S = \{n_k\}$ of the given switching law in Lemma 2.2 is not necessarily associated with a set of strict contractions although the composite $K(n_k, n_{k+1})$, for $n_k \in S$, defines a composite strict contraction of non-necessarily strictly contractive self-mappings. The composite sequences $K(n, n_k)$ for $n \in [n_k + 1, n_{k+1})$, where $n_k \in S$, are not necessarily associated with a composite strict contraction. Note also that the testing sequence $S = \{n_k\}$ is not unique for a given switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$

since the only requirement is that it be strictly increasing with a maximum prescribed, but arbitrary, separation in-between any two adjacent elements of such a sequence.

Remark 2.4 Lemma 2.2 can be fulfilled, in general, by non-unique sequences $S = \{n_k\}$ of a given switching law in Lemma 2.2 as well as non-unique associated $\{\rho_k\}$, μ and μ^*. A typical case occurs when the switching law consists of strict contractions or converges to a sequence of strict contractions. Those ones can be grouped individually in the limit $\hat{T}^*(0, \mu^*)$ or this one may be a composite self-mapping of strict limit contractions so that the next result follows.

Theorem 2.5 *Assume an iterative procedure* (2.1) *built from a given switching law* $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$. *If the self-mapping* $\hat{T}^*(0, \mu^*)$ *on* X *is a strict contraction such that* $\hat{T}(n_k, n_{k+1}) \to \hat{T}^*(0, \mu^*)$ *uniformly in* X *as* $k \to \infty$ *with the strictly increasing testing sequence* $S = \{n_k\}$ *being subject to* $\max_{k \in \mathbf{Z}_{0+}}(n_{k+1} - n_k) \le \mu < +\infty$ *and* $\lim_{k \to \infty}(n_{k+1} - n_k) = \mu^* \le \mu$. *Then the following properties hold*:

(i) *There is a* (*in general, non-unique*) *decomposition of the composite* $\hat{T}^*(0, \mu^*) : X \to X$ *in a maximum number of* $p \in [1, \mu^*]$ *strict contractions* $\hat{T}_i^*(\bar{\mu}_{i-1}^*, \bar{\mu}_i^*) : X \to X$, $\forall i \in \bar{p}$, *with* $\mu_0^* = \bar{\mu}_0^* = 0$ *for the testing sequence* $S = \{n_k\}$, *such that* $\mu^* = \bar{\mu}^* = \sum_{i=1}^p \mu_i^*$ *so that*

$$\hat{T}^*(0, \mu^*) = \hat{T}^*(0, \bar{\mu}^*) = \hat{T}_p^*(\bar{\mu}_{p-1}^*, \bar{\mu}_p^*) \cdot \hat{T}_{p-1}^*(\bar{\mu}_{p-2}^*, \bar{\mu}_{p-1}^*) \cdot \ldots \cdot \hat{T}_1^*(0, \bar{\mu}_1^*), \qquad (2.20)$$

where $\bar{\mu}_i^* = \sum_{j=0}^i \mu_i^*$, $\forall i \in \bar{p}$.

(ii) *The decomposition* (2.20) *in a maximum number of strict contractions is unique if and only if the positive integer numbers* μ_i^*, $\forall i \in \bar{p}$ *such that* $\mu^* = \sum_{i=1}^p \mu_i^*$ *are unique. In particular, the decomposition* (2.20) *is unique if* $\mu_i^* = 1$, $\forall i \in \bar{p}$.

(iii) *Assume that, furthermore,* $(T_{\sigma(kn+j)} - T_{\sigma((k+1)n+j)}) \to 0$ *uniformly in* X *as* $n \to \infty$ *for some* $k(\ge \mu^*) \in \mathbf{Z}_{0+}$, $\forall j \in \overline{k-1} \cup \{0\}$ *so that* $\{\sigma(kn+j)\} \to \{\sigma_j^*\}$ *and* $\{K_{\sigma(kn+j)}\} \to \{K_j^*\}$, $\forall j \in \bar{\mu}^*$. *Then the decomposition* (2.20) *is unique in a maximum number of strict contractions if and only if*

$$\mu_i^* = \min_{z \ge \mu_{i-1}^*}\left(z \in \mathbf{Z}_+ : \left[\left(K_z^* < \frac{1}{\prod_{j=0}^{z-1}[K_j^*]} \right) \wedge \left(\prod_{j=z}^{\mu^*}[K_j^*] < 1 \right) \right] \right), \qquad (2.21a)$$

$$\mu_p^* = \min_{z \ge \mu_{p-1}^*}\left(z \in \mathbf{Z}_+ : K_z^* < \frac{1}{\prod_{j=0}^{z-1}[K_j^*]} \right), \qquad (2.21b)$$

$\forall i \in \overline{p-1}$ *since* $\mu_0^* = \bar{\mu}_0^* = 0$, *where* $p = (\max z \in \mathbf{Z}_+ : \mu^* = \sum_{i=1}^z \mu_i^*)$. *Then* $\mu^* = \sum_{i=1}^p \mu_i^*$ *with* $\mu_i^*(\le \mu^*) \in \mathbf{Z}_+$, $\forall i \in \bar{p}$ *being unique as well.*

Proof Since the switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$ is given, any strictly increasing sequence $S = \{n_k\}$ satisfying the constraints $\max_{k \in \mathbf{Z}_{0+}}(n_{k+1} - n_k) \le \mu < +\infty$, $\lim_{k \to \infty}(n_{k+1} - n_k) = \mu^* \le \mu$ and $\hat{T}(n_k, n_{k+1}) \to \hat{T}^*(0, \mu^*)$ uniformly in X as $k \to \infty$ for $n_k, n_{k+1} \in S$, with $\mu = \mu(S)$ and $\mu^* = \mu^*(S)$ being finite, imply that the composite-self-mapping $\hat{T}^*(0, \mu^*) : X \to X$ is a composite self-mapping of at least one and at most μ^* strict contractions. So, the maximum number p of strict limit contractions $p \in [1, \mu^*]$ in the composite $\hat{T}^*(0, \mu^*)$ for the sequence $\{T_{n_k}\}$ for $n_k \in S$ for a given sequence S exists and satisfies (2.20). Property (i) follows. Uniqueness of the decomposition (2.20) holds if there is a unique μ_i^*, $\forall i \in \bar{p}$ such

that $\mu^* = \sum_{i=1}^p \mu_i^*$ and $\hat{T}_i^*(\bar{\mu}_{i-1}^*, \bar{\mu}_i^*) : X \to X$, $\forall i \in \bar{p}$ are strict contractions. In particular, the decomposition (2.16) is trivially unique if $\mu_i^* = 1$; $\forall i \in \bar{p}$ so that $p = \mu^*$. Property (ii) has been proven.

The proof of property (iii) is split into two parts as follows:

- $p = 1$ and $\mu_1^* = \bar{\mu}_1^* = \mu^*$. Since (2.21a)-(2.21b) hold, $\mu_1^* \in \mathbf{Z}_+$, subject to $1 \le \mu_1^* \le \mu^*$ exists so that $\hat{T}(0, \mu_1^*)$ is a unique strict contraction on X. Thus, either (2.21b) holds with $1 \le \mu_1^* = \mu^*$ and the sufficiency part of the property is already proven for $p = 1$ with a strict contraction $\hat{T}(0, \mu_1^*) = \hat{T}(0, \mu^*)$ on X for $p = 1$ or $p > 1$ and (2.21a) holds for some integer $i \ge 1$. Necessity follows since if $p = 1$ and $\hat{T}(0, \mu^*)$ is a strict contraction then $\hat{T}(0, \mu_1^*) = \hat{T}(0, \mu^*)$ for $\mu_1^* = \mu^*$. Uniqueness is trivial since there is a single self-mapping in the decomposition of the composite self-mapping $\hat{T}(0, \mu^*)$ on X.

- $p \ge 2$. The proof of sufficiency is proven by complete induction. Assume that (2.21a)-(2.21b) hold so that $\mu_1^* \in \mathbf{Z}_+$, subject to $1 \le \mu_1^* \le \mu^*$, has to exist so that $\hat{T}(0, \mu_1^*)$ is a unique strict contraction on X since $\mu^* \in \mathbf{Z}_+$ exists such that $\hat{T}(0, \mu^*)$ exists, being a strict contraction on X. Set $\bar{\mu}_0^* = \mu_0^* = 0$ and note that $\mu_j^* = \bar{\mu}_j^* - \bar{\mu}_{j-1}^*$ and $\bar{\mu}_j^* = \mu_j + \bar{\mu}_{j-1}^* = \mu_j + \sum_{i=0}^{j-1} \mu_i$, $\forall j \in \bar{p}$, so that $\hat{T}(\bar{\mu}_{j-1}^*, \bar{\mu}_j^*)$, then $\mu_j^* = \bar{\mu}_j^* - \bar{\mu}_{j-1}^*$ is a unique strict contraction on X for all $j(\le i - 1) \in \mathbf{Z}_+$ and some $i(\le p) \in \mathbf{Z}_+$ such that the set of positive integers $\{\mu_1^*, \mu_2^*, \ldots, \mu_{i-1}^*\}$, subject to $\bar{\mu}_{i-1}^* = \sum_{j=1}^{i-1} \mu_j^* < \mu^*$, is unique. Then the composite $\hat{T}(0, \bar{\mu}_{i-1}^*) = \hat{T}(\bar{\mu}_{i-2}^*, \bar{\mu}_{i-1}^*) \cdot \ldots \cdot \hat{T}(0, \mu_1^*)$ is a unique strict contraction, with $\bar{\mu}_i^* = \sum_{j=0}^i \mu_j^*$, $\forall i \in \bar{p}$, and the positive integer μ_i^* is unique so that the set $\{\mu_1^*, \mu_2^*, \ldots, \mu_i^*\}$, subject to $\sum_{j=1}^i \mu_j^* < \mu^*$, is also unique and $\hat{T}(0, \mu_j^*)$, for all $j(\le i) \in \mathbf{Z}_+$ and $i(\le p) \in \mathbf{Z}_+$, and then the composite $\hat{T}(0, \bar{\mu}_i^*) = \hat{T}(0, \mu_i^*) \cdot \ldots \cdot \hat{T}(0, \mu_1^*)$ are unique and strictly contractive on X. The proof follows by complete induction, $\forall i \in \bar{p}$ with the existence of a unique positive integer $p = p(\mu^*) = (\max z \in \mathbf{Z}_+ : \mu^* = \sum_{i=1}^z \mu_i^*)$. Then $\mu^* = \sum_{i=1}^p \mu_i^*$ with $\mu_i^*(\le \mu^*) \in \mathbf{Z}_+$, $\forall i \in \bar{p}$ being unique. Necessity follows since if (2.20) holds and (2.22) fails for some $i(\ge 2) \in \bar{p}$ for a given $p \ge 2$ then the factorization of the composite self-mapping $\hat{T}(0, \mu^*)$ on X, subject to $\mu^* = \sum_{i=1}^p \mu_i^*$, does not consist of strict contractions. Property (iii) has been proven. □

A particular case of the decomposition of the composite self-mapping (2.20) is when such a decomposition becomes invariant and equal to

$$\hat{T}^*(0, \mu^*) = \hat{T}_p^*(0, \mu_p^*) \cdot \hat{T}_{p-1}^*(0, \mu_{p-1}^*) \cdot \ldots \cdot \hat{T}_1^*(0, \mu_1^*)$$

under the invariance identities $\hat{T}_i^*(0, \mu_i^*) = \hat{T}_i^*(\bar{\mu}_{i-1}^*, \bar{\mu}_i^*)$, implied by the identities $\mu_i = \bar{\mu}_i^* - \bar{\mu}_{i-1}^*$, $\forall i \in \bar{p}$. If the collection of all the sequences $\hat{S} = \hat{S}(\sigma) = \{S = \{n_k\}\}$ for given $x_0 \in X$ and $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$ is considered under the given convergence hypotheses, it follows easily from the above reasoning that there is also a maximum number $\hat{p}$ of strict limit contractions $\hat{p} \in [1, \hat{\mu}^*]$, where

$$\hat{\mu}^* = \max\left(\mu^*(S) : S \in \hat{S}\right) = \max_{S \in \hat{S}}\left(\lim_{k \to \infty}(n_{k+1} - n_k) : n_k, n_{k+1} \in S\right) = \max_{S \in \hat{S}} \mu^*(S) \qquad (2.22)$$

of the (in general, non-unique) decomposition:

$$\hat{T}^*(0, \hat{\mu}^*) = \hat{T}_{\hat{p}}^*(\bar{\hat{\mu}}_{\hat{p}-1}^*, \bar{\hat{\mu}}_{\hat{p}}^*) \cdot \hat{T}_{\hat{p}-1}^*(\bar{\hat{\mu}}_{\hat{p}-2}^*, \bar{\hat{\mu}}_{\hat{p}-1}^*) \cdot \ldots \cdot \hat{T}_1^*(0, \bar{\hat{\mu}}_1^*). \qquad (2.23)$$

Note that the decomposition (2.20) of the composite $\hat{T}^*(0, \mu^*)$ on X is not unique, in general, and so it is not unique, in general, the decomposition (2.23). The decomposition

(2.20) is unique if the testing S converges to a finite subsequence of strict contractions so that $\mu^* = \sum_{i=1}^{p} \mu_i^*$ where p is the maximum number of strict contractions of the decomposition (2.20) and the numbers μ_i^*, $\forall i \in \bar{p}$ are unique. Uniqueness holds in the case that $p = \mu^*$ and $\mu_i^* = 1$, $\forall i \in \bar{p}$.

Note that the assumption $(T_{\sigma(kn+j)} - T_{\sigma((k+1)n+j)}) \to 0$ uniformly in X as $n \to \infty$ for some $k(\geq \mu^*) \in \mathbf{Z}_{0+}$, $\forall j \in \overline{k-1} \cup \{0\}$ in Theorem 2.5(iii) implies that $\{T_{\sigma(kn+j)}\} \to T_j^* = \hat{T}_j^*(0,1)$, $\forall j \in \overline{k-1} \cup \{0\}$. Note also that $k \geq \mu^*$ is a consequence of the fact that (2.20) is a decomposition of strict contraction while some of the limit self-mappings on X reached as a result of the uniform convergence constraint $\{T_{\sigma(kn+j)}\} \to T_j^* = \hat{T}_j^*(0,1)$, $\forall j \in \overline{k-1} \cup \{0\}$ can be expansive.

Theorem 2.6 *Consider the iterative process* (2.1) *under the assumptions of Lemma* 2.2. *Then the following properties hold*:

(i) *If Lemma* 2.2(ii) *holds then* $\{x_{n_k}\}$ *and* $\{x_{n_k}, x_{n_k+1}\}$, *with* $n_k \in S$, $\forall k \in \mathbf{Z}_{0+}$, *are Cauchy sequences, so bounded, and then convergent in X if, in addition,* (X,d) *is complete.*

(ii) *If Lemma* 2.2(iii) *holds with* $\rho^* \leq \rho/(\sum_{i=0}^{\mu-1} M_0^i)$, $\forall k \in \mathbf{Z}_{0+}$, *for some real* $\rho \in (0,1)$ *then property* (i) *holds.*

(iii) *If both Lemma* 2.2(ii) *and Theorem* 2.5(i) *hold then* $\{x_{n_k+\sum_{i=0}^{j} \mu_i^*}\}$ *are Cauchy sequences, so bounded, and then convergent in X if, in addition,* (X,d) *is complete for all* $j \in \bar{p} \cup \{0\}$ *and* $n_k \in S$, $\forall k \in \mathbf{Z}_{0+}$ *and* $\forall S \in \hat{S}$, *where* $\hat{S} = \hat{S}(\sigma) = \{S = \{n_k\}\}$.

(iv) *Assume that* (X,d) *is a compact metric space and that Lemma* 2.2(ii) *and Theorem* 2.5(i) *both hold. Then Fix* $\hat{T}(0,\mu^*) = \{z\}$ *for some* $z \in X$ *and* $\hat{T}(0,\mu^*): X \to X$ *is a strict Picard self-mapping and* $\{x_{n_k+\bar{\mu}_j^*}\} \to z$ *for any initial* $x_0 \in X$, *where* $\bar{\mu}_j^* = \sum_{i=1}^{j} \mu_i^*$, $\forall j \in \bar{p}$. *If* $q_{0c} = \emptyset$ *then the above result holds if* (X,d) *is a complete metric space.*

Proof If Lemma 2.2(i) holds then $\{x_{n_k}\}$ is a Cauchy sequence from (2.16), then bounded and, in addition, convergent in X if (X,d) is complete. The properties are extendable to $\{x_{n_k}, x_{n_k+1}\}$ from (2.11). Thus, property (ii) has been proven. Property (ii) follows from property (i) since $\rho_k \to \rho^* \in (0,1)$ as $k \to \infty$ and $\rho^* \leq \rho/(\sum_{i=0}^{\mu-1} M_0^i)$, $\forall k \in \mathbf{Z}_{0+}$, for some real $\rho, \rho^* \in (0,1)$ then for any given $\varepsilon \in \mathbf{R}_+$ there are $\bar{k} = \bar{k}(\varepsilon) \in \mathbf{Z}_{0+}$ and $\bar{k}^* = \bar{k}^*(\varepsilon)(\geq \bar{k}) \in \mathbf{Z}_{0+}$ such that for $k(> \bar{k}) \in \mathbf{Z}_{0+} \rho_k \leq \rho/(\sum_{i=0}^{\mu-1} M_0^i)$, $\forall k \geq \bar{k}$ and $\rho_k \leq \rho^*/(\sum_{i=0}^{\mu-1} M_0^i)$, $\forall k \geq \bar{k}^*$. To prove property (iii), note that if, in addition, Theorem 2.5(i) holds then $\{x_{n_{k+1}+\sum_{i=0}^{j} \mu_i^*} - x_{n_k+\sum_{i=0}^{j} \mu_i^*}\} \to 0$ since $\hat{T}^*(0,\mu^*) = \hat{T}^*(0,\bar{\mu}^*) = \hat{T}_p^*(\bar{\mu}_{p-1}^*, \bar{\mu}_p^*) \cdot \hat{T}_{p-1}^*(\bar{\mu}_{p-2}^*, \bar{\mu}_{p-1}^*) \cdots \hat{T}_1^*(0,\bar{\mu}_1^*)$ are strict contractions from (2.20), $\forall j \in \bar{p}$ and $\{x_{n_k}\}$ is a Cauchy sequence from property (i) since by defining $\bar{\mu}_j^* = \sum_{i=1}^{j} \mu_i^*$, $\bar{\rho}_j^* = K(0,\bar{\mu}_j^*) < 1$, $\forall j \in \bar{p}$, one gets

$$d(x_{n_{k+1}+\bar{\mu}_j^*}, x_{n_k+\bar{\mu}_j^*}) \leq d(x_{n_k}, x_{n_k+\bar{\mu}_j^*}) + d(x_{n_k}, x_{n_k+1}) + d(x_{n_{k+1}}, x_{n_{k+1}+\bar{\mu}_j^*})$$

$$\leq d(x_{n_k}, x_{n_k+1}) + \frac{2}{1-\bar{\rho}_j^*} \max\big(d(x_{n_k}, x_{n_k+1}), d(x_{n_{k+1}}, x_{n_{k+1}+1})\big), \quad (2.24)$$

$\forall j \in \bar{p}$, $\forall k \in \mathbf{Z}_{0+}$, and, from property (i), one gets

$$\lim_{k \to \infty} d(x_{n_{k+1}+\bar{\mu}_j^*}, x_{n_k+\bar{\mu}_j^*}) = \lim_{k \to \infty} d(x_{n_{k+1}+\bar{\mu}_j^*}, x_{n_k+\bar{\mu}_j^*}),$$

$$\lim_{k\to\infty} d\big(\hat{T}\big(n_{k+1} + \bar{\mu}_j^*, n_k + \bar{\mu}_j^*\big)x_{n_k+\bar{\mu}_j^*}, x_{n_k+\bar{\mu}_j^*}\big)$$

$$= d\Big(\lim_{k\to\infty} \big(\hat{T}\big(n_{k+1} + \bar{\mu}_j^*, n_k + \bar{\mu}_j^*\big)x_{n_k+\bar{\mu}_j^*}\big), \lim_{k\to\infty} x_{n_k+\bar{\mu}_j^*} \Big)$$

$$= d\Big(\lim_{k\to\infty} \big(\hat{T}\big(n_{k+1} + \bar{\mu}_j^*, n_k + \bar{\mu}_j^*\big)x_{n_k+\bar{\mu}_j^*}\big), \lim_{k\to\infty} \big(\hat{T}\big(0, \bar{\mu}_j^*\big)x_{n_k}\big) \Big)$$

$$= d\Big(\lim_{k\to\infty} \big(\hat{T}\big(\bar{\mu}_j^*, \bar{\mu}_j^* + \mu^*\big)\hat{T}\big(0, \bar{\mu}_j^*\big)x_{n_k}\big), \lim_{k\to\infty} \big(\hat{T}\big(0, \bar{\mu}_j^*\big)x_{n_k}\big) \Big)$$

$$= d\Big(\hat{T}\big(\bar{\mu}_j^*, \bar{\mu}_j^* + \mu^*\big)\hat{T}\big(0, \bar{\mu}_j^*\big) \lim_{k\to\infty} x_{n_k}, \hat{T}\big(0, \bar{\mu}_j^*\big) \lim_{k\to\infty} x_{n_k} \Big)$$

$$= d\big(\hat{T}\big(\bar{\mu}_j^*, \bar{\mu}_j^* + \mu^*\big)\hat{T}\big(0, \bar{\mu}_j^*\big)z, \hat{T}\big(0, \bar{\mu}_j^*\big)z \big) = 0, \quad \forall j \in \bar{p} \tag{2.25}$$

since $\{x_{n_k}\} \to z$ from property (i) and the interchange of the limit and the distance function from the Lipschitz-continuity of the contraction self-mappings $\hat{T}(0, \bar{\mu}_j^*)$ and $\hat{T}(\bar{\mu}_j^*, \bar{\mu}_j^* + \mu^*)\hat{T}(0, \bar{\mu}_j^*)$ on $\forall j \in \bar{p}$. Then $\{x_{n_k+\bar{\mu}_j^*}\} \to z_j = \hat{T}(0, \bar{\mu}_j^*)z$ and $\{x_{n_{k+1}+\bar{\mu}_j^*}\} \to z_j = \hat{T}(\bar{\mu}_j^*, \bar{\mu}_j^* + \mu^*)\hat{T}(0, \bar{\mu}_j^*)z$ are Cauchy sequences, $\forall j \in \bar{p}$, and then bounded. If, in addition, (X, d) is complete then $z_j \in X$, $\forall j \in \bar{p}$. Property (iii) has been proven for a sequence $S = \{n_k\}$. The proof for $\hat{S} = \{\{n_k\}\}$ is similar. On the other hand, one gets from (2.25)

$$d\big(\hat{T}\big(0, \mu^*\big)z_j, z_j\big) = 0, \quad \forall j \in \bar{p} \tag{2.26}$$

then $\hat{T}(0, \mu^*)z_j = z_j$ so that $z_j \in \operatorname{Fix} \hat{T}(0, \mu^*)$, $\forall j \in \bar{p}$. Since $\hat{T}(0, \mu^*) : X \to X$ is a strict contraction from Lemma 2.2(ii), it has a unique fixed point from Banach contraction principle since (X, d) is a compact metric space (*i.e.* totally bounded and complete, equivalently, if it very family of closed subsets of X with finite intersection property has a nonempty intersection). Then $z_j = z \in \operatorname{Fix} \hat{T}(0, \mu^*) = \{z\}$, $\forall j \in \bar{p}$ and $\hat{T}(0, \mu^*)z = z$ for some $z \in X$. As a result, $\{x_{n_k+\bar{\mu}_j^*}\} \to z$ for any initial condition $x_0 \in X$, $\forall j \in \bar{p}$. If there is no contractive self-mapping on X not being a strict contraction in the switching law, the above holds if (X, d) is just a complete metric space. □

Note that Theorem 2.6(iv) holds even if $\hat{T}_i^*(\bar{\mu}_{i-1}^*, \bar{\mu}_i^*) : X \to X$ for some $i \in \bar{p}$ is not contractive. However, the composite self-mapping $\hat{T}(0, \mu^*) : X \to X$ is a strict Picard self-mapping and $\{x_{n_k+\bar{\mu}_j^*}\} \to z$, $\forall j \in \bar{p}$, [1]. The error estimates and convergence rate are characterized in the subsequent result.

Theorem 2.7 *Assume that (X, d) is a compact (complete if $q_{0c} = \emptyset$) metric space and that Lemma 2.2(ii) and Theorem 2.5(i) jointly hold. Then the following respective a priori and a posteriori error estimates and convergence rate hold for the iterative process (2.1) for any $x_0 \in X$ if $\operatorname{Fix} \hat{T}(0, \mu^*) = \{z\}$:*

$$d(x_n, z) \le \left(\frac{K\rho^{*^n}}{1 - \rho^*} + o\big(|\varepsilon_n|\big) \right) d(x_0, x_1), \tag{2.27}$$

$$d(x_n, z) \le \left(\frac{K\rho^*}{1 - \rho^*} + o\big(|\varepsilon_n|\big) \right) d(x_{n-1}, x_n), \tag{2.28}$$

$$d(x_n, z) \le \big(K\rho^{*^n} + o\big(|\varepsilon_n|\big) \big) d(x_0, z), \tag{2.29}$$

$\forall n \geq n_0$ for any given $\varepsilon \in (0, 1 - \rho^*) \cap \mathbf{R}_+$, some real convergent sequence $\{\varepsilon_n\} \to 0$, with $|\varepsilon_n| \leq \varepsilon$, $\forall n \geq n_0$, and some $n_0 = n_0(\varepsilon) \in \mathbf{Z}_+$ and $K(\geq 1) = K(\varepsilon) \in \mathbf{R}_+$, where $\rho_k \leq \rho/(\sum_{i=0}^{\mu} M_0^i)$ converges $\{\rho_k\} \to \rho^* \in (0, 1)$.

Proof Given any $\varepsilon \in (0, 1 - \rho^*) \cap \mathbf{R}_+$, and since $\{\rho_{n_k}\} \to \rho^*$ from Lemma 2.2(ii), there are $n_{0i} = n_{0i}(\varepsilon) \in \mathbf{Z}_+$ $(i = 1, 2, 3)$ and $K_a(\geq 1) = K_a(\varepsilon) \in \mathbf{R}_+$ and $K_b(\geq 1) = K_b(\varepsilon) \in \mathbf{R}_+$ such that $\rho_{n_k} = \rho^* + \varepsilon_{n_k} \leq \rho^* + \varepsilon$ with $|\varepsilon_{n_k}| \leq \varepsilon$, $\forall n_k \geq n_{01}$ (since $\{\varepsilon_{n_k}\} \to 0$) and

$$d(x_{n_{02}}, x_{n_{02}+1}) \leq K_a \rho^{*n_{02}} d(x_0, x_1), \qquad d(x_{n_{03}}, z) \leq K_b \rho^{*n_{03}} d(x_0, z), \tag{2.30}$$

where $z \in X$ is the unique element in the set Fix $\hat{T}(0, \mu^*)$, since n_{02} and n_{03} are finite and $\min(K_a, K_b) \geq 1$, one gets for the testing sequence $S = S(\sigma) = \{n_k\}$ of a solution of the iterative scheme (2.1)

$$\max\big(d(x_{n_{02}}, x_{n_{02}+1}), d(x_{n_k-1}, x_{n_k}), d(x_{n_k-1}, z)\big)$$
$$\leq \max(K_a, K_b) \max\big(\rho^{*n_{02}} d(x_0, x_1), d(x_{n_k-1}, x_{n_k}), \rho^{*n_{03}} d(x_{n_k-1}, z)\big), \tag{2.31}$$

$\forall n \geq n_{02}$ since $\{x_{n_k}\} \to z \ (\in X)$ from Theorem 2.5(i) and Theorem 2.6(iv) which holds since Lemma 2.2(ii) and Theorem 2.5(i) hold and (X, d) is a compact (it suffices it be complete if $q_{0c} = \emptyset$) metric space. Then one gets by taking $K' = \max(K_a, K_b)$ and $n_0 = \max(n_{01}, n_{02}, n_{03})$:

$$d(x_{n_k}, z) \leq \left(\frac{\rho^{*n_k - n_{02}} + o(|\varepsilon_{n_k}|)}{1 - \rho^* - \varepsilon}\right) d(x_{n_{02}}, x_{n_1}) \leq K_a \left(\frac{\rho^{*n_k} + \rho^{*n_{02}} o(|\varepsilon_{n_k}|)}{1 - \rho^* - \varepsilon}\right) d(x_0, x_1)$$
$$\leq \left(\frac{K_a \rho^{*n_k}}{1 - \rho^* - \varepsilon} + o(|\varepsilon_{n_k}|)\right) d(x_0, x_1) \leq \left(\frac{K' \rho^{*n_k}}{1 - \rho^*} + o(|\varepsilon_{n_k}|)\right) d(x_0, x_1), \tag{2.32}$$

$$d(x_{n_k}, z) \leq \left(\frac{K' \rho^{*n_k}}{1 - \rho^*} + o(|\varepsilon_{n_k}|)\right) d(x_0, z) \leq \left(\frac{\rho^*}{1 - \rho^* - \varepsilon} + o(|\varepsilon_{n_k}|)\right) d(x_{n_k-1}, x_{n_k})$$
$$\leq \left(\frac{K' \rho^*}{1 - \rho^*} + o(|\varepsilon_{n_k}|)\right) d(x_{n_k-1}, x_{n_k}), \tag{2.33}$$

$$d(x_{n_k}, z) \leq \left(\frac{\rho^{*n_k - n_{03}} + \rho^{*n_{03}} o(|\varepsilon_{n_k}|)}{1 - \rho^* - \varepsilon}\right) d(x_{n_{03}}, z) \leq \left(\frac{K_b \rho^{*n_k}}{1 - \rho^* - \varepsilon} + o(|\varepsilon_{n_k}|)\right) d(x_0, z)$$
$$\leq \left(\frac{K' \rho^{*n_k}}{1 - \rho^*} + o(|\varepsilon_{n_k}|)\right) d(x_0, z), \tag{2.34}$$

since $\frac{1}{1-\rho^*-\varepsilon} < \frac{1}{1-\rho^*}$, $\frac{K_a \rho^{*n_{02}} o(|\varepsilon_{n_k}|)}{1-\rho^*-\varepsilon} = o(|\varepsilon_{n_k}|)$ and $\frac{K_b \rho^{*n_{03}} o(|\varepsilon_{n_k}|)}{1-\rho^*-\varepsilon} = o(|\varepsilon_{n_k}|)$. Now, denote in (2.20) the Lipschitz constants of the limit self-mappings on X which define the composite limit self-mapping $\hat{T}^*(0, \mu^*)$ on X by $K_j^* = K_j(0, \mu_j^*)$, $\forall j \in \bar{p}$. Now, note from (2.32) and the triangle inequality that the sequence $\{x_n\}$, which contains the testing subsequence $\{x_{n_k}\}$, satisfies for any $j \in \overline{n_{k+1} - n_k - 1}$, $\forall k(\geq n_0) \in \mathbf{Z}_+$:

$$d(x_{n_k+j}, z) \leq d(x_{n_k}, z) + \sum_{i=1}^{j} d(x_{n_k+i}, x_{n_k+i-1})$$
$$\leq \big(1 + K_1(n_k, n_k + 1) + K_1(n_k, n_k + 1) K_2(n_k + 1, n_k + 2) + \cdots$$

$$+ K_1(n_k, n_k + 1)K_2(n_k + 1, n_k + 2) \cdots K_j(n_k + j - 1, n_k + j))d(x_{n_k}, z)$$

$$\leq \big[\big(1 + K_1^* + K_1^* K_2^* + \cdots + K_1^* K_2^* K_3^* \cdots K_j^* \big) + \big(K_1(n_k, n_k + 1) - K_1^* \big)$$

$$+ \big(K_1(n_k, n_k + 1)K_2(n_k + 1, n_k + 2) - K_1^* K_2^* \big) + \cdots$$

$$+ \big(K_1(n_k, n_k + 1)K_2(n_k + 1, n_k + 2) \cdots K_j(n_k + j - 1, n_k + j)$$

$$- K_1^* K_2^* K_3^* \cdots K_j^* \big) \big]$$

$$\times \rho^{*(n_k + j - n)} \left[\left(\frac{K' \rho^{*^n}}{1 - \rho^*} + o(\varepsilon_n) \right) + \big| o(|\varepsilon_{n_k}|) - o(|\varepsilon_n|) \big| \right] d(x_0, x_1)$$

$$\leq \Big[1 + p \left(\max_{1 \leq j \leq p} \big(K_j^*, K_j^{*p} \big) + \max_{1 \leq j \leq p} \left(K(n_k, n_k + j) - \prod_{i=1}^{j} [K_i^*] \right) \right) + \cdots$$

$$+ \big(K_1(n_k, n_k + 1)K_2(n_k + 1, n_k + 2) \cdots K_j(n_k + j - 1, n_k + j)$$

$$- K_1^* K_2^* K_3^* \cdots K_j^* \big) \Big] \rho^{*(n_k + j - n)}$$

$$\times \left[\left(\frac{K' \rho^{*^n}}{1 - \rho^*} + o(|\varepsilon_n|) \right) + \big| o(|\varepsilon_{n_k}|) - o(|\varepsilon_n|) \big| \right] d(x_0, x_1)$$

$$\leq M(n_k) \left[\left(\frac{K' \rho^{*^n}}{1 - \rho^*} + o(|\varepsilon_n|) \right) \right] d(x_0, x_1)$$

$$+ M(n_k) \big| o(|\varepsilon_{n_k}|) - o(|\varepsilon_n|) \big| d(x_0, x_1) \tag{2.35}$$

for any $j \in \overline{n_{k+1} - n_k - 1}$, where $\varepsilon_j = \rho_j - \rho^*$ with $\rho^* \in (0, 1)$, $\forall j \in \mathbf{Z}_{0+}$, and

$$M(n_k) = \max_{1 \leq j \leq p} \left(\Big[1 + p \left(\max_{1 \leq j \leq p} \big(K_j^*, K_j^{*p} \big) + \max_{1 \leq j \leq p} \left(K(n_k, n_k + j) - \prod_{i=1}^{j} [K_i^*] \right) \right) + \cdots \right.$$

$$+ \big(K_1(n_k, n_k + 1)K_2(n_k + 1, n_k + 2) \cdots K_j(n_k + j - 1, n_k + j)$$

$$\left. - K_1^* K_2^* K_3^* \cdots K_j^* \big) \Big] \right) < +\infty \tag{2.36}$$

and one gets from (2.35) for any integer $n = n_k + j$, $\forall j \in \overline{n_{k+1} - n_k - 1}$, $\forall k \in \mathbf{Z}_{0+}$ with $K = K' \sup_{k \geq n_0} M(n_k) = \max(K_a, K_b) \sup_{k \geq n_0} M(n_k)$ that the *a priori* error estimate satisfies

$$d(x_n, z) \leq \left[\left(\frac{K' M(n_k) \rho^{*^n}}{1 - \rho^*} \right) + o(|\varepsilon_n| + |\varepsilon_{n_k}|) \right] d(x_0, x_1), \quad \forall n \in [n_k, n_{k+1}),$$

$$\forall k \in \mathbf{Z}_{0+} \tag{2.37}$$

and, since $\{\varepsilon_{n_k} : (n_k \geq n_0 \wedge k \in \mathbf{Z}_{0+})\}$ and $\{\varepsilon_n : n \geq n_0\}$ subject to $\{\varepsilon_{n_k} : (n_k \geq n_0 \wedge k \in \mathbf{Z}_{0+})\} \subseteq \{\varepsilon_n : n \geq n_0\}$, with both sequences converging to zero,

$$d(x_n, z) \leq \left[\left(\frac{K \rho^{*^n}}{1 - \rho^*} \right) + o(|\varepsilon_n|) \right] d(x_0, x_1), \quad \forall n \in [n_k, n_{k+1}), \forall k \in \mathbf{Z}_{0+} \tag{2.38}$$

and $(d(x_n, z)/d(x_0, x_1)) \to 0$ as $n \to \infty$ at exponential rate so that (2.27) is proven. Closely analogous proofs to that of (2.27) follow directly for (2.28) and (2.29). □

A discussion of cases of interest concerning the above result follows.

Remark 2.8 (1) Theorem 2.7 refers to the case when $\{\hat{T}(n_k, n_{k+1})\} \to \hat{T}^*(0, \mu^*)$ with $\hat{T}^*(0, \mu^*)$ being a composite strictly contractive self-mapping on X of the form (2.20), *i.e.* possessing p (non-necessarily strictly contractive) fixed configurations which are the limit of the switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$. $\hat{T}^*(0, \mu^*)$ is a strict Picard self-mapping on X as a result. Also, $\{\hat{T}(n+1, 0)x_0\} \to z$, satisfying (2.27)-(2.29), and $\{\hat{T}^{*^n}(0, \mu^*)x_0\} \to z$, satisfying (2.27)-(2.29) with $K = 1$ and $o(|\varepsilon_n|)$ being replaced with 0, for any given $x_0 \in X$, where Fix $\hat{T}^*(0, \mu^*) = \{z\}$.

(2) A particular case of interest of Theorem 2.7 is that when $\hat{T}^*(0, \mu^*) = \hat{T}^*(0, 1) = \hat{T}_j^*(0, 1) = T_j^*$, so that $\{T_{\sigma(n)}\} \to T_j^*$, with $\mu^* = 1$ and some $j \in \bar{q}$ (one of the configurations of the switching law) so that the limit self-mapping T_j^* on X is a strict contraction, and then a strict Picard self-mapping, and the switching law $\sigma : \mathbf{Z}_{0+} \to \bar{q} \subseteq \mathbf{Z}_+$ is such that $\{T_{\sigma(n)}\} \to T_j^*$ uniformly in X. Since the testing switching $\{n_k\}$ has the property $\lim_{k\to\infty}(n_{k+1} - n_k) = 1$, $\lim_{k\to\infty} M(n_k) = \lim_{n\to\infty} M(n) = 1$ so that an admissible choice of $K = \max(K_a, K_b)$ can be made in (2.27)-(2.29). The interpretation of the presence of the real constant $K \neq 1$ in the error estimates and convergence rate is due to the fact that the sequence of composite self-mappings $\{\hat{T}(0, n)\}$ governed by the switching law to build the iterative scheme $x_{n+1} = T_n x_n = \hat{T}(n+1, 0)x_0$ for any given $x_0 \in X$ is not of the form $\{T_j^{*^n}\}$, while $\{T_{\sigma(n)}\}$ converges to a uniform limit strictly contractive T_j^* on X with a unique fixed point $z_j \in X$ subject to $\lim_{k\to\infty}(n_{k+1} - n_k) = 1$ and $\{\hat{T}(n+1, 0)x_0\} \to z_j$ and $\{T_j^{*^n} x_0\} \to z_j$ for any given $x_0 \in X$ according to (2.27)-(2.29) with the replacement $z \to z_j$.

(3) Particular cases of interest of Remark 2.8(1)-(2) are when fixed points $z = 0$, respectively, $z_j = 0$ (which are also, in particular, globally asymptotically stable equilibrium points if the iterative scheme refers to a dynamic system). Thus, the iterative scheme converges exponentially fast to zero. If the formalism is concerned with a Banach space $(X, \|\cdot\|)$ endowed with the norm $\|\cdot\|$, with X being a linear space, we can use as metric the norm-induced metric so that (2.27)-(2.29) become

$$\|x_n\| \leq \left(\frac{K\rho^{*^n}}{1-\rho^*} + o(|\varepsilon_n|)\right)\|x_1 - x_0\| \leq \left(\frac{K\|T_{\sigma(0)} - 1\|\rho^{*^n}}{1-\rho^*} + o(|\varepsilon_n|)\right)\|x_0\|, \qquad (2.39)$$

$$\|x_n\| \leq \left(\frac{K\rho^*}{1-\rho^*} + o(|\varepsilon_n|)\right)\|x_n - x_{n-1}\|$$

$$\leq \left(\frac{K\|\hat{T}(n, 0) - \hat{T}(n-1, 0)\|\rho^*}{1-\rho^*} + o(|\varepsilon_n|)\right)\|x_0\|, \qquad (2.40)$$

$$\|x_n\| \leq \left(K\rho^{*^n} + o(|\varepsilon_n|)\right)\|x_0\|, \qquad (2.41)$$

$\forall n \geq n_0$. If a fixed point of $z \neq 0$ then (2.39)-(2.41) hold in closed forms under the replacements $\|x_n\| \to \|x_n - z\|$, $\forall n \geq n_0$ and $\|x_0\| \to \|x_0 - z\|$.

(4) Note that in the case that $\rho^* = 1$, the limit self-mapping $\hat{T}(0, \mu^*)$ is only guaranteed to be non-expansive. Then (2.39) and (2.40) do not hold. However, one gets from Theorem 2.7 and (2.41), with the replacements $\|x_n\| \to \|x_n - z\|$, $\forall n \geq n_0$ and $\|x_0\| \to \|x_0 - z\|$,

$$\|x_n - z\| \leq \left(K + o(|\varepsilon_n|)\right)\|x_0 - z\|, \quad \forall n \geq n_0 \qquad (2.42)$$

provided that the non-expansive self-mapping $\hat{T}^*(0, \mu^*)$ on X has a fixed point $z \in X$. Then one has $|\limsup_{n \to \infty} \|x_n\| - \|z\|| \leq \limsup_{n \to \infty} \|x_n - z\| \leq K\|x_0 - z\| + \|z\|$ so that $\limsup_{n \to \infty} \|x_n\| \leq K\|x_0\| + (1 + K)\|z\|$ and $\{x_n\}$ is bounded for any initial $x_0 \in X$ although convergence to the fixed point is not guaranteed. This is a well-known result from fixed point theory for non-expansive self-mappings and a well-known (non-asymptotic) global stability result in related problems of stability of dynamic systems which can have a global attractor which can be either a fixed point, which is also an equilibrium point which is not asymptotically stable, or a stable limit cycle. If $z = 0$ then the above result takes the simpler form $\limsup_{n \to \infty} \|x_n\| \leq K\|x_0\|$.

Remark 2.9 The given convergence properties to fixed points also hold if $\{\hat{T}(n_k, n_{k+1})\} \to \hat{T}(0, \mu^*)$ point-wisely in X and $\hat{T}(0, \mu^*)$ is a strict contraction on X. Assume that $\{\hat{T}(n_k, n_{k+1})\} \to \hat{T}(0, \mu^*)$ with $\{\hat{T}(n_k, n_{k+1})x_{n_k}\} \to x$ and $\{\hat{T}(0, \mu^*)x_0\} \to z$ so that

$$
\begin{aligned}
d(x, z) \leq\ & d\big(x, \hat{T}(n_k, n_{k+1})x_{n_k}\big) + d\big(\hat{T}(n_k, n_{k+1})x_{n_k}, \hat{T}(0, \mu^*)x_{n_k}\big) \\
& + d\big(\hat{T}(0, \mu^*)x_{n_k}, z\big) + d\big(\hat{T}(0, \mu^*)z, z\big) \\
\leq\ & d\big(x, \hat{T}(n_k, n_{k+1})x_0\big) + \big(\hat{T}(n_k, n_{k+1})x_{n_k}, \hat{T}(n_k, n_{k+1})x_0\big) \\
& + d\big(\hat{T}(n_k, n_{k+1})x_{n_k}, \hat{T}(0, \mu^*)x_{n_k}\big) \\
& + d\big(\hat{T}(0, \mu^*)x_{n_k}, z\big) + d\big(\hat{T}(0, \mu^*)z, z\big), \quad \forall k \in \mathbf{Z}_{0+}.
\end{aligned}
\tag{2.43}
$$

Then $\{d(\hat{T}(n_k, n_{k+1})x_{n_k}, \hat{T}(0, \mu^*)x_{n_k})\} \to 0$ from point-wise convergence, $\{d(\hat{T}(0, \mu^*)x_{n_k}, z)\} \to 0$, since z is a fixed point of $\hat{T}(0, \mu^*)$, $\{d(x, \hat{T}(n_k, n_{k+1})x_0)\} \to x$ and $\{d(\hat{T}(n_k, n_{k+1})x_{n_k}, \hat{T}(n_k, n_{k+1})x_0)\} \to x$. Taking the limit as $k \to \infty$ in (2.43) yields $x = z$.

Example 2.10 Retaking the simple Example 2.1, the following conclusions arise. Assume that $\{T_{\sigma(n)}\} \to \hat{T}(0, \mu^*) = (T_1 T_3 T_2)^\omega T_2^g$ which is a composite self-mapping composed of three, in general, composite self-mappings, subject to nonzero integers $\omega > 1$ and $g > \omega - 1$ with $\alpha\beta < 1$. In this case, there is a limit self-mapping $\hat{T}(0, \mu^*)$ which is a strict contraction and then a strict Picard mapping and has a unique fixed point $\{0\}$ to which any iteration built with (2.1) for any initial real condition converges. However, the decomposition in a maximum number of strict contractions is not unique since $0 \leq g_1 \leq g$ can be chosen arbitrarily in the subsequent decomposition by taking into account that $T_i : X \to X$, $i = 1, 2, 3$, commute:

$$
\begin{aligned}
\hat{T}(0, \mu^*) &= (T_1 T_3 T_2)^{\omega-1} \cdot \big(T_1 T_3 T_2^{g+1-\omega}\big) = \hat{T}_1\big(0, \mu_1^*\big) \cdot \hat{T}_2\big(\mu_1^*, \mu_1^* + \mu_2^*\big) \\
&= \big(T_1 T_3 T_2^{g+1-\omega}\big) \cdot (T_1 T_3 T_2)^{\omega-1} = \hat{T}_1\big(0, \mu_{01}^*\big) \cdot \hat{T}_2\big(\mu_{01}^*, \mu_{01}^* + \mu_{02}^*\big) \\
&= \big(T_1 T_3 T_2^g\big) \cdot (T_1 T_3)^{\omega-1} = \hat{T}_1\big(0, \mu_{11}^*\big) \cdot \hat{T}_2\big(\mu_{11}^*, \mu_{11}^* + \mu_{12}^*\big)
\end{aligned}
\tag{2.44}
$$

with $\omega^* = 3\omega + g$, $\mu_2^* = \omega^* - \mu_1^*$ and $\mu_1^* = 3(\omega - 1) \neq \mu_{01}^* = g + 3 - \omega < g + 2$. But if $g = 4\omega - 6$ then $\mu_{11}^* = g + 2 \neq \mu_{01}^* < g + 2$ (since $\omega > 1$) if $g \neq 4\omega - 6$.

3 Numerical examples
This section contains some numerical examples regarding the theoretical results stated in the previous section. Two examples are discussed. The first one considers a scalar time-
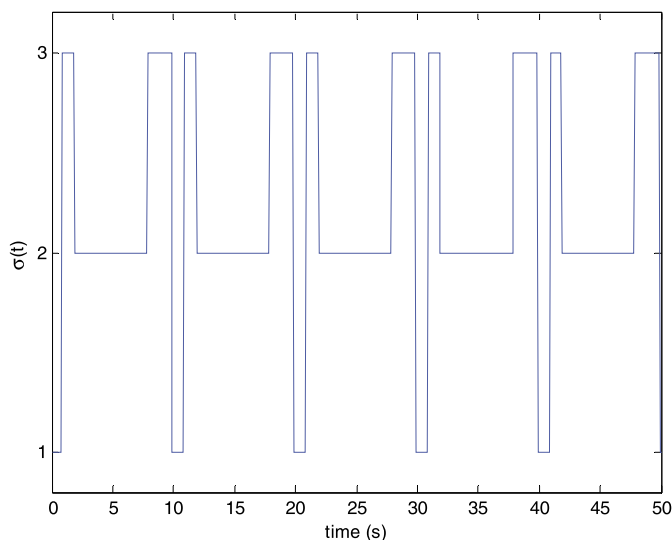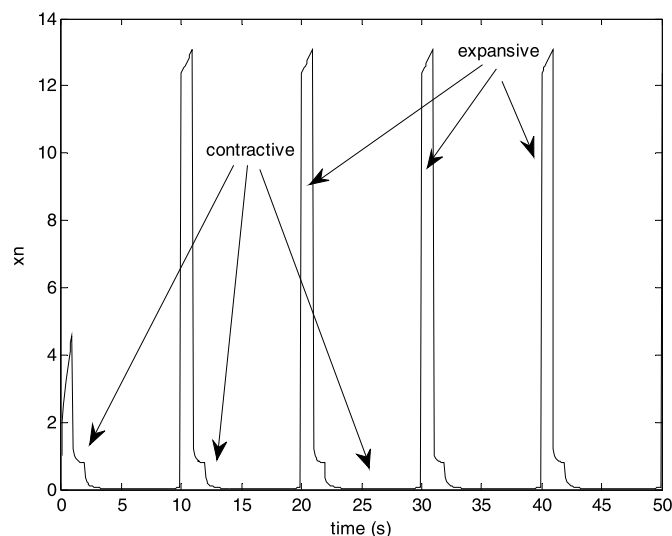
Figure 1 Switching law.

Figure 2 Solution trajectory for the switching rule depicted in Figure 1.

invariant nonlinear switched system while the second one deals with a linear time-varying switched system.

3.1 Scalar nonlinear switched system

Consider the nonlinear discrete-time dynamic system given by (2.1) with $\sigma(t) \in \{1, 2, 3\}$ and $T_1(x_n) = x_n + \frac{1}{x_n}$, $T_2(x_n) = 0.95 x_n e^{-x_n}$ and $T_3(x_n) = 1.2 \tanh(x_n)$. Note that $T_1(x_n)$ is a mixed type operator, being contractive for large x_n while expansive for small values of x_n (it is also non-strictly contractive in $[1, \infty)$), $T_2(x_n)$ is strictly contractive with $K_2 = 0.95$ and $T_3(x_n)$ is expansive with $K_3 = 1.2$. The stability of the nonlinear switched system (2.1) depends on the switching law. For instance, for the switching law depicted in Figure 1, the solution of the discrete-time system is shown in Figure 2.

Figure 3 Switching rule.

Figure 4 Solution trajectory for the switching rule depicted in Figure 3.
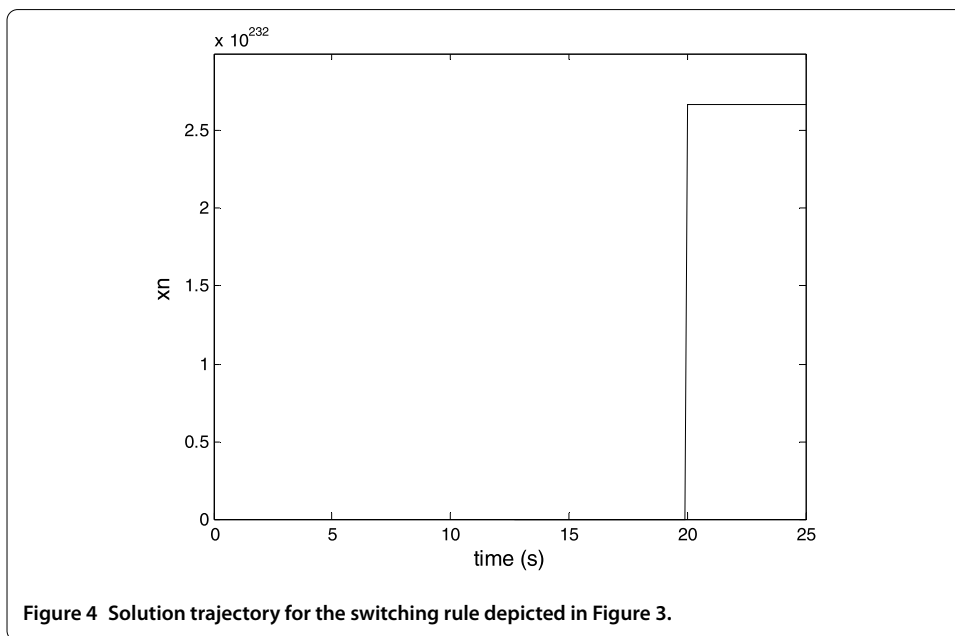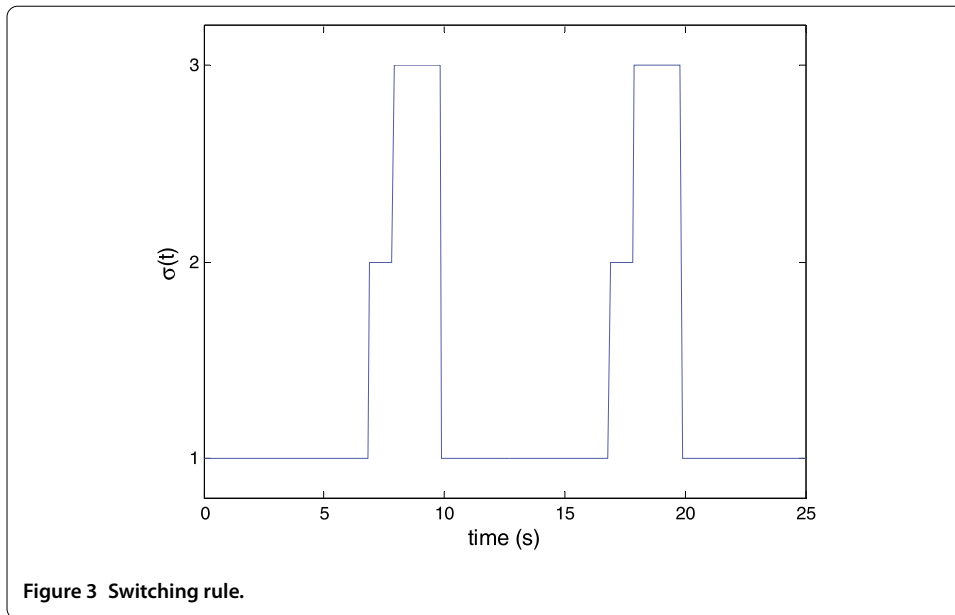
The contractive and expansive phases generated by the switching rule can be clearly appreciated in Figure 2. In this way, when the contractive operator, $T_2(x_n)$, is activated by the switching rule, the solution trajectory converges to zero while when the non-contractive operators are activated, the solution enlarges. Nevertheless, the combination of these operators according to the switching rule provides a globally stable system, with a bounded trajectory at all time. On the other hand, if the switching rule is given by Figure 3, the solution trajectory is displayed in Figure 4.

In this case, the trajectory enlarges very much as Figure 4 shows. The time scale in this figure has been reduced to handle the large values of the signal. Hence, a serious problem arises in this example: How can the stability of the switched system be proved under dif-

ferent switching rules? The results stated in Section 2 can be used to solve this question as the following example shows.

3.2 Linear time-varying switched dynamic system

This example shows how the previously presented theoretical results can be applied to study the asymptotic stability of time-varying dynamical systems by means of their discretization. This problem is of great importance in practice since many control systems are currently designed in a discrete-time set-up which leads to iterative schemes of the form (2.1) despite the system being originally in continuous-time. Therefore, the results introduced above might be used in a highly practical context. For this purpose, a linear time-varying switched dynamical system will be considered. These types of systems are described by a number of different parameterizations which are themselves time-varying. The switching rule selects the active parameterization during each time interval, as discussed in Section 2. The stability of time-varying systems is far from being trivial and offers a variety of behaviors different from those of linear time-invariant ones. For instance, a linear time-varying system may possess constant stable eigenvalues (*i.e.* constant eigenvalues with negative real part) and, nevertheless, to be asymptotically unstable in the Lyapunov sense, [51]. The tools provided by fixed point theory are useful in this situation where other techniques become intricate. Therefore, consider a linear time-varying system of the form

$$\dot{x}(t) = A_{\sigma(t)}(t)x(t), \tag{3.1}$$

where $\sigma(t) \in \{1, 2, 3\}$ and the dynamics matrices are given by

$$A_{\sigma(t)}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \qquad B_{\sigma(t)} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \tag{3.2}$$

$$B_1 = \begin{pmatrix} -1 & -4 \\ 0 & -1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & 4 \\ 1 & -1 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}.$$

It is easy to verify that matrices (3.2) are in the form $A_{\sigma(t)} = e^{\Omega t} B_{\sigma(t)} e^{-\Omega t}$, [52], with $\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In this way, the solution of (3.1) is given by

$$x(t) = e^{\Omega t} e^{(-\Omega + B_{\sigma(t_i)})t} x(t_i) \tag{3.3}$$

[52], with $\sigma(t_i^-) \neq \sigma(t_i^+) \equiv \sigma(t_i)$ and $\sigma(t) = \sigma(t_i)$ for $t \in [t_i, t_{i+1})$, $\forall i \in \mathbf{Z}_{0+}$. The eigenvalues of Ω are $\pm j$ implying that the stability of (3.3) is directly influenced by that of the matrices $(-\Omega + B_{\sigma(t)})$. Thus, we have

$$\text{spec}(-\Omega + B_1) = \{-1 - j\sqrt{3}, -1 + j\sqrt{3}\},$$

$$\text{spec}(-\Omega + B_2) = \{-1, -1\},$$

$$\text{spec}(-\Omega + B_3) = \{-1, 0\}.$$

Therefore, the first parameterization leads to an unstable system, the second to a stable one while the last one is marginally stable. Under these circumstances, the stability of the switched system relies directly on the switching law. That is, there will be switching
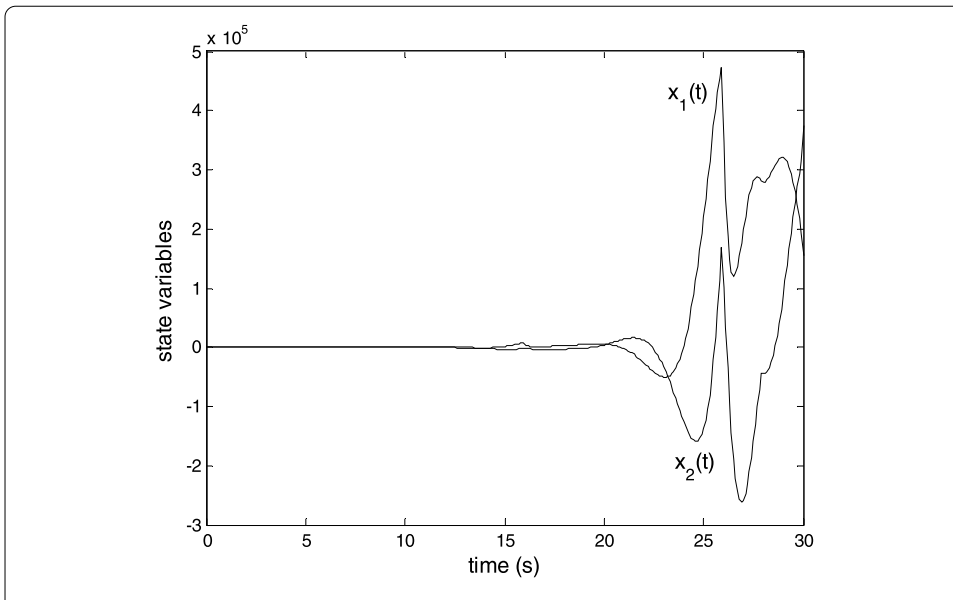
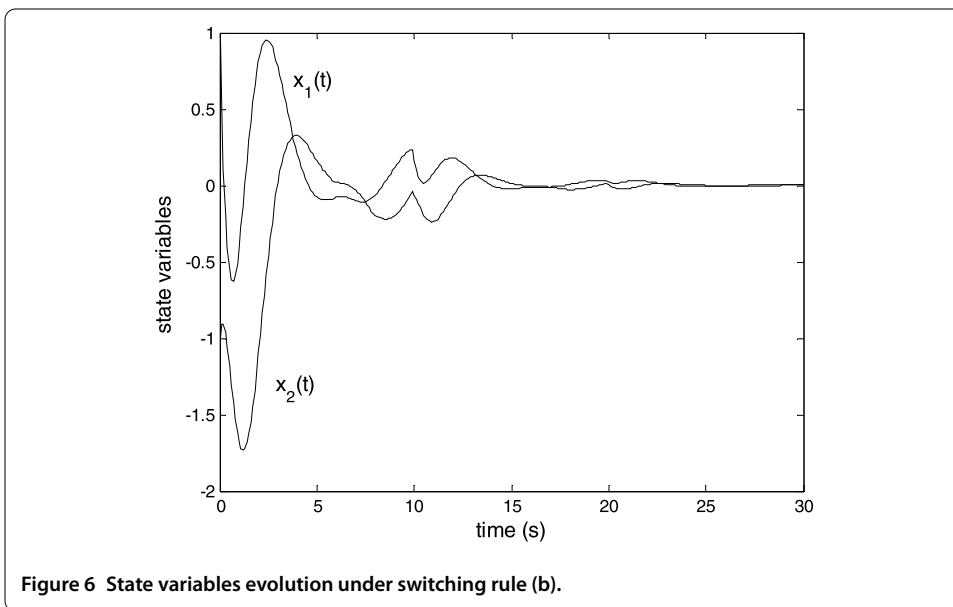**Figure 5 State variables evolution under switching rule (a).**

Figure 6 State variables evolution under switching rule (b).

laws under which the switched system is asymptotically stable while for others it may be unstable. To verify the stability of (3.1) under different switching laws, we will discretize the system so as to generate an iterative scheme of the form (2.1). Thus, from (3.3) if the sampling period is denoted by h and $x(nh) = x_n$, we have after a little bit of algebra:

$$x_{n+1} = e^{\Omega nh} e^{\Omega h} e^{(-\Omega + B_{\sigma(nh)})h} e^{-\Omega nh} x_n,$$

which fits into the iterative structure (2.1) with operators $T_{\sigma(n)} = e^{\Omega nh} e^{\Omega h} e^{(-\Omega + B_{\sigma(nh)})h} e^{-\Omega nh}$. It is remarkable the complexity of these operators for the time-varying system (3.1)-(3.2). Also, notice that all $e^{\Omega nh}$, $e^{\Omega h}$ and $e^{-\Omega nh}$ have eigenvalues with absolute value unity.

Figure 7 Switching rule (a).

Figure 8 Switching rule (b).
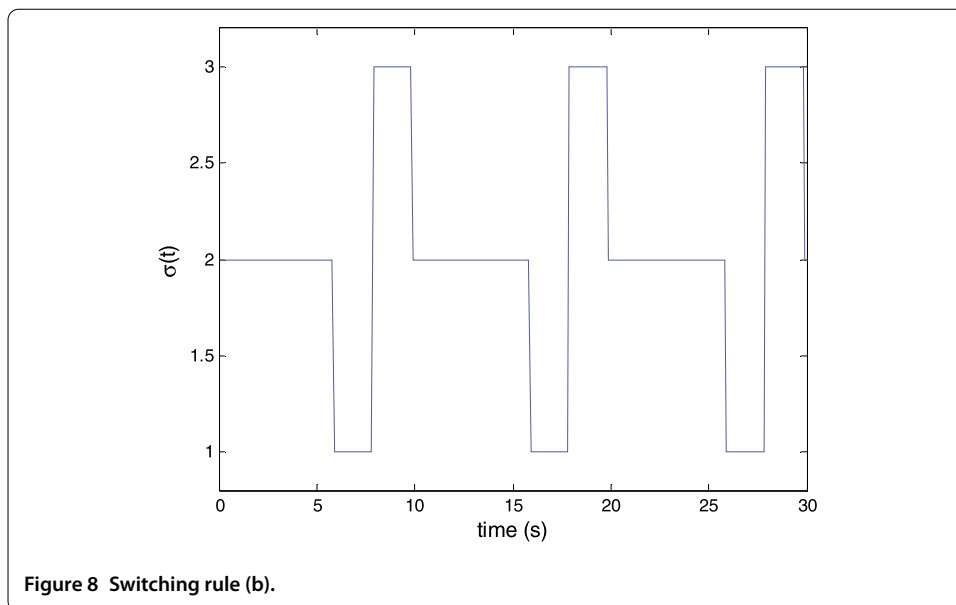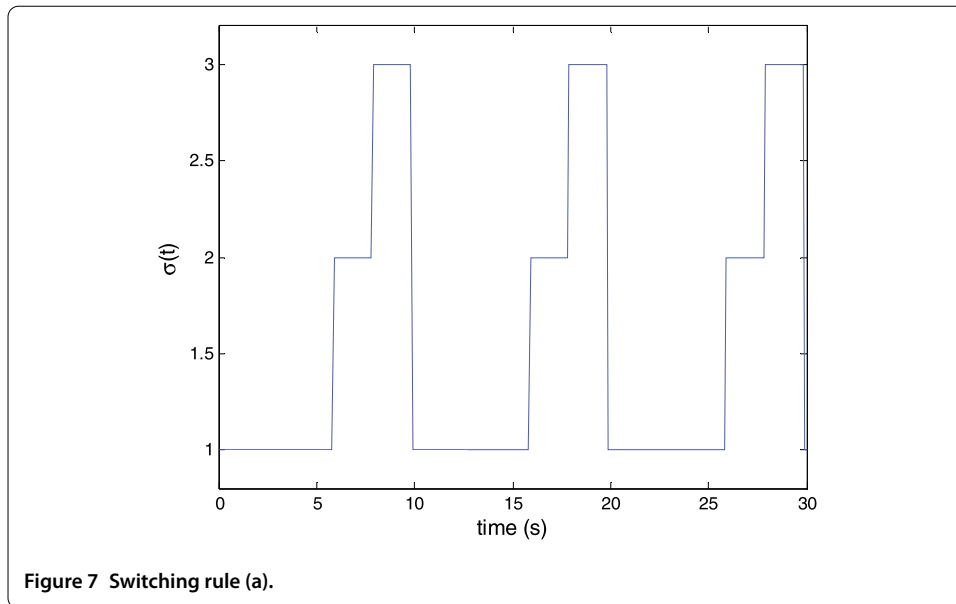
Therefore, the dynamics of the system is essentially controlled by $e^{(-\Omega+B_1)h}$, $e^{(-\Omega+B_2)h}$, and $e^{(-\Omega+B_3)h}$ according to the switching rule (*i.e.* there are only three different acting operators). For simulation purposes consider $h = 0.1$ seconds. Equations (2.2) are fulfilled for these operators with $K_1 = 1.076$, $K_2 = 0.905$ and $K_3 = 1$ corresponding to the expansive, strictly contractive and non-expansive (and not strictly contractive) cases (for the Euclidean distance). Now, Lemma 2.2 can be applied to establish the asymptotic stability of the origin for the switched system under different switching rules. For this purpose, assume that the switching signal is synchronized with the sampling period (*i.e.* the change in system's parameterization is at the same time that sampling). Let's consider these two cases: (a) the switching rule is periodic with period of 10 seconds, *i.e.* the switching rule is

the periodic extension of the following signal:

$$\sigma(t) = \begin{cases} 1, & 0 \le t < 6, \\ 2, & 6 \le t < 8, \\ 3, & 8 \le t < 10 \end{cases}$$

and (b) the switching rule is periodic with period:

$$\sigma(t) = \begin{cases} 2, & 0 \le t < 6, \\ 1, & 6 \le t < 8, \\ 3, & 8 \le t < 10. \end{cases}$$

Consider now the sequence of numbers $\{n_k\}$ satisfying $(n_{k+1} - n_k)h = 10$, implying that these numbers are selected when a period of the switching signal is completed. For case (a) we have $K(n_k, n_{k+1}) = 1.076^{60} \cdot 0.905^{20} \cdot 1^{20} = 10.94 > 1$ therefore, the switching rule (a) leads to an unstable switched system according to Lemma 2.2(iii). On the other hand, for case (b) we have $K(n_k, n_{k+1}) = 1.076^{20} \cdot 0.905^{60} \cdot 1^{20} = 0.01 < 1$ which leads to an asymptotically stable system according to Lemma 2.2(i). These results are corroborated by the numerical simulations shown in Figures 5 and 6 for initial condition $x_0 = [1\ -1]^T$.

It can be appreciated in Figure 5 that the state variables absolute values enlarge with time if switching rule (a) is employed while Figure 6 shows the convergence to zero of the state variables (and the asymptotic stability of the origin) when switching rule (b) is used. Switching rules (a) and (b) are depicted for convenience in Figures 7 and 8.

Hence, the stability of the linear time-varying switched system is analyzed by just calculating the product of some constants, easing off the determination of the stability properties of the system.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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