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A fixed point theorem for nonautonomous type superposition operators and integrable solutions of a general nonlinear functional integral equation

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Dedicated to Professor Shih-sen Chang on his 80th birthday

Abstract

We first establish a new fixed point theorem for nonautonomous type superposition operators. After that, we prove the existence of integrable solutions for a general nonlinear functional integral equation in an L^1 space on an unbounded interval by using our theorem. Our main tool is the measure of weak noncompactness.

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1 Introduction

It is well known that the class of nonlinear operator equations of various types has many useful applications in describing numerous problems of the real world. A number of equations which include several given operators have arisen in many branches of science such as the theory of optimal control, economics, biological, mathematical physics and engineering. The present paper is concerned with the solvability of the following quite general nonlinear functional integral equation:

$$x(t) = f\left(t, x(t), \int_0^t k(t,s)u\big(s, x(s)\big)\, ds\right), \quad t \in \mathbb{R}^+ := [0, \infty), \tag{1.1}$$

in $L^1 := L^1(\mathbb{R}^+)$, the space of Lebesgue integrable functions on $\mathbb{R}^+$. Here, $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ are two given functions, while k is a given real function defined on $\mathbb{R}^+ \times \mathbb{R}^+$.

Among nonlinear operators, there is a distinguished class called superposition operators. The solvability of Eq. (1.1) is closely related to the fixed points of the nonautonomous type superposition operator, which is asked to prove that there exists $x \in \mathcal{D}$ satisfying the following operator equation:

$$x = F(x, Ax) \tag{1.2}$$

for two given operators $F : X \times Y \to X$ and $A : \mathcal{D} \subset X \to X$, where X and Y are two Banach spaces. Our goal in this paper is to study under what conditions Eq. (1.1) is solvable in an L^1 space. To this end, we establish a fixed point theorem for the solvability of Eq. (1.2) in advance.

The organization of this paper is as follows. In Section 2, we gather some notions and preliminary facts, including the concepts and properties of the measure of weak noncompactness, which will be needed in our further considerations. In Section 3, we establish a new fixed point theorem for Eq. (1.2). In Section 4, we prove the existence of integrable solutions for Eq. (1.1) by virtue of the measure of weak noncompactness.

2 Preliminaries

Definition 2.1 Let I be an interval in $\mathbb{R}$. A function $f : I \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if:

(a) for each fixed $x \in \mathbb{R}$, the function $f(\cdot, x)$ is Lebesgue measurable in I;
(b) for almost everywhere (a.e., for short) fixed $t \in I$, the function $f(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

Let $m(I)$ be a set of all measurable functions $\psi : I \to \mathbb{R}$. If f is a Carathéodory function, then f defines a mapping $\mathcal{N}_f : m(I) \to m(I)$ by $(\mathcal{N}_f \psi)(t) = f(t, \psi(t))$. This mapping is called the superposition operator (or Nemytskii operator) associated to f. The theory concerning superposition operators is presented in [1].

For a given measurable function $\psi : I \to \mathbb{R}$, the composite operator $\mathcal{N}_f \circ \psi(\cdot) := f(\cdot, \psi(\cdot))$ which maps I into $\mathbb{R}$ is said to be a nonautonomous type superposition operator. By generalizing this concept, the solvability of Eq. (1.2) may be thought the existence of fixed points of the nonautonomous type superposition operator $\mathcal{N}_F \circ A$ on $\mathcal{D}$.

The following theorem was proved by Krasnosel'skii [2] (see also [3]) in the case when I is a bounded interval and has been extended to an unbounded interval by Appell and Zabrejko [1].

Theorem 2.2 (see [1, Theorem 3.1, pp.93]) *Let I be a* (*bounded or unbounded*) *interval in $\mathbb{R}$. The superposition operator $\mathcal{N}_f$ maps $L^1(I)$ into $L^1(I)$ if and only if there exist a function $L^1_+(I)$ and a constant $b > 0$ such that*

$$\left|f(t,x)\right| \le a(t) + b|x|,$$

where $L^1_+(I)$ denotes a positive cone of the space $L^1(I)$.

In this case, the operator $\mathcal{N}_f$ is continuous and bounded in the sense that it maps bounded sets in bounded sets.

The following Scorza-Dragoni theorem explains the structure of the Carathéodory functions on a bounded interval.

Theorem 2.3 (see [4, Theorem 3]) *Let I be a bounded interval of $\mathbb{R}$, and let $f : I \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Then, for each $\varepsilon > 0$, there exists a closed subset D_ε of the interval I such that* $\operatorname{meas}(I \backslash D_\varepsilon) < \varepsilon$ *and $f : D_\varepsilon \times \mathbb{R} \to \mathbb{R}$ is continuous.*

Next, we gather together some notations and preliminary facts of some weak topology feature which will be needed in our further considerations. Let $\mathfrak{B}(X)$ be a collection of all nonempty bounded subsets of a Banach space X, and let $\mathfrak{W}(X)$ be a subset of $\mathfrak{B}(X)$ consisting of all weakly compact subsets of X. Also, let $\mathcal{U}_r$ denote a closed ball in X centered in 0 and with radius r.

In what follows, we accept the following definition [5].

Definition 2.4 Let X be a Banach space; let M, M_1 and M_2 be in $\mathfrak{B}(X)$. A function $\mu : \mathfrak{B}(X) \to \mathbb{R}^+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:

(1) the family $\ker(\mu) := \{M \in \mathfrak{B}(X) : \mu(M) = 0\}$ is nonempty and $\ker(\mu)$ is contained in the set of relatively weakly compact sets of X;
(2) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$;
(3) $\mu(\overline{\mathrm{conv}}(M)) = \mu(M)$, where $\overline{\mathrm{conv}}(M)$ refers to the closed convex hull of M;
(4) $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$ for $\lambda \in [0,1]$;
(5) if $(M_n)_{n=1}^{\infty}$ is a decreasing sequence of nonempty, bounded and weakly closed subsets of X with $\lim_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker(\mu)$ described in (1) is called the kernel of the measure of weak noncompactness μ. Note that the intersection set M_∞ from (5) belongs to $\ker(\mu)$ since $\mu(M_\infty) \leq \mu(M_n)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} M_n = 0$.

The first important example of a measure of weak noncompactness has been defined by De Blasi [6] as follows:

$$\omega(M) = \inf\{r > 0 : \exists W \in \mathfrak{W}(X) \text{ such that } M \subset W + \mathcal{U}_r\}.$$

The De Blasi measure of weak noncompactness has some interesting properties. It plays a significant role in nonlinear analysis and has many applications.

Nevertheless, it is rather difficult to express the De Blasi measure of weak noncompactness with the help of a convenient formula in a concrete Banach space. Such a formula is only known in the case of the space of $L^1(I)$, where I is a bounded subinterval of $\mathbb{R}$. In [7], Appell and De Pascale gave to ω the following simple form in spaces:

$$\omega(M) = \lim_{\varepsilon\to 0} \sup_{\psi\in M} \left\{ \sup\left\{ \int_D |\psi(t)|\, dt : D \subset I, \mathrm{meas}(D) \leq \varepsilon \right\} \right\}$$

for all bounded subsets M of $L^1(I)$, where $\mathrm{meas}(\cdot)$ denotes the Lebesgue measure.

For a nonempty and bounded subset M of the space $L^1(\mathbb{R}^+)$, Banas and Knap [8] constructed a useful measure of weak noncompactness as follows:

$$c(M) := \lim_{\varepsilon\to 0} \sup_{\psi\in M} \left\{ \sup\left\{ \int_D |\psi(t)|\, dt : D \subset \mathbb{R}^+, \mathrm{meas}(D) \leq \varepsilon \right\} \right\},$$

$$d(M) := \lim_{T\to\infty} \sup_{\psi\in M} \left\{ \int_T^\infty |\psi(t)|\, dt \right\}.$$

Finally, let us put

$$\mu(M) = c(M) + d(M).$$

Based on the following criterion for weak noncompactness due to Dieudonné [9], it was shown that the function μ is a measure of weak noncompactness in the space $L^1(\mathbb{R}^+)$.

Theorem 2.5 *A bounded set N is relatively weakly compact in $L^1(\mathbb{R}^+)$ if and only if the following two conditions are satisfied*:

(1) *for any $\varepsilon > 0$, there exists $\delta > 0$ such that if* $\operatorname{meas}(D) \le \delta$ *then $\int_D |\psi(t)|\,dt \le \varepsilon$ for all $\psi \in N$,*
(2) *for any $\varepsilon > 0$, there exists $T > 0$ such that $\int_T^\infty |\psi(t)|\,dt \le \varepsilon$ for all $\psi \in N$.*

The nonlinear contractive property of the operators plays some important roles in our subsequent considerations.

Definition 2.6 Let $\mathcal{D}$ be a subset of the Banach space X. An operator $T : \mathcal{D} \to X$ is said to be nonlinear contractive (or a φ-nonlinear contraction) if there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(r) < r$ for $r > 0$ such that

$$\|Tx_1 - Tx_2\| \le \varphi\big(\|x_1 - x_2\|\big)$$

for all $x_1, x_2 \in \mathcal{D}$.

Remark 2.7 If we take $\varphi(r) = \lambda r$ with $0 \le \lambda < 1$, then such a φ-nonlinear contraction is also said to be λ-contraction.

Lemma 2.8 *Let X and Y be two Banach spaces, and let $\mathcal{D}$ be a subset of Y. If $F : X \times Y \to X$ is continuous, and for any $y \in \mathcal{D}$ the operator $F(\cdot, y)$ is a φ-nonlinear contraction, then there exists a continuous map $J : \mathcal{D} \to X$ such that $Jy = F(Jy, y)$ for any $y \in \mathcal{D}$.*

Proof For arbitrary fixed $y \in \mathcal{D}$, the mapping $F(\cdot, y)$ defined by $x \mapsto F(x, y)$ is a φ-nonlinear contraction and maps X into X, so it has a unique fixed point by [10, Theorem 1]. Let us denote by $J : \mathcal{D} \to X$ the map which assigns to each $y \in \mathcal{D}$ the unique point in X such that $Jy = F(Jy, y)$. Thus, the map J is well defined.

For any $y_0 \in \mathcal{D}$ and a sequence $(y_n)_{n\in\mathbb{N}} \subset \mathcal{D}$ which converges to y_0, we have

$$\begin{aligned}
\|Jy_n - Jy_0\| &= \big\|F(Jy_n, y_n) - F(Jy_0, y_0)\big\| \\
&\le \big\|F(Jy_n, y_n) - F(Jy_0, y_n)\big\| + \big\|F(Jy_0, y_n) - F(Jy_0, y_0)\big\| \\
&\le \varphi\big(\|Jy_n - Jy_0\|\big) + \big\|F(Jy_0, y_n) - F(Jy_0, y_0)\big\|,
\end{aligned}$$

which implies

$$\|Jy_n - Jy_0\| - \varphi\big(\|Jy_n - Jy_0\|\big) \le \big\|F(Jy_0, y_n) - F(Jy_0, y_0)\big\|.$$

Let $r_n := \|Jy_n - Jy_0\|$. Since the operator F is continuous, we obtain $r_n - \varphi(r_n) \to 0$ as $n \to \infty$. The properties of the function φ show that $r_n \to 0$, that is, $Jy_n \to Jy_0$. The continuity of J is proved. $\square$

3 Fixed point theorem of nonautonomous type superposition operators

Theorem 3.1 *Let X and Y be two Banach spaces, and let $\mathcal{D}$ be a nonempty subset of X. Suppose that the operators $A : \mathcal{D} \to Y$ and $F : X \times Y \to X$ satisfy the following*:

(1) *A and F is continuous*,

(2) *for any $y \in A(\mathcal{D})$, the operator $F(\cdot, y)$ is a φ-nonlinear contraction*,

(3) *there exists a nonempty, compact and convex subset $\mathcal{P}$ of $\mathcal{D}$ such that*

$$x = F(x, Az) \quad \Rightarrow \quad x \in \mathcal{P} \quad \text{for all } z \in \mathcal{P}.$$

Then there is a point x in $\mathcal{D}$ such that $x = F(x, Ax)$.

Proof Let us denote by $J : A(\mathcal{D}) \to X$ the map which assigns to each $y \in A(\mathcal{D})$ the unique point in X such that $Jy = F(Jy, y)$. From Lemma 2.8, the map J is well defined and continuous on $A(\mathcal{D})$.

By assumption (3), for any $z \in \mathcal{P}$, we infer that there is $x = (J \circ A)z \in \mathcal{P}$ such that $x = F(x, Az)$. This shows that $(J \circ A)(\mathcal{P}) \subset \mathcal{P}$. Since A and J are all continuous, the composite operator $J \circ A$ is continuous on $\mathcal{P}$. Now applying the Schauder fixed point theorem, we conclude that $J \circ A$ has at least one fixed point $x \in \mathcal{P} \subset \mathcal{D}$ such that $(J \circ A)x = x$, which implies that

$$F(x, Ax) = F\big((J \circ A)x, Ax\big) = (J \circ A)x = x.$$

This completes the proof. □

Remark 3.2 There are some fixed point theorems, which involve several operators such as the operators $Tx := Ax + Bx$ in a Banach space, or $Tx := AxBx + Cx$ in Banach algebras *etc.*, and they may be formulated by Theorem 3.1 in a coincident form.

For example, let $F(x, Ax) := Ax + Bx$ and $\mathcal{D}$ be a nonempty, convex and closed set of a Banach space X in Theorem 3.1, where $A : \mathcal{D} \to X$ is compact and continuous, $B : \mathcal{D} \to X$ is a contraction mapping and $Ax + By \in \mathcal{D}$ for all $x, y \in \mathcal{D}$. Then we immediately obtain the celebrated Krasnosel'skii fixed point theorem (see [11, Theorem 4.4.1, pp.31]), which implies that Theorem 3.1 is a generalization of the Krasnosel'skii fixed point theorem. In fact, if we take $\mathcal{P} = \overline{\operatorname{conv}}((I - B)^{-1}A(M))$, then $\mathcal{P}$ satisfies the assumption (3) of Theorem 3.1 (see the proof of [11, Lemma 4.4.2, pp.32]).

As another example, let $F(x, Ax) := AxBx + Cx$ and $\mathcal{D}$ be a closed convex and bounded subset of the Banach algebra X in Theorem 3.1, where $B : \mathcal{D} \to X$ is completely continuous, and $A, C : X \to X$ satisfy

$$\|Ax - Ay\| \le \phi_A\big(\|x - y\|\big), \qquad \|Cx - Cy\| \le \phi_C\big(\|x - y\|\big), \quad \forall x, y \in X,$$

where $\phi_A, \phi_C : \mathbb{R}^+ \to \mathbb{R}^+$ are two continuous nondecreasing functions satisfying $\phi_A(0) = \phi_C(0) = 0$ and

$$M\phi_A(r) + \phi_C(r) < r, \quad \forall r > 0 \ \big(M := \sup\big\{\|Bx\| : x \in \mathcal{D}\big\}\big).$$

It is obvious that $F(\cdot, y)$ is a φ-nonlinear contraction for any $y \in A(\mathcal{D})$ with $\varphi(r) = M\phi_A(r) + \phi_C(r)$, and if we take $\mathcal{P} = \overline{\operatorname{conv}}((\frac{I-C}{A})^{-1}B(\mathcal{D}))$, it is also easily proved that $\mathcal{P}$ satisfies the assumption (3) of Theorem 3.1. Thus, we obtain the fixed point theorem for the operator $AB + C$ in Banach algebras, which implies that Theorem 3.1 is a generalization of [12, Theorem 1.5].

4 The solvability of general nonlinear integral equations in L^1 space

In this section, we study the existence of integrable solutions for Eq. (1.1). A number of functional integral equations, such as the following:

$$x(t) = f_1\left(t, \int_0^t k(t,s)f_2\bigl(s,x(s)\bigr)\,ds\right), \quad t \in \mathbb{R}; \tag{4.1}$$

$$x(t) = g\bigl(t,x(t)\bigr) + f_1\left(t, \int_0^t k(t,s)f_2\bigl(s,x(s)\bigr)\,ds\right), \quad t \in \mathbb{R}; \tag{4.2}$$

$$x(t) = f_1\bigl(t,x(t)\bigr) + f_2\bigl(t,x(t)\bigr)\int_0^t k(t,s)u\bigl(s,x(s)\bigr)\,ds, \quad t \in \mathbb{R}, \tag{4.3}$$

may all be illustrated as special examples of Eq. (1.1).

Solutions to Eq. (1.1) will be sought in $L^1 := L^1(\mathbb{R}^+)$, the space of Lebesgue integrable functions on $\mathbb{R}^+$ with values in $\mathbb{R}$, endowed with the standard norm $\|x\| := \int_0^\infty |x(t)|\,dt$. Here are some hypotheses on the nonlinear functions involved in Eq. (1.1).

Assumption 4.1 Assume that

($\mathcal{H}1$) $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and there exist a function $a \in L^1_+$ and a constant $b > 0$ such that $|u(t,x)| \le a(t) + b|x|$;

($\mathcal{H}2$) $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a Carathéodory function, and $\operatorname{ess\,sup}_{s\in\mathbb{R}^+} \int_s^\infty |k(t,s)|\,dt < \infty$;

($\mathcal{H}3$) $f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function; there exist two positive numbers α, β and a function $g \in L^1_+$ such that $|f(t,x(t),y(t))| \le g(t) + \alpha|x(t)| + \beta|y(t)|$ for a.e. $t \in \mathbb{R}^+$;

($\mathcal{H}4$) $\alpha + b\beta\|K\| + \|g\| \le 1$ if $g \not\equiv 0$, otherwise $\alpha + b\beta\|K\| < 1$, where $\|K\|$ denotes the norm of the linear Volterra integral operator K generated by the function k;

($\mathcal{H}5$) for an arbitrary fixed $y(t) = \int_0^t k(t,s)u(s,z(s))\,ds$ with $z \in \mathcal{U}_{r_0}$, where r_0 satisfies

$$r_0 \ge \frac{\|g\| + \beta\|k\|\|a\|}{1 - \alpha - b\beta\|K\|},$$

there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(r) < r$ for $r > 0$ such that

$$\int_0^\infty \bigl|f\bigl(t,x_1(t),y(t)\bigr) - f\bigl(t,x_2(t),y(t)\bigr)\,dt\bigr| \le \varphi\bigl(\|x_1 - x_2\|\bigr), \quad \forall x_1, x_2 \in L^1.$$

Remark 4.2 First notice that Eq. (1.1) may be written in an abstract form by Eq. (1.2), where F is the superposition operator associated to the function f ($F := \mathcal{N}_f$, the superposition operator of double variables type was proposed by [13]):

$$F : L^1 \times L^1 \to L^1,$$

$$(x,y) \mapsto F(x,y) : \mathbb{R}^+ \to \mathbb{R}; \qquad F(x,y)(t) = f\bigl(t,x(t),y(t)\bigr),$$

and $A := K \circ \mathcal{N}_u$ appears as the composition of the superposition operator associated to u with the linear Volterra integral operator defined by

$$\begin{aligned} &K : L^1 \times L^1 \to L^1, \\ &\psi \mapsto K\psi : \mathbb{R}^+ \to \mathbb{R}; \qquad K\psi(t) = \int_0^t k(t,s)\psi(s)\,ds. \end{aligned}$$

Our aim is now to prove that the nonautonomous type superposition operator $\mathcal{N}_F \circ A$ has a fixed point in $L^1(\mathbb{R}^+)$. Before starting to study this problem, we give some remarks to illustrate that the operators A and F are well defined as follows.

(1) It should be noted that assumption $(\mathcal{H}2)$ leads to the estimate

$$\begin{aligned} \left\| \int_0^t k(t,s)\psi(s)\,ds \right\| &= \int_s^\infty \left|k(t,s)\right| dt \int_0^\infty \left|\psi(s)\right| ds \\ &\le \left(\operatorname*{ess\,sup}_{s\in\mathbb{R}^+} \int_s^\infty \left|k(t,s)\right| dt \right) \|\psi\|, \quad \psi \in L^1, \end{aligned}$$

which shows that the linear Volterra integral operator K is continuous from an L^1 space into itself, and $\|K\| \le \operatorname{ess\,sup}_{s\in\mathbb{R}^+} \int_s^\infty |k(t,s)|\,dt$.

(2) Assumption $(\mathcal{H}1)$ shows that the superposition operator $\mathcal{N}_u$ is continuous and maps bounded sets of L^1 into bounded sets of L^1 by Theorem 2.2.

(3) Note that $\alpha|x| + \beta|y|$ being an equivalent norm of (x,y) in $\mathbb{R}^2$, according to the Lucchetti-Patrone theorem (see [14] or [15, Theorem 1]), assumption $(\mathcal{H}3)$ shows that the superposition operator $\mathcal{N}_f$ is continuous and maps bounded sets of $L^1 \times L^1$ into bounded sets of L^1.

Theorem 4.3 *If Assumption* 4.1 *is verified, then the equation*

$$x(t) = f\left(t, x(t), \int_0^t k(t,s)u\bigl(s, x(s)\bigr)\,ds\right), \quad t \in \mathbb{R},$$

that is, Eq. (1.1) *has at least a solution* $x \in L^1$.

Proof It is clear that the solutions of the operator equation $x = F(x, Ax)$ satisfy Eq. (1.1). We will use Theorem 3.1 to prove the present theorem, thus the assumptions of Theorem 3.1 have to be checked. Our proving is divided into several steps.

(1) By Remark 4.2, the operators $A : L^1 \to L^1$, $F : L^1 \times L^1 \to L^1$ are well defined and continuous, and then the assumption (1) of Theorem 3.1 is fulfilled.

(2) By $(\mathcal{H}5)$, for arbitrary fixed $y = Az$ with $z \in \mathcal{U}_{r_0}$, there exists a continuous and non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\begin{aligned} \left\|F(x_1, y) - F(x_2, y)\right\| &= \int_0^\infty \left|f\bigl(t, x_1(t), y(t)\bigr) - f\bigl(t, x_2(t), y(t)\bigr)\right| dt \\ &\le \varphi\bigl(\|x_1 - x_2\|\bigr) \end{aligned}$$

for any $x_1, x_2 \in L^1$, and then the assumption (2) of Theorem 3.1 is fulfilled.

(3) If there is $x \in L^1$ such that $x(t) = f(t, x(t), Az(t))$ for $z \in \mathcal{U}_{r_0}$, then by $(\mathcal{H}3)$ we have

$$\bigl|f\bigl(t, x(t), Az(t)\bigr)\bigr| \le g(t) + \alpha\bigl|x(t)\bigr| + \beta\bigl|Az(t)\bigr| = g(t) + \alpha\bigl|x(t)\bigr| + \beta\bigl|(K \circ \mathcal{N}_u \circ z)(t)\bigr|.$$

It follows that

$$\|x\| = \int_0^\infty \bigl|f\bigl(t, x(t), Az(t)\bigr)\bigr|\,dt \le \|g\| + \alpha\|x\| + \beta\|K\|\bigl(\|a\| + b\|z\|\bigr),$$

that is,

$$\begin{aligned}\|x\| &\le (1-\alpha)^{-1}\bigl(\|g\| + \beta\|K\|\bigl(\|a\| + b\|z\|\bigr)\bigr)\\ &\le (1-\alpha)^{-1}\bigl(\|g\| + \beta\|K\|\|a\| + b\beta\|K\|r_0\bigr) \le r_0\end{aligned}$$

since $\|g\| + \beta\|K\|\|a\| \le r_0(1 - \alpha - b\beta\|K\|)$ by $(\mathcal{H}5)$. This shows that the nonautonomous type superposition operator $\mathcal{N}_F \circ A$ maps $\mathcal{U}_{r_0}$ into itself.

(4) Let $\mathcal{P}_0 := \mathcal{U}_{r_0}$, and let

$$\mathcal{P}_n := \overline{\operatorname{conv}}\left\{x \in L^1 : x(t) = f\left(t, x(t), \int_0^t k(t,s)u\bigl(s, z(s)\bigr)\,ds\right), z \in \mathcal{P}_{n-1}\right\}, \quad n \in \mathbb{N}.$$

Then $\mathcal{P}_n$ $(n = 0, 1, 2, \ldots)$ are all nonempty closed convex, and then they are weakly closed. Moreover, we have $\mathcal{P}_1 \subset \mathcal{U}_{r_0} = \mathcal{P}_0$ from step (3), and by the induction we may infer that $\mathcal{P}_n \subset \mathcal{P}_{n-1}$ for all $n \in \mathbb{N}$.

On the other hand, for each $\varepsilon > 0$ and a nonempty measurable subset D of $\mathbb{R}^+$ such that $\operatorname{meas}(D) \le \varepsilon$, we know that if there exist $z \in \mathcal{P}_{n-1}$ and $x \in \mathcal{P}_n$ such that

$$x(t) = f\left(t, x(t), \int_0^t k(t,s)u\bigl(s, z(s)\bigr)\,ds\right),$$

then

$$\begin{aligned}\int_D \bigl|x(t)\bigr|\,dt &= \int_D \left|f\left(t, x(t), \int_0^t k(t,s)u\bigl(s, z(s)\bigr)\,ds\right)\right|\,dt\\ &\le \int_D g(t)\,dt + \alpha\int_D \bigl|x(t)\bigr|\,dt + \beta\|K\|\left(\int_D a(t)\,dt + b\int_D \bigl|z(t)\bigr|\,dt\right),\end{aligned}$$

which implies that

$$\int_D \bigl|x(t)\bigr|\,dt \le (1-\alpha)^{-1}\left(\int_D g(t)\,dt + \beta\|K\|\int_D a(t)\,dt + b\beta\|K\|\int_D \bigl|z(t)\bigr|\,dt\right).$$

Taking into account the fact that the set consisting of one element is weakly compact, the use of Theorem 2.5 leads to

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sup\left\{\int_D g(t)\,dt : D \subset \mathbb{R}^+, \operatorname{meas}(D) \le \varepsilon\right\} = 0,\\ &\lim_{\varepsilon\to 0}\sup\left\{\int_D a(t)\,dt : D \subset \mathbb{R}^+, \operatorname{meas}(D) \le \varepsilon\right\} = 0.\end{aligned}$$

As a result,

$$\begin{aligned}
c(\mathcal{P}_n) &= \lim_{\varepsilon\to 0}\sup_{x\in\mathcal{P}_n}\left\{\sup\left[\int_D |x(t)|\,dt : D\subset\mathbb{R}^+, \operatorname{meas}(D)\le\varepsilon\right]\right\}\\
&\le (1-\alpha)^{-1}b\beta\|K\|\lim_{\varepsilon\to 0}\sup_{z\in\mathcal{P}_{n-1}}\left\{\sup\left\{\int_D |z(t)|\,dt : D\subset\mathbb{R}^+, \operatorname{meas}(D)\le\varepsilon\right\}\right\}\\
&= \lambda c(\mathcal{P}_{n-1}),
\end{aligned}\tag{4.4}$$

where $\lambda := (1-\alpha)^{-1}b\beta\|K\| < 1$ by $(\mathcal{H}4)$.

In the sequel let us fix arbitrarily the number $T > 0$. Then, for $z\in\mathcal{P}_{n-1}$ and $x\in\mathcal{P}_n$ with

$$x(t) = f\left(t, x(t), \int_0^t k(t,s)u\big(s,z(s)\big)\,ds\right),$$

we have

$$\begin{aligned}
\int_T^\infty |x(t)|\,dt &= \int_T^\infty \left|f\left(t,x(t),\int_0^t k(t,s)u\big(s,z(s)\big)\,ds\right)\right|\,dt\\
&\le \int_T^\infty g(t)\,dt + \alpha\int_T^\infty |x(t)|\,dt + \beta\|K\|\left(\int_T^\infty a(t)\,dt + b\int_T^\infty |z(t)|\,dt\right),
\end{aligned}$$

which implies that

$$\int_T^\infty |x(t)|\,dt \le (1-\alpha)^{-1}\left(\int_T^\infty g(t)\,dt + \beta\|K\|\int_T^\infty a(t)\,dt + b\beta\|K\|\int_T^\infty |z(t)|\,dt\right),$$

and the use of Theorem 2.5 leads to

$$\lim_{T\to\infty}\int_T^\infty g(t)\,dt = 0,\quad\text{and}\quad \lim_{T\to\infty}\int_T^\infty a(t)\,dt = 0.$$

As a result,

$$\begin{aligned}
d(\mathcal{P}_n) &= \lim_{\varepsilon\to 0}\sup_{x\in\mathcal{P}_n}\left\{\int_T^\infty |x(t)|\,dt\le\varepsilon\right\}\\
&\le \lambda\lim_{\varepsilon\to 0}\sup_{z\in\mathcal{P}_{n-1}}\left\{\int_T^\infty |z(t)|\,dt\le\varepsilon\right\} = \lambda d(\mathcal{P}_{n-1}).
\end{aligned}\tag{4.5}$$

Thus, combining estimates (4.4) and (4.5), we obtain that

$$\mu(\mathcal{P}_n)\le\lambda\mu(\mathcal{P}_{n-1}).$$

Further, from $\mu(\mathcal{P}_n)\le\lambda\mu(\mathcal{P}_{n-1})\le\cdots\le\lambda^n\mu(\mathcal{P}_0)$ for $n\in\mathbb{N}$, we obtain that

$$\lim_{n\to\infty}\mu(\mathcal{P}_n) = 0.$$

Setting $\mathcal{P} := \bigcap_{n=0}^\infty \mathcal{P}_n$, we see that $\mathcal{P}$ is nonempty and weakly compact by Definition 2.4. Moreover, we infer that for any $z\in\mathcal{P}$ if $x = F(x, Az)$ holds, then $x\in\mathcal{P}$.

(5) In this final step, let us prove that $\mathcal{P}$ is compact. To this end, for any sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ with

$$x_n(t) = f\left(t, x_n(t), \int_0^t k(t,s)u\bigl(s, z_n(s)\bigr)\,ds\right), \quad (z_n)_{n\in\mathbb{N}} \subset \mathcal{P}, \tag{4.6}$$

we shall show that $(x_n)_{n\in\mathbb{N}}$ possesses a convergent subsequence in L^1.

Since $\mathcal{P}$ is weakly compact, for an arbitrary fixed $\varepsilon > 0$, by Theorem 2.5 there exists $T > 0$ such that

$$\int_T^\infty \bigl|x_m(t) - x_n(t)\bigr|\,dt \le \frac{\varepsilon}{2} \tag{4.7}$$

for all $m, n \in \mathbb{N}$.

Moreover, for the sequence $(z_n)_{n\in\mathbb{N}}$ in (4.6), let

$$y_n(t) := \int_0^t k(t,s)u\bigl(s, z_n(s)\bigr)\,ds, \quad n \in \mathbb{N}.$$

According to Theorem 2.3, there exists a closed subset D_ε of $[0,T]$ such that the functions k and u are continuous on $D_\varepsilon \times [0,T]$ and $D_\varepsilon \times \mathbb{R}$, respectively, where $\operatorname{meas}([0,T]\backslash D_\varepsilon) \le \varepsilon$. By taking $t_1, t_2 \in D_\varepsilon$ with $t_1 < t_2$, we have

$$\begin{aligned}
\bigl|y_n(t_2) - y_n(t_1)\bigr| &= \left|\int_0^{t_2} k(t_2,s)u\bigl(s,z_n(s)\bigr)\,ds - \int_0^{t_1} k(t_1,s)u\bigl(s,z_n(s)\bigr)\,ds\right| \\
&\le \int_0^{t_1} \bigl|k(t_2,s) - k(t_1,s)\bigr|\bigl|u\bigl(s,z_n(s)\bigr)\bigr|\,ds + \left|\int_{t_1}^{t_2} k(t_2,s)u\bigl(s,z_n(s)\bigr)\,ds\right| \\
&\le \omega^T\bigl(k, |t_2 - t_1|\bigr)\int_0^T \bigl(a(s) + b\bigl|z_n(s)\bigr|\bigr)\,ds + \widetilde{k}\int_{t_1}^{t_2} \bigl(a(s) + b\bigl|z_n(s)\bigr|\bigr)\,ds \\
&\le \omega^T\bigl(k, |t_2 - t_1|\bigr)\bigl(\|a\| + br\bigr) + \widetilde{k}\int_{t_1}^{t_2} a(s)\,ds + b\widetilde{k}\int_{t_1}^{t_2} \bigl|z_n(s)\bigr|\,ds,
\end{aligned} \tag{4.8}$$

where $\widetilde{k} := \max\{|k(t,s)| : (t,s) \in D_\varepsilon \times [0,T]\}$, and $\omega^T(k, |t_2 - t_1|)$ denotes the modulus of continuity of the function k on the set $D_\varepsilon \times [0,T]$.

Since $(z_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ is relatively weakly compact and the set consisting of one element is also weakly compact, by Theorem 2.5 we infer that the terms $\int_{t_1}^{t_2} |z_n(s)|\,ds$ and $\int_{t_1}^{t_2} a(s)\,ds$ in (4.8) may all be arbitrarily small provided that the number $|t_1 - t_2|$ is small enough. Thus, we obtain that the sequence $(y_n(t))_{n\in\mathbb{N}}$ is equicontinuous on $\mathcal{C}(D_\varepsilon)$ (the space of all continuous functions defined on D_ε).

On the other hand, we have

$$\begin{aligned}
\bigl|y_n(t)\bigr| &= \left|\int_0^t k(t,s)u\bigl(s,z_n(s)\bigr)\,ds\right| \le \int_0^t \bigl|k(t,s)\bigr|\bigl(a(s) + b\bigl|z_n(s)\bigr|\bigr)\,ds \\
&\le \widetilde{k}\bigl(\|a\| + b\|z_n\|\bigr) \le \widetilde{k}\bigl(\|a\| + br\bigr) := \overline{Y},
\end{aligned}$$

which implies the sequence $(y_n(t))_{n\in\mathbb{N}}$ is uniformly bounded on $\mathcal{C}(D_\varepsilon)$.

Since the map J, which signs each $y \in A(\mathcal{P})$ the unique point $x \in \mathcal{P}$ such that $x(t) = f(t, x(t), y(t))$, is well defined and uniformly continuous on $D_\varepsilon \times [-\overline{Y}, \overline{Y}]$, by Lemma 2.8 we infer that the sequence $(x_n(t))_{n\in\mathbb{N}}$ with $x_n(t) = f(t, x_n(t), y_n(t))$ is uniformly bounded and equicontinuous on $\mathcal{C}(D_\varepsilon)$. Hence, by applying the Arzéla-Ascoli theorem, we obtain that the set $\{x_n : n \in \mathbb{N}\}$ forms a relatively compact set in $\mathcal{C}(D_\varepsilon)$.

Note that our reasoning does not depend on the choice of ε. Thus we can construct a sequence $(D_{1/k})_{k\in\mathbb{N}}$ of closed subsets of the interval $[0, T]$ such that $\operatorname{meas}([0, T]\backslash D_{1/k}) \to 0$ as $k \to \infty$, and the sequence $(x_n)_{n\in\mathbb{N}}$ is relatively compact in every space $\mathcal{C}(D_{1/k})$. Passing to subsequences if necessary, we can assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in each space $\mathcal{C}(D_{1/k})$ for $k \in \mathbb{N}$.

In what follows, by virtue of the fact that $\mathcal{P}$ is weakly compact, let us choose a number $\delta > 0$ such that for each closed subset D_δ of the interval $[0, T]$ with $\operatorname{meas}([0, T]\backslash D_\delta) \le \delta$ satisfies

$$\int_{[0,T]\backslash D_\delta} \left|x_m(t) - x_n(t)\right| dt \le \frac{\varepsilon}{4} \quad \text{for all } m, n \in \mathbb{N}. \tag{4.9}$$

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in each space $\mathcal{C}(D_{1/k})$, there is $k_0 \in \mathbb{N}$ such that $\operatorname{meas}([0, T]\backslash D_{1/k_0}) \le \delta$ and for $m, n \ge k_0$

$$\|x_m - x_n\|_{\mathcal{C}(D_{1/k_0})} \le \frac{\varepsilon}{4(1 + \operatorname{meas}(D_{1/k_0}))}. \tag{4.10}$$

Consequently, (4.9) and (4.10) imply that for $m, n \ge k_0$ we have

$$\begin{aligned} &\int_0^T \left|x_m(t) - x_n(t)\right| dt \\ &\quad = \int_{D_{1/k_0}} \left|x_m(t) - x_n(t)\right| dt + \int_{[0,T]\backslash D_{1/k_0}} \left|x_m(t) - x_n(t)\right| dt < \frac{\varepsilon}{2}. \end{aligned} \tag{4.11}$$

Now, combining (4.7) and (4.11) for $m, n \ge k_0$, we obtain that

$$\|x_m - x_n\| = \int_0^T \left|x_m(t) - x_n(t)\right| dt + \int_T^\infty \left|x_m(t) - x_n(t)\right| dt < \varepsilon,$$

which shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in an L^1 space. Thus, the sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{P}$ has a convergent subsequence, which implies that the closed set $\mathcal{P}$ is compact.

This shows that the assumptions of Theorem 3.1 are all fulfilled, which completes the proof. □

Remark 4.4 The techniques of the proof of Theorem 4.3 based on Carathéodory conditions and the Scorza-Dragoni theorem were already used in [16–19] for proving the solvability of Eq. (4.1), (4.2), *etc.*

Finally, we provide an example, which is not included in Eq. (4.1)-(4.3), and which may be treated by our Theorem 4.3.

Example 4.5 Consider the following nonlinear integral equation:

$$\psi(t) = \frac{t}{e^{2t}} - \frac{\arctan\psi(t)}{4 + t} + \frac{\sin t}{2 + |\psi(t)|} \sin\left(\int_0^t \frac{\sqrt{(1+t)^{-4} + \psi^2(s)}}{e^{t+s}}\, ds\right) \tag{4.12}$$

for $t \in \mathbb{R}^+$. In order to show that such an equation admits a solution in L^1, we are going to check that the conditions of Theorem 4.3 are satisfied.

Define the functions as follows:

$$\begin{aligned}
&k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}, \qquad k(t,s) = e^{-(t+s)};\\
&u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}, \qquad u(t,x) = \sqrt{(1+t)^{-4} + x^2};\\
&f : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}, \qquad f(t,x,y) = te^{-2t} - \frac{\arctan x}{4+t} + \frac{\sin t \sin y}{2+|x|}.
\end{aligned}$$

It obvious that u, k and f are all Carathéodory functions. Taking $a(t) = (1+t)^{-2}$ and $b = 1$, we have

$$\left|u(t,x)\right| = \sqrt{(1+t)^{-4} + x^2} \leq (1+t)^{-2} + |x| = a(t) + b|x|.$$

So, u satisfies $(\mathcal{H}_1)$. Taking $g(t) = te^{-2t}$, $\alpha = 1/4$ and $\beta = 1/2$, we have

$$\left|f(t,x,y)\right| = \left|te^{-2t} - \frac{\arctan x}{4+t} + \frac{\sin t \sin y}{2+|x|}\right| \leq te^{-2t} + \frac{1}{4}|x| + \frac{1}{2}|y| = g(t) + \alpha|x| + \beta|y|.$$

It follows that f satisfies $(\mathcal{H}_3)$. By a simple calculation, we obtain that

$$\|g\| = \int_0^\infty te^{-2t}\,dt = \frac{1}{4}, \qquad \|K\| \leq \sup_{s\geq 0}\int_s^\infty \left|k(t,s)\right|dt = \sup_{s\geq 0}\int_s^\infty e^{-(t+s)}\,dt = 1.$$

It follows that

$$\alpha + b\beta\|K\| + \|g\| \leq \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1,$$

which shows that $(\mathcal{H}_2)$ and $(\mathcal{H}_4)$ are satisfied.

From the inequality

$$\begin{aligned}
&\left|f\bigl(t,x_1(t),y(t)\bigr) - f\bigl(t,x_2(t),y(t)\bigr)\right|\\
&\quad \leq \frac{1}{4+t}\left|\arctan x_1(t) - \arctan x_2(t)\right| + \left|\frac{\sin t \sin y(t)}{2+|x_1(t)|} - \frac{\sin t \sin y(t)}{2+|x_2(t)|}\right|,\\
&\quad \leq \frac{1}{4+t}\left|x_1(t) - x_2(t)\right| + \frac{|x_1(t) - x_2(t)|}{4 + |x_1(t)| + |x_2(t)| + |x_1(t)x_2(t)|}\\
&\quad \leq \frac{1}{2}\left|x_1(t) - x_2(t)\right|,
\end{aligned}$$

it follows that

$$\int_0^\infty \left|f\bigl(t,x_1(t),y(t)\bigr) - f\bigl(t,x_2(t),y(t)\bigr)\right|dt \leq \frac{1}{2}\|x_1 - x_2\|, \quad \forall x_1, x_2 \in L^1$$

for all $y \in L^1$. So $(\mathcal{H}_5)$ is satisfied for $\varphi(r) := \frac{1}{2}r$.

Since the assumptions $(\mathcal{H}_1)$-$(\mathcal{H}_5)$ are all satisfied, we apply Theorem 4.3 to derive the existence of a solution to Eq. (4.12).

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The author declares that they have no competing interests.

Author's contributions
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