Muradov *Journal of Inequalities and Applications* 2014, 2014:495
http://www.journalofinequalitiesandapplications.com/content/2014/1/495

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On bases from perturbed system of exponents in Lebesgue spaces with variable summability exponent

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Abstract
In this paper the perturbed system of exponents with some asymptotics is considered. Basis properties of this system in Lebesgue spaces with variable summability exponent are investigated.

Keywords: system of exponents; perturbation; generalized Lebesgue space; variable exponent

1 Introduction

Consider the following system of exponents:

$$\{e^{i\lambda_n t}\}_{n\in Z},\tag{1}$$

where $\{\lambda_n\} \subset R$ is a sequence of real numbers, Z is a set of integer numbers. It is the aim of this paper to investigate basis properties (basicity, completeness, and minimality) of the system (1) in Lebesgue space L_{p_t} with variable summability index $p(t)$, when $\{\lambda_n\}$ has the asymptotics

$$\lambda_n = n - \alpha \operatorname{sign} n + O(|n|^{-\beta}), \quad n \to \infty,\tag{2}$$

where $\alpha, \beta \in R$ are some parameters.

Many authors have investigated the basicity properties of system of exponents of the form (1), beginning with the well-known result of Paley and Wiener [1] on Riesz basicity. Some of the results in this direction have been obtained by Young [2]. The criterion of basicity of the system (1) in $L_p \equiv L_p(-\pi,\pi)$, $1 < p < +\infty$, when $\lambda_n = n - \alpha \operatorname{sign} n$, has been obtained earlier in [3, 4].

Recently in connection with consideration of some specific problems of mechanics and mathematical physics [5, 6], interest in the study of the various questions connected with Lebesgue L_{p_t} and Sobolev $W_{p_t}^k$ spaces with variable summability index $p(t)$ has increased [5–9].

Many questions of the theory of operators (for example, theory of singular operators, theory of potentials and *etc.*) are studied in spaces L_{p_t} [7]. These investigations have allowed one to consider questions of basicity of some system of functions (for example, the classical system of exponents $\{e^{int}\}_{n\in Z})$ in L_{p_t}. In [9] the basicity of system $\{e^{int}\}_{n\in N}$ in L_{p_t} has been established. The special case of the system (1) is considered in [10–12], when $\lambda_n = n - \alpha \operatorname{sign} n$, $n \in Z$.

In this paper basis properties of the system (1) in L_{p_t} spaces are investigated. Under certain conditions on the parameters α and β equivalence of the basis properties (completeness, minimality, ω-linearly independence, basicity) of the system (2) in L_{p_t} are proved.

2 Necessary notion and facts

Let $p : [-\pi,\pi] \to [1,+\infty)$ be a Lebesgue measurable function. By L_0 we denote the class of all functions measurable on $[-\pi,\pi]$ with respect to Lebesgue measure. We choose the notation

$$I_p(f) \stackrel{\text{def}}{\equiv} \int_{-\pi}^{\pi} \left|f(t)\right|^{p(t)} dt.$$

Let $L \equiv \{f \in L_0 : I_p(f) < +\infty\}$. Let $p^- = \inf \operatorname{vrai}_{[-\pi,\pi]} p(t)$, $p^+ = \sup \operatorname{vrai}_{[-\pi,\pi]} p(t)$. For $p^+ < +\infty$, with respect to ordinary linear operations of addition of functions and multiplication by number, L turns into a linear space. If we define in L_{p_t} the norm

$$\|f\|_{p_t} \stackrel{\text{def}}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

then L is a Banach space and we denote it by L_{p_t}. Denote

$$H^{\ln} \stackrel{\text{def}}{\equiv} \left\{p : p(\pi) = p(-\pi) \text{ and } \exists C > 0, \forall t_1, t_2 \in [-\pi,\pi], |t_1 - t_2| \le \frac{1}{2} \right.$$
$$\left. \Rightarrow \left|p(t_1) - p(t_2)\right| \le \frac{C}{-\ln|t_1 - t_2|}\right\}.$$

Throughout this paper, $q(t)$ denotes the function conjugate to function $p(t)$, that is, $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$.

We have Hölder's generalized inequality,

$$\int_{-\pi}^{\pi} \left|f(t)g(t)\right| dt \le C\left(p^-;p^+\right) \|f\|_{p_t} \|g\|_{q_t},$$

where $C(p^-;p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

For our investigation we need some basic concepts of the theory of close bases, given as follows.

We adopt the standard notation: B-space is a Banach space; X^* is the conjugate to space X; $f(x)$, $f \in X^*$, and $x \in X$ means the value of functional f on x; $L[M]$ is a linear span of a set M. The system $\{x_n\}_{n\in N} \subset X$ is called ω-linear independent in B-space X, if $\sum_{n=1}^{\infty} \alpha_n x_n = 0$ true for $\alpha_n = 0$, $\forall n \in N$.

The following lemma is true.

Lemma 1 *Let X be a Banach space with basis $\{x_n\}_{n\in N} \subset X$ and $F : X \to X$ be a Fredholm operator. Then the following properties of the system $\{y_n = Fx_n\}_{n\in N}$ in X are equivalent*:

(1) $\{y_n\}_{n\in N}$ *is complete*;

(2) $\{y_n\}_{n\in N}$ *is minimal*;
(3) $\{y_n\}_{n\in N}$ *is* ω*-linear independent*;
(4) $\{y_n\}_{n\in N}$ *is isomorphic to* $\{x_n\}_{n\in N}$ *basis.*

We also need the following easily provable lemma.

Lemma 2 *Let* X *be a Banach space with basis* $\{x_n\}_{n\in N}$ *and* $\{y_n\}_{n\in N} \subset X : \operatorname{card}\{n : x_n \neq y_n\} < +\infty$. *Then the expression*

$$Fx = \sum_{n=1}^{\infty} x_n^*(x) y_n$$

generates the Fredholm operator $F : X \to X$, *where* $\{x_n^*\}_{n\in N} \subset X^*$ *is conjugate to* $\{x_n\}_{n\in N}$ *system*.

The following lemma is also true.

Lemma 3 *Let* $\{x_n\}_{n\in N}$ *be complete and minimal in B-space* X *and* $\{y_n\}_{n\in N} \subset X : \operatorname{card}\{n : x_n \neq y_n\} < +\infty$. *Then the following properties of system* $\{y_n\}_{n\in N}$ *in* X *are equivalent*:
(1) $\{y_n\}_{n\in N}$ *is complete*;
(2) $\{y_n\}_{n\in N}$ *is minimal.*

These and other results are obtained in [13, 14].

We will use the following statement, which has a proof similar to the proof of Levinson [15].

Statement 1 Let system $\{e^{i\lambda_n t}\}_{n\in Z}$ be complete in L_{p_t}. If from the system we remove n any functions and add instead of them n other functions $e^{i\mu_j t}$, $j = 1, \dots, n$, where $\mu_1, \dots, \mu_n$ are any, mutually different complex numbers not equal to any of numbers λ_k, then the new system will be complete.

We shall also need the following theorem of Krein-Milman-Rutman.

Theorem 1 (Krein-Milman-Rutman [13]) *Let* X *be a Banach space with norm* $\|\cdot\|$, $\{x_n\}_{n\in N} \subset X$ *be normed basis in* X (*i.e.* $\|x_n\| = 1$, $\forall n \in N$) *with conjugate system* $\{x_n^*\}_{n\in N} \subset X^*$, *and* $\{y_n\}_{n\in N} \subset X$ *be a system satisfying the inequality*

$$\sum_{n=1}^{\infty} \|x_n - y_n\| < \gamma^{-1},$$

where $\gamma = \sup_n \|x_n^*\|$. *Then* $\{y_n\}_{n\in N}$ *also forms a basis isomorphic to the basis* $\{x_n\}_{n\in N}$ *in* X.

3 Basic results

Before giving the basic results we will prove the following auxiliary theorem.

Theorem 2 *Let* $p \in H^{\ln}$ *and* $p^- > 1$. *If the system*

$$\{e^{i(n-\alpha \operatorname{sign} n)t}\}_{n\in Z}, \tag{3}$$

forms a basis in $L_{p_t} \equiv L_{p_t}(-\pi,\pi)$, *then this system is isomorphic to the classical system of exponents* $\{e^{int}\}_{n\in Z}$, *where the isomorphism is given by*

$$Sf = e^{-i\alpha t}\sum_{0}^{\infty}\bigl(f, e^{inx}\bigr)e^{int} + e^{i\alpha t}\sum_{1}^{\infty}\bigl(f, e^{-inx}\bigr)e^{-int}, \tag{4}$$

where

$$(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt.$$

Proof Consider the operator (4). From the basicity of system $\{e^{int}\}_{n\in Z}$ in L_{p_t} it follows that S is a bounded operator on L_{p_t} into itself. It is easy to see that $\operatorname{Ker} S = 0$. Actually, let $Sf = 0$. From the basicity of the system (3) in L_{p_t} and from (4) we obtain $(f, e^{inx}) = 0$, $\forall n \in Z$. Also, from the basicity of system $\{e^{int}\}_{n\in Z}$ in L_{p_t} it follows that $f = 0$. We show that for all $g \in L_{p_t}$, the equation $Sf = g$ in L_{p_t} is solved. Let us assume that

$$f = \sum_{n\in Z} g_n e^{int},$$

where $\{g_n\}_{n\in Z}$ are the biorthogonal coefficients of the function g by the system (3).

It is clear that $f \in L_{p_t}$, and so

$$\begin{aligned} Sf &= e^{-i\alpha t}\sum_{n=0}^{\infty}\bigl(f, e^{inx}\bigr)e^{int} + e^{i\alpha t}\sum_{n=1}^{\infty}\bigl(f, e^{-inx}\bigr)e^{-int} \\ &= e^{-i\alpha t}\sum_{n=0}^{\infty} g_n e^{int} + e^{i\alpha t}\sum_{n=1}^{\infty} g_{-n} e^{-int} = g, \end{aligned}$$

as by the condition of the theorem, the system (3) forms a basis in L_{p_t}.

This means that for all $g \in L_{p_t}$ the equation $Sf = g$ is solved in L_{p_t}. Then by the Banach theorem the operator S has a bounded inverse. It is obvious that $S[e^{int}] = A(t)e^{int}$, $n \geq 0$, and $S[e^{-int}] = B(t)e^{-int}$, $n \geq 1$. This completes the proof. □

Now we study some basis properties of the system (1). Firstly, we recall the following theorem.

Theorem 3 ([11]) *Let* $p \in H^{\ln}$ *and* $p^- > 1$. *If parameter* $\alpha \in R$ *satisfies the condition* $-\frac{1}{2p(\pi)} < \alpha < \frac{1}{2q(\pi)}$, *then the system* $\{e^{i\mu_n t}\}$ *forms a basis in* L_{p_t}.

Let the asymptotics (2) occur. Let us assume $\mu_n = n - \alpha \operatorname{sign} n$ and $\delta_n = \lambda_n - \mu_n$, $\forall n \in Z$. It is easy to see that the inequality

$$\left|e^{i\lambda_n t} - e^{i\mu_n t}\right| \leq c|n|^{-\beta}, \quad \forall n \neq 0, \tag{5}$$

is fulfilled, where c is some constant. Let us assume that the following inequalities are satisfied:

$$-\frac{1}{2p(\pi)} < \alpha < \frac{1}{2q(\pi)}, \qquad \beta > \frac{1}{\tilde{p}}, \tag{6}$$

where $\tilde{p} = \min\{p^-; 2\}$. Then, from Theorem 3, the system of exponents $\{e^{i\mu_n t}\}_{n\in Z}$ forms a basis in L_{p_t}. By Theorem 1, it is isomorphic to the classical system of exponents $\{e^{int}\}_{n\in Z}$ in L_{p_t}. Therefore the spaces of coefficients of the bases $\{e^{i\mu_n t}\}_{n\in Z}$ and $\{e^{int}\}_{n\in Z}$ coincide.

Let $T : L_{p_t} \to L_{p_t}$ be a natural automorphism

$$T\left[e^{i\mu_n t}\right] = e^{int}, \quad \forall n \in Z.$$

For all $f \in L_{p_t}$, let $\{f_n\}_{n\in Z}$ be biorthogonal coefficients of f by the system $\{e^{i\mu_n t}\}_{n\in Z}$, and let $g = Tf$. Therefore, $\{f_n\}_{n\in Z}$ are the Fourier coefficients of the function g by the system $\{e^{int}\}_{n\in Z}$. From (4) and (5), it directly follows that

$$\sum_{n\in Z} \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\|_{p_t}^{\tilde{p}} < +\infty.$$

Consider the following expression:

$$\sum_n \left(e^{i\lambda_n t} - e^{i\mu_n t}\right) f_n.$$

We have

$$\begin{aligned} \left\| \sum_{n\in Z} \left(e^{i\lambda_n t} - e^{i\mu_n t}\right) f_n \right\|_{p_t} &\le \sum_{n\in Z} \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\| |f_n| \\ &\le \left(\sum_n \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\|_{p_t}^{\tilde{p}} \right)^{1/\tilde{p}} \left(\sum_n |f_n|^{\tilde{q}} \right)^{1/\tilde{q}}, \end{aligned}$$

where $\frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1$. By the Hausdorff-Young theorem [16], we have

$$\left(\sum_n |f_n|^{\tilde{q}} \right)^{1/\tilde{q}} \le m_1 \|g\|_{\tilde{p}},$$

where m_1 is some constant. From $\tilde{p} \le p^-$ and the continuous embedding $L_{p_t} \subset L_{\tilde{p}}$, it follows that, $\exists m_2 > 0$,

$$\|g\|_{\tilde{p}} \le m_2 \|g\|_{p_t} \le m_2 \|T\| \|f\|_{p_t}.$$

As a result, we obtain

$$\left\| \sum_n \left(e^{i\lambda_n t} - e^{i\mu_n t}\right) f_n \right\|_{p_t} \le m_1 m_2 \|T\| \left(\sum_n \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\|_{p_t}^{\tilde{p}} \right)^{1/\tilde{p}} \|f\|_{p_t}. \tag{7}$$

Let us take $n_0 \in N$ such that

$$\delta = m_1 m_2 \|T\| \left(\sum_{|n|>n_0} \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\|_{p_t}^{\tilde{p}} \right)^{1/\tilde{p}} < 1.$$

Assume that

$$\omega_n = \begin{cases} \lambda_n, & |n| > n_0, \\ \mu_n, & |n| \le n_0. \end{cases}$$

It is clear that the following inequality is satisfied:

$$\left\| \sum_n \left(e^{i\omega_n t} - e^{i\mu_n t}\right) f_n \right\|_{p_t} \leq \delta \|f\|_{p_t}. \tag{8}$$

It follows immediately from (7) that the expression $\sum_n (e^{i\omega_n t} - e^{i\mu_n t}) f_n$ represents a function from L_{p_t} and it can be denoted by $T_0 f$. Drawing attention to (8) we obtain $\|T_0\| \leq \delta < 1$. Thus, the operator $F = I + T_0$ is invertible, and it is easy to see that $F[e^{i\mu_n t}] = e^{i\omega_n t}$, $\forall n \in Z$. Hence, the system $\{e^{i\omega_n t}\}_{n\in Z}$ forms a basis in L_{p_t} isomorphic to $\{e^{i\mu_n t}\}_{n\in Z}$. Systems $\{e^{i\lambda_n t}\}_{n\in Z}$ and $\{e^{i\omega_n t}\}_{n\in Z}$ differ in a finite number of elements. Therefore, by Statement 1, the system $\{e^{i\lambda_n t}\}_{n\in Z}$ is complete in L_{p_t}, if $\lambda_i \neq \lambda_j$ for $i \neq j$. In the following it is necessary to apply Lemmas 1 and 2.

As a result we obtain the following theorem.

Theorem 4 *Let the asymptotics* (2) *occur and the inequalities*

$$-\frac{1}{2p(\pi)} < \alpha < \frac{1}{2q(\pi)}, \qquad \beta > \frac{1}{\tilde{p}}, \tag{9}$$

be fulfilled, where $\tilde{p} = \min\{p^-; 2\}$. *Then the following properties of the system* (1) *are equivalent in* L_{p_t}:

(1) *the system* (1) *is complete*;
(2) *the system* (1) *is minimal*;
(3) *the system* (1) *is* ω*-linear independent*;
(4) *the system* (1) *is isomorphic to* $\{e^{int}\}_{n\in N}$ *basis*;
(5) $\lambda_i \neq \lambda_j$ *for* $i \neq j$.

Let us consider the case $\alpha = -\frac{1}{2p(\pi)}$. In this case, by the results of [11], the system $\{e^{i\mu_n t}\}_{n\in Z}$ is complete and minimal in L_{p_t}, but it does not form a basis in it. Then from the previous considerations it follows that the system (1) cannot form a basis in L_{p_t}. Because otherwise, by Theorem 2, it will be isomorphic to system $\{e^{int}\}_{n\in Z}$ in L_{p_t}, and as a result the system $\{e^{i\mu_n t}\}_{n\in Z}$ should form a basis in L_{p_t}. This gives a contradiction.

By $\{\nu_n\}_{n\in Z} \subset L_{q_t}$ we denote the system biorthogonal to $\{e^{i\mu_n t}\}_{n\in Z}$. It is clear that using the estimates from [4], for ν_n, $n \in Z$, we establish that the following relation is true:

$$\gamma = \sup_n \|\nu_n\|_{q_t} < +\infty.$$

Let $\beta > 1$. Then it is clear that the following inequality is satisfied:

$$\sum_n \left\| e^{i\lambda_n t} - e^{i\mu_n t} \right\|_{p_t} < +\infty.$$

Similarly to the previous case, we can show that the operator

$$\tilde{T} f = \sum_n \nu_n(f) \left(e^{i\lambda_n t} - e^{i\mu_n t}\right), \quad \forall f \in L_{p_t},$$

is bounded in L_{p_t}. Introducing the new system $\{e^{i\omega_n t}\}_{n\in Z}$ in the same manner we establish the completeness of the system (1) in L_{p_t}, if $\lambda_i \neq \lambda_j$ for $i \neq j$. Minimality of the system (1) in L_{p_t} follows from Lemma 3. Thus, if $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\beta > 1$, then the system (1) is complete and minimal in L_{p_t} if the condition $-\frac{1}{2p(\pi)} \leq \alpha < \frac{1}{2q(\pi)}$ is satisfied.

Consider the case $\alpha \notin [-\frac{1}{2p(\pi)}, \frac{1}{2q(\pi)})$. Let, for example, $\alpha \in [\frac{1}{2q(\pi)}, \frac{1}{2q(\pi)} + \frac{1}{2})$. Multiplication of each member of the system (1) by $e^{i\frac{t}{2}}$ does not affect its basis properties in L_{p_t}. After appropriate transformations we obtain the system

$$e^{i[\tilde{\alpha}+\tilde{\alpha}_0]t} \bigcup \{e^{i\tilde{\lambda}_n t}\}_{n\in Z}, \tag{10}$$

where $\tilde{\alpha} = \alpha - \frac{1}{2}$ and

$$\tilde{\lambda}_n = n - \tilde{\alpha} \operatorname{sign} n + O(|n|^{-\beta}), \quad n \to \infty.$$

Denote by $\tilde{\alpha}_0$ the member of $O(|n|^{-\beta})$ in (2), corresponding to $n = 0$. It is easy to see that condition $\lambda_i \neq \lambda_j$ is equivalent to $\tilde{\lambda}_i \neq \tilde{\lambda}_j$. It is clear that $-\frac{1}{2p(\pi)} \leq \tilde{\alpha} < \frac{1}{2q(\pi)}$. Then, by the previous results, the system $\{e^{i\tilde{\lambda}_n t}\}_{n\in Z}$ is complete and minimal in L_{p_t}, and therefore the system (10), and at the same time the system (1), is complete, but it is not minimal in L_{p_t}. Continuing this process we find that the system (1) is not complete, but it is minimal for $\alpha < -\frac{1}{2p(\pi)}$; and the system (1) is complete, but it is not minimal in L_{p_t} for $\alpha \geq \frac{1}{2q(\pi)}$. Thus, the following theorem is proved.

Theorem 5 *We have*:

(I) *Let the asymptotics* (2) *occur and the inequalities* (9) *be fulfilled, where* $\tilde{p} = \min\{p^-; 2\}$. *Then the following properties of the system* (1) *are equivalent in* L_{p_t}:
- (1.1) *the system* (1) *is complete*;
- (1.2) *the system* (1) *is minimal*;
- (1.3) *the system* (1) *is* ω*-linear independent*;
- (1.4) *the system* (1) *is isomorphic to* $\{e^{int}\}_{n\in N}$ *basis*;
- (1.5) $\lambda_i \neq \lambda_j$ *for* $i \neq j$.

(II) *Let* $\beta > 1$ *and* $\alpha = -\frac{1}{2p(\pi)}$. *Then the following properties of the system* (1) *in* L_{p_t} *are equivalent*:
- (2.1) *the system* (1) *is complete*;
- (2.2) *the system* (1) *is minimal*;
- (2.3) $\lambda_i \neq \lambda_j$, *for* $i \neq j$.

Moreover, in this case the system (1) *does not form a basis in* L_{p_t}.

(III) *Let* $\beta > 1$ *and* $\lambda_i \neq \lambda_j$, *for* $i \neq j$. *Then the system* (1) *is complete and minimal in* L_{p_t} *for* $-\frac{1}{2p(\pi)} \leq \alpha < \frac{1}{2q(\pi)}$, *and for* $\alpha < -\frac{1}{2\pi}$ *it is not complete, but it is minimal*; *and for* $\alpha \geq \frac{1}{2q(\pi)}$ *it is complete, but it is not minimal in* L_{p_t}.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
I wish to expresses my thanks to Prof. Bilal T Bilalov, Institute of Mathematics and Mechanics of National Academy of Sciences, Baku, Azerbaijan, for his kind help, careful reading, and making useful comments on the earlier version of the paper.

Received: 26 September 2014 Accepted: 26 November 2014 Published: 12 Dec 2014

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10.1186/1029-242X-2014-495
Cite this article as: Muradov: On bases from perturbed system of exponents in Lebesgue spaces with variable summability exponent. *Journal of Inequalities and Applications* 2014, 2014:495