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Product of differentiation and composition operators on the logarithmic Bloch space

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Abstract
We obtain a criterion for the boundedness and compactness of the products of differentiation and composition operators $C_\varphi D^m$ on the logarithmic Bloch space in terms of the sequence $\{z^n\}$. An estimate for the essential norm of $C_\varphi D^m$ is given.

MSC: 47B38; 30H30

1 Introduction

Denote by $H(\mathbb{D})$ the space of all analytic functions on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ in the complex plane. Let $H^\infty = H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space $\mathcal{B}$ if

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)| \big(1 - |z|^2\big) < \infty.$$

The logarithmic-Bloch space, denoted by $\mathcal{LB}$, consists of all $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{\log} = \sup_{z\in\mathbb{D}} \big(1 - |z|\big) \big|f'(z)\big| \log \frac{e}{1 - |z|} < \infty.$$

$\mathcal{LB}$ is a Banach space with the norm $\|f\|_{\mathcal{LB}} = |f(0)| + \|f\|_{\log}$. It is well known that $\mathcal{LB} \cap H^\infty$ is the space of multipliers of the Bloch space $\mathcal{B}$ (see [1, 2]). For some results on logarithmic-type spaces and operators on them, see, for example, [3–10].

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

The differentiation operator D is defined by $Df = f'$, $f \in H(\mathbb{D})$. For a nonnegative integer $m \in \mathbb{N}$, we define

$$D^m f = f^{(m)}, \quad f \in H(\mathbb{D}).$$

The product of differentiation and composition operators $C_\varphi D^m$ is defined as follows:

$$C_\varphi D^m f = f^{(m)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

A basic problem concerning concrete operators on various Banach spaces is to relate the operator theoretic properties of the operators to the function theoretic properties of their symbols, which attracted a lot of attention recently, the reader can refer to [4–37].

It is a well-known consequence of the Schwarz-Pick lemma that the composition operator is bounded on $\mathcal{B}$. See [21–24, 27, 33–35, 37] for the study of composition operators and weighted composition operators on the Bloch space. The product-type operators on or into Bloch type spaces have been studied in many papers recently; see [12–20, 26, 28–32, 34, 36] for example.

Let X and Y be two Banach spaces. Recall that a linear operator $T: X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. The essential norm of an operator T between X and Y is the distance to the compact operators K, that is, $\|T\|_e^{X\to Y} = \inf\{\|T-K\| : K \text{ is compact}\}$, where $\|\cdot\|$ is the operator norm. It is easy to see that $\|T\|_e^{X\to Y} = 0$ if and only if T is compact. For some results in the topic, see, for example, [11, 20, 22, 24, 26, 28, 37].

In [34], Wu and Wulan obtained a characterization for the compactness of the product of differentiation and composition operators acting on the Bloch space as follows:

Theorem A *Let φ be an analytic self-map of $\mathbb{D}$, $m \in \mathbb{N}$. Then $C_\varphi D^m : \mathcal{B} \to \mathcal{B}$ is compact if and only if*

$$\lim_{n\to\infty} \left\| C_\varphi D^m\left(z^n\right)\right\|_{\mathcal{B}} = 0.$$

The purpose of the paper is to extend Theorem A to the case of $\mathcal{LB}$. We will characterize the boundedness and compactness of $C_\varphi D^m$ in terms of the sequence $\{z^n\}$. Moreover, an estimate for the essential norm of $C_\varphi D^m$ will be given. The main results are given in Sections 3 and 4.

In the paper, we say that a real sequence $\{a_n\}_{n\in\mathbb{N}}$ is asymptotic to another real sequence of $\{b_n\}_{n\in\mathbb{N}}$ and write '$a_n \sim b_n$' if and only if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1.$$

In addition, we say that $A \preceq B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \preceq B \preceq A$.

2 Auxiliary lemmas

In this section, we state and prove some auxiliary results which will be used to prove the main results in this paper.

Lemma 2.1 *For $m, n \in \mathbb{N}$, define the function $H_{m,n} : [0,1) \to [0,\infty)$ by*

$$H_{m,n}(x) = \frac{n!}{(n-m-1)!} x^{n-m-1}(1-x)^{m+1} \log \frac{e}{1-x}. \tag{2.1}$$

Then the following statements hold:

(i) *For $n, m \in \mathbb{N}$ and $n \geq m+1$, there is a unique $x_{m,n} \in [0,1)$ such that $H_{m,n}(x_{m,n})$ is the absolute maximum of $H_{m,n}$.*

(ii)

$$\lim_{n\to\infty} x_{m,n} = 1 \tag{2.2}$$

and

$$\lim_{n\to\infty}\bigl[n(1-x_{m,n})\bigr] = m+1. \tag{2.3}$$

(iii)

$$\lim_{n\to\infty}\frac{\max_{0<t<1} H_{m,n}(t)}{\log(n+1)} = \left(\frac{m+1}{e}\right)^{m+1}. \tag{2.4}$$

Proof Directly computing we have

$$H'_{m,n}(x) = \frac{n!}{(n-m-1)!}x^{n-m-2}(1-x)^m\left((n-m-1-nx)\log\frac{e}{1-x}+x\right).$$

Define

$$g_{m,n}(x) = (n-m-1-nx)\log\frac{e}{1-x}+x, \quad x\in[0,1). \tag{2.5}$$

It is easy to see that $g_{m,n}$ is continuous on [0,1) and $g_{m,n}(0) = n-m-1 \geq 0$, $\lim_{x\to 1^-} g_{m,n}(x) = -\infty$. Furthermore,

$$g'_{m,n}(x) = -n\log\frac{e}{1-x}+n-\frac{m+1}{1-x}+1<0, \quad x\in[0,1).$$

Then $g_{m,n}$ is decreasing on $[0,1)$. When $n = m+1$, we get $\max_{0\leq x<1} H_{m,n}(x) = H_{m,n}(0)$. When $n > m+1$, the intermediate value theorem of continuous function gives the result that there exists a unique $x_{m,n}\in(0,1)$ such that $g_{m,n}(x_{m,n}) = 0$. So we have

$$\max_{0<t<1} H_{m,n}(x) = H_{m,n}(x_{m,n}).$$

(i) has been proved. By (2.5), we have $g_{m,n}(x_{m,n}) = 0$. Thus

$$\left(\frac{n-m-1}{n}-x_{m,n}\right)\log\frac{e}{1-x_{m,n}} = -\frac{x_{m,n}}{n}.$$

It follows from $\lim_{n\to\infty}\frac{x_{m,n}}{n} = 0$ and $\log\frac{e}{1-x_{m,n}} \geq 1$ that (2.2) holds. Also, $g_{m,n}(x_{m,n}) = 0$ gives the result that

$$\frac{n-m-1}{n}-x_{m,n} = -\frac{x_{m,n}}{n\log\frac{e}{1-x_{m,n}}}.$$

So we have

$$n(1-x_{m,n})-m-1 = -\frac{x_{m,n}}{\log\frac{e}{1-x_{m,n}}}.$$

This gives the result (2.3). The proof of (ii) is complete.

Note that

$$n\log x_{m,n} \sim n\log\bigl[1+(x_{m,n}-1)\bigr] \sim n(x_{m,n}-1) \to -m-1 \quad \text{as } n\to\infty.$$

This and (2.2) give

$$\lim_{n\to\infty} x_{m,n}^{n-m-1} = e^{-m-1}. \tag{2.6}$$

By (2.3) and (2.6) we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{H_{m,n}(x_{m,n})}{\log(n+1)} &= \lim_{n\to\infty}\frac{n!x_{m,n}^{n-m-1}(1-x_{m,n})^{m+1}\log\frac{e}{1-x_{m,n}}}{(n-m-1)!\log(n+1)} \\ &= e^{-m-1}\lim_{n\to\infty}\frac{n!((m+1)/n)^{m+1}\log\frac{en}{m+1}}{(n-m-1)!\log(n+1)} = \left(\frac{m+1}{e}\right)^{m+1},\end{aligned}$$

which shows that (iii) hold. The proof is complete. □

Lemma 2.2 *Let $m,n\in\mathbb{N}$ and $n-m-1>0$. Let $r_{m,n}=(n-m-1)/n$. Then $H_{m,n}$ is increasing on $[r_{m,n-m},r_{m,n}]$ and*

$$\min_{r_{m,n-m}\le x\le r_{m,n}} H_{m,n}(x) = H_{m,n}(r_{m,n-m}) \sim \left(\frac{m+1}{e}\right)^{m+1}\log(n+1) \quad \textit{as } n\to\infty. \tag{2.7}$$

Consequently,

$$\min_{r_{m,n-m}\le x\le r_{m,n}} \frac{H_{m,n}(x)}{\|z^n\|_{\mathcal{LB}}} = \frac{H_{m,n}(r_{m,n-m})}{\|z^n\|_{\mathcal{LB}}} \sim \frac{(m+1)^{m+1}}{e^m} \quad \textit{as } n\to\infty. \tag{2.8}$$

Proof Since $n-m-1>0$, we have

$$H'_{m,n}(r_{m,n}) = \frac{n!}{(n-m-1)!}\left(\frac{n-m-1}{n}\right)^{n-m-2}\left(\frac{m+1}{n}\right)^{m}\left(\frac{n-m-1}{n}\right) > 0.$$

By Lemma 2.1, we have $r_{m,n} < x_{m,n}$, where $x_{m,n}$ is given as in Lemma 2.1. Since $H'_{m,n}(x) > 0$ for $x\in(0,x_{m,n})$, we see that $H_{m,n}$ is increasing on $[r_{m,n-m},r_{m,n}]$. Thus

$$\begin{aligned}\min_{r_{m,n-m}\le x\le r_{m,n}} H_{m,n}(x) &= H_{m,n}(r_{m,n-m}) \\ &= \frac{n!}{(n-m-1)!}\left(\frac{n-2m-1}{n-m}\right)^{n-m-1}\left(\frac{m+1}{n-m}\right)^{m+1}\log\frac{e(n-m)}{m+1}.\end{aligned}$$

Applying the important limit $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ we obtain the result that (2.7) holds.

By Lemma 2.1 we have

$$\|z^n\|_{\mathcal{LB}} = \sup_{|z|<1} n|z|^{n-1}\bigl(1-|z|\bigr)\log\frac{e}{1-|z|} = H_{0,n}(x_{0,n}), \tag{2.9}$$

where $x_{0,n}$ is given in Lemma 2.1. By Lemma 2.1 we have

$$\begin{aligned}\lim_{n\to\infty}\frac{H_{m,n}(r_{m,n-m})}{\|z^n\|_{\mathcal{LB}}} &= \lim_{n\to\infty}\frac{H_{m,n}(r_{m,n-m})}{\log(n+1)}\frac{\log(n+1)}{\|z^n\|_{\mathcal{LB}}}\\ &= \lim_{n\to\infty}\frac{H_{m,n}(r_{m,n-m})}{\log(n+1)}\lim_{n\to\infty}\frac{\log(n+1)}{\|z^n\|_{\mathcal{LB}}} = \frac{(m+1)^{m+1}}{e^m}.\end{aligned}$$

This gives (2.8). The proof is complete. □

Lemma 2.3 [3] *For $m\in\mathbb{N}$. Then $f\in\mathcal{LB}$ if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|)^m\left|f^{(m)}(z)\right|\log\frac{e}{1-|z|}<\infty.$$

Moreover,

$$\|f\|_{\mathcal{LB}}\approx\sum_{j=0}^{m-1}\left|f^{(j)}(0)\right|+\sup_{z\in\mathbb{D}}(1-|z|)^m\left|f^{(m)}(z)\right|\log\frac{e}{1-|z|}.$$

3 The boundedness of $C_\varphi D^m$ on $\mathcal{LB}$

In this section, we will state the boundedness criterion for the operator $C_\varphi D^m$ on $\mathcal{LB}$. Since the boundedness of $C_\varphi D^m$ on $\mathcal{LB}$ gives $\varphi\in\mathcal{LB}$, we may always assume that $\varphi\in\mathcal{LB}$. The main result of this section is stated as follows.

Theorem 3.1 *Let $m\in\mathbb{N}$ and φ be an analytic self-map of $\mathbb{D}$ such that $\varphi\in\mathcal{LB}$. Then $C_\varphi D^m$ is bounded on $\mathcal{LB}$ if and only if*

$$\sup_{n\in\mathbb{N}}\frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}<\infty. \tag{3.1}$$

Proof ⇒) Assume that $C_\varphi D^m$ is bounded on $\mathcal{LB}$, that is, $\|C_\varphi D^m\|_{\mathcal{LB}\to\mathcal{LB}}<\infty$. Since the sequence $\{z^n/\|z^n\|_{\mathcal{LB}}\}$ is bounded in the logarithmic Bloch space $\mathcal{LB}$, we have

$$\frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}\le\left\|C_\varphi D^m\right\|_{\mathcal{LB}\to\mathcal{LB}}\left\|\frac{z^n}{\|z^n\|_{\mathcal{LB}}}\right\|_{\mathcal{LB}}\le\left\|C_\varphi D^m\right\|_{\mathcal{LB}\to\mathcal{LB}}<\infty,$$

for any $n\in\mathbb{N}$, from which the implication follows.

⇐) We now assume that the condition (3.1) holds. On the one hand, for the case $\sup_{z\in\mathbb{D}}|\varphi(z)|<1$, there is an $r\in(0,1)$ such that $|\varphi(z)|<r$. By (3.1), for any given $f\in\mathcal{LB}$, we have

$$\begin{aligned}\left\|C_\varphi D^m f\right\|_{\mathcal{LB}} &= \sup_{z\in\mathbb{D}}(1-|z|)\log\frac{e}{1-|z|}\left|f^{(m+1)}(\varphi(z))\varphi'(z)\right|\\ &\le\sup_{z\in\mathbb{D}}\|\varphi\|_{\mathcal{LB}}\frac{|f^{(m+1)}(\varphi(z))|(1-|\varphi(z)|)^{m+1}\log\frac{e}{1-|\varphi(z)|}}{(1-|\varphi(z)|)^{m+1}\log\frac{e}{1-|\varphi(z)|}}\\ &\preceq\sup_{z\in\mathbb{D}}\frac{\|\varphi\|_{\mathcal{LB}}\|f\|_{\mathcal{LB}}}{(1-r)^{m+1}\ln\frac{e}{1-r}}<\infty.\end{aligned}$$

The last estimate shows that the operator C_φ is bounded on $\mathcal{LB}$.

On the other hand, for the case $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$. Let N be the smallest positive integer such that $\mathbb{D}_N$ is not empty, where

$$\mathbb{D}_n = \{z \in \mathbb{D} : r_{m,n-m} \le |\varphi(z)| \le r_{m,n}\}$$

and $r_{m,n}$ is given in Lemma 2.2. Note that $H_{m,n}(|\varphi(z)|) > 0$, when $z \in \mathbb{D}_n$, $n \ge N$, by (2.8) we obtain

$$\epsilon := \inf_{z\in\mathbb{D}_n} \frac{H_{m,n}(|\varphi(z)|)}{\|z^n\|_{\mathcal{LB}}} > 0.$$

For any given $f \in \mathcal{LB}$, by Lemma 2.3 we have

$$\begin{aligned}
\|C_\varphi D^m f\|_{\mathcal{LB}} &= \sup_{z\in\mathbb{D}}(1-|z|)\log\frac{e}{1-|z|}\left|f^{(m+1)}(\varphi(z))\varphi'(z)\right| \\
&= \sup_{n\ge N}\sup_{z\in\mathbb{D}_n}(1-|z|)\log\frac{e}{1-|z|}\left|f^{(m+1)}(\varphi(z))\varphi'(z)\right| \\
&= \sup_{n\ge N}\sup_{z\in\mathbb{D}_n}(1-|z|)\log\frac{e}{1-|z|}\left|f^{(m+1)}(\varphi(z))\varphi'(z)\right|\frac{\|z^n\|_{\mathcal{LB}}}{H_{m,n}(|\varphi(z)|)}\frac{H_{m,n}(|\varphi(z)|)}{\|z^n\|_{\mathcal{LB}}} \\
&\preceq \frac{\|f\|_{\mathcal{LB}}}{\epsilon}\sup_{n\ge N}\sup_{z\in\mathbb{D}_n}\frac{n!}{(n-m-1)!}(1-|z|)\log\frac{e}{1-|z|}\left|\varphi'(z)\right|\frac{|\varphi(z)|^{n-m-1}}{\|z^n\|_{\mathcal{LB}}} \\
&\le \frac{\|f\|_{\mathcal{LB}}}{\epsilon}\sup_{n\ge N}\frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}.
\end{aligned}$$

The proof is complete. □

4 The essential norm of $C_\varphi D^m$ on $\mathcal{LB}$

Denote $K_r f(z) = f(rz)$ for $r \in (0,1)$. Then K_r is a compact operator on the space $\mathcal{LB}$. It is easy to see that $\|K_r\| \le 1$. We denote by I the identity operator.

In order to give the lower and upper estimate for the essential norm of $C_\varphi D^m$ on $\mathcal{LB}$, we need the following result.

Lemma 4.1 *There is a sequence* $\{r_k\}$, *with* $0 < r_k < 1$ *tending to* 1, *such that the compact operator*

$$L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$$

on $\mathcal{LB}$ *satisfies*:

(i) *for any* $t \in (0,1)$, $\lim_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le t}\sup_{|z|\le t}|((I-L_n)f)'(z)| = 0$,
(iia) $\lim_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{|z|<1}|(I-L_n)f(z)| \le 1$,
(iib) $\lim_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{|z|<s}|(I-L_n)f(z)| = 0$, *for any* $s \in (0,1)$,
(iii) $\limsup_{n\to\infty}\|I-L_n\| \le 1$.

Proof (i) follows from (iib) by Cauchy's formula. The proof of (iii) is similar to the proof of Proposition 8 in [25]. Hence we omit it. Next we prove (iia) and (iib). The argument is much like that given in the proof of Proposition 2.1 of [25] or Lemmas 1 and 2 in [22]. For any $0 < s < 1$, we choose an increasing sequence r_k tending to 1 such that $r_k \geq 1 - \frac{1-s}{k^2}$. For any given $z \in \mathbb{D}$ and r_k, $k = 1, 2, 3, \ldots$, there exists an $s_k \in (r_k, 1)$ such that

$$\left|f(z) - f_{r_k}(z)\right| = zf'(s_k z)(z - r_k z). \tag{4.1}$$

For any $f \in \mathcal{LB}$ with $\|f\|_{\mathcal{LB}} \leq 1$, we have

$$\begin{aligned}\left|(I - L_n)f(z)\right| &\leq \frac{1}{n}\sum_{k=1}^{n}\left|f(z) - f_{r_k}(z)\right| \leq \frac{1}{n}\sum_{k=1}^{n}\left|f'(s_k z)\right|(1 - r_k) \\ &\leq \frac{1}{n}\sum_{k=1}^{n}\frac{1 - r_k}{(1 - |r_k z|)\log\frac{e}{1-|r_k z|}} \leq \frac{1}{n}\sum_{k=1}^{n}1 = 1.\end{aligned}$$

Thus

$$\limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\leq 1}\sup_{|z|<1}\left|(I - L_n)f(z)\right| \leq 1.$$

This shows that (iia) holds.

If $|z| \leq s$, by the equality (4.1), we have

$$\begin{aligned}\left|(I - L_n)f(z)\right| &\leq \frac{1}{n}\sum_{k=1}^{n}\frac{1 - r_k}{(1 - |sz|)\log\frac{e}{1-|sz|}} \\ &\leq \frac{1}{n}\sum_{k=1}^{n}\frac{1 - r_k}{(1 - s)} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k^2} \leq \frac{\pi^2}{6n}.\end{aligned}$$

The above estimate gives (iib). The proof is complete. □

The following lemma can be proved in a standard way; see, for example Proposition 3.11 in [11].

Lemma 4.2 *Let $m \in \mathbb{N}$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi D^m$ is compact on $\mathcal{LB}$ if and only if $C_\varphi D^m$ is bounded on $\mathcal{LB}$ and for any bounded sequence $\{f_n\}$ in $\mathcal{LB}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi D^m f_n\|_{\mathcal{LB}} \to 0$ as $n \to \infty$.*

Theorem 4.3 *Let $m \in \mathbb{N}$ and φ be an analytic self-map of $\mathbb{D}$. Suppose that $C_\varphi D^m$ is bounded on $\mathcal{LB}$. Then the estimate for the essential norm of $C_\varphi D^m$ on $\mathcal{LB}$ is*

$$\left\|C_\varphi D^m\right\|_e^{\mathcal{LB}\to\mathcal{LB}} \approx \limsup_{n\to\infty}\frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}. \tag{4.2}$$

Proof We first give the lower estimate for the essential norm. Without loss of generality, we assume that $n \geq m+1$. Choose the sequence of function $f_n(z) = z^n/\|z^n\|_{\mathcal{LB}}$, $n \in \mathbb{N}$. Then $\|f_n\|_{\mathcal{LB}} = 1$, and $\{f_n\}$ converges to zero weakly on $\mathcal{LB}$ as $n \to \infty$. Thus we have

$$\lim_{n\to\infty}\|Kf_n\|_{\mathcal{LB}} = 0$$

for any given compact operator K on $\mathcal{LB}$. The basic inequality gives

$$\|C_\varphi D^m - K\|^{\mathcal{LB}\to\mathcal{LB}} \geq \|(C_\varphi D^m - K)f_n\|_{\mathcal{LB}} \geq \|C_\varphi D^m f_n\|_{\mathcal{LB}} - \|Kf_n\|_{\mathcal{LB}}.$$

Thus we obtain

$$\|C_\varphi D^m - K\|^{\mathcal{LB}\to\mathcal{LB}} \geq \limsup_{n\to\infty}\|C_\varphi D^m f_n\|_{\mathcal{LB}} \geq \limsup_{n\to\infty}\|C_\varphi D^m f_n\|_{\mathcal{LB}}.$$

So we have

$$\|C_\varphi D^m\|_e^{\mathcal{LB}\to\mathcal{LB}} = \inf_K \|C_\varphi D^m - K\| \geq \limsup_{n\to\infty}\frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}.$$

Now we give the upper estimate for the essential norm. For the case of $\sup_{z\in\mathbb{D}}|\varphi(z)| < 1$, there is a number $\delta \in (0,1)$ such that $\sup_{z\in\mathbb{D}}|\varphi(z)| < \delta$. In this case, the operator $C_\varphi D^m$ is compact on $\mathcal{LB}$. In fact, choose a bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{LB}$ which converges to zero uniformly on compact subset of $\mathbb{D}$. From Cauchy's integral formula, $\{f_n^{(m+1)}\}$ converges to zero on compact subsets of $\mathbb{D}$ as $n \to \infty$. It follows that

$$\begin{aligned}
\lim_{n\to\infty}\|C_\varphi D^m f_n\|_{\mathcal{LB}} &= \lim_{n\to\infty}\left(\left|f_n^{(m)}(\varphi(0))\right| + \|C_\varphi D^m f_n\|_{\log}\right)\\
&= \lim_{n\to\infty}\sup_{z\in\mathbb{D}}\left(1-|z|\right)\log\frac{e}{1-|z|}\left|f_n^{(m+1)}(\varphi(z))\varphi'(z)\right|\\
&\leq \|\varphi\|_{\mathcal{LB}}\lim_{n\to\infty}\sup_{z\in\mathbb{D}}\left|f_n^{(m+1)}(\varphi(z))\right|\\
&= \|\varphi\|_{\mathcal{LB}}\lim_{n\to\infty}\sup_{|w|\leq\delta}\left|f_n^{(m+1)}(w)\right| = 0.
\end{aligned}$$

Then the operator $C_\varphi D^m$ is compact on $\mathcal{LB}$ by Lemma 4.2. This gives

$$\|C_\varphi D^m\|_e^{\mathcal{LB}\to\mathcal{LB}} = 0. \tag{4.3}$$

On the other hand, by Lemma 2.1 and (2.9) we obtain

$$\|z^n\|_{\mathcal{LB}} = H_{0,n}(x_{0,n}) \geq H_{0,n}(r_{0,n}) \geq \frac{1}{2}\log(en),$$

which implies that

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}\\
&\quad\leq e\limsup_{n\to\infty}\sup_{z\in\mathbb{D}}\left(1-|z|\right)\log\frac{e}{1-|z|}\frac{n!}{(n-m-1)!}\left|\varphi(z)\right|^{n-m-1}\left|\varphi'(z)\right|\\
&\quad\leq e\|\varphi\|_{\mathcal{LB}}\lim_{n\to\infty}n^m\delta^{n-m-1} = 0.
\end{aligned}$$

Combining the last inequality with (4.3), we get the desired result.

Next, we assume that $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$. Let L_n be the sequence of operators given in Lemma 4.1. Since L_n is compact on $\mathcal{LB}$ and $C_\varphi D^m$ is bounded on $\mathcal{LB}$, then $C_\varphi D^m L_n$ is also compact on $\mathcal{LB}$. Hence

$$\begin{aligned}\|C_\varphi D^m\|_e^{\mathcal{LB}\to\mathcal{LB}} &\le \limsup_{n\to\infty}\|C_\varphi D^m - C_\varphi D^m L_n\|_{\mathcal{LB}\to\mathcal{LB}} \\ &= \limsup_{n\to\infty}\|C_\varphi D^m(I-L_n)\|_{\mathcal{LB}\to\mathcal{LB}} \\ &= \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\|C_\varphi D^m(I-L_n)f\|_{\mathcal{LB}} \\ &= \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\|((I-L_n)f)^{(m)}\circ\varphi\|_{\mathcal{LB}} \\ &\le I_1 + I_2,\end{aligned}$$

where

$$I_1 = \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\left|\big((I-L_n)f\big)^{(m)}\big(\varphi(0)\big)\right|$$

and

$$I_2 = \limsup_{n\to\infty}\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{z\in\mathbb{D}}\left|\big((I-L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\varphi'(z)\right|\big(1-|z|\big)\log\frac{e}{1-|z|}.$$

It follows from Lemma 4.1(iib) and Cauchy's integral formula that $I_1 = 0$.

For each positive integer $n \ge m+1$, we define

$$\mathbb{D}_n = \{z\in\mathbb{D} : r_{m,n-m} \le |\varphi(z)| < r_{m,n}\},$$

where $r_{m,n}$ is given in Lemma 2.1. Let k be the smallest positive integer such that $\mathbb{D}_k \ne 0$. Since $\sup_{z\in\mathbb{D}}|\varphi(z)| = 1$, $\mathbb{D}_n$ is not empty for every integer $n \ge k$ and $\mathbb{D} = \bigcup_{n=k}^{\infty}\mathbb{D}_n$, we have

$$\sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{z\in\mathbb{D}}\left|\big((I-L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\varphi'(z)\right|\big(1-|z|\big)\log\frac{e}{1-|z|} = I_{21} + I_{22},$$

where

$$I_{21} = \sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{k\le i\le N-1}\sup_{z\in\mathbb{D}_i}\left|\big((I-L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\varphi'(z)\right|\big(1-|z|\big)\log\frac{e}{1-|z|}$$

and

$$I_{22} = \sup_{\|f\|_{\mathcal{LB}}\le 1}\sup_{N\le i}\sup_{z\in\mathbb{D}_i}\left|\big((I-L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\varphi'(z)\right|\big(1-|z|\big)\log\frac{e}{1-|z|}.$$

Here N is a positive integer determined as follows.

By (2.8),

$$\lim_{i\to\infty}\frac{\|z^i\|_{\mathcal{LB}}}{H_{m,i}(r_{m,i-m})} = \frac{e^m}{(m+1)^{m+1}}.$$

Hence, for any given $\varepsilon > 0$, there exists an N such that

$$\frac{\|z^i\|_{\mathcal{LB}}}{H_{m,i}(r_{m,i-m})} \le \frac{e^m}{(m+1)^{m+1}} + \varepsilon$$

when $i \geq N$. For such N it follows that

$$\begin{aligned}
I_{22} &= \sup_{\|f\|_{\mathcal{LB}} \leq 1} \sup_{N \leq i} \sup_{z \in \mathbb{D}_i} \left| \big((I - L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\varphi'(z)\right| \big(1 - |z|\big) \log \frac{e}{1 - |z|} \\
&= \sup_{\|f\|_{\mathcal{LB}} \leq 1} \sup_{N \leq i} \sup_{z \in \mathbb{D}_i} \left| \big((I - L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\varphi'(z)\right| \\
&\quad \cdot \big(1 - |z|\big) \log \frac{e}{1 - |z|} \frac{H_{m,i}(|\varphi(z)|)}{\|z^i\|_{\mathcal{LB}}} \frac{\|z^i\|_{\mathcal{LB}}}{H_{m,i}(|\varphi(z)|)} \\
&\preceq \left(\frac{e^m}{(m+1)^{m+1}} + \varepsilon \right) \sup_{\|f\|_{\mathcal{LB}} \leq 1} \big\| (I - L_n)f \big\|_{\mathcal{LB}} \sup_{N \leq i} \sup_{z \in \mathbb{D}_i} \left| \varphi'(z) \right| \\
&\quad \cdot \big(1 - |z|\big) \log \frac{e}{1 - |z|} \frac{i!}{(i - m - 1)!} \frac{|\varphi(z)|^{i-m-1}}{\|z^i\|_{\mathcal{LB}}} \\
&\leq \left(\frac{e^m}{(m+1)^{m+1}} + \varepsilon \right) \|I - L_n\| \sup_{N \leq i} \frac{\|C_\varphi D^m(z^i)\|_{\mathcal{LB}}}{\|z^i\|_{\mathcal{LB}}}.
\end{aligned}$$

Thus

$$\limsup_{n \to \infty} I_{22} \preceq \left(\frac{e^m}{(m+1)^{m+1}} + \varepsilon \right) \sup_{N \leq i} \frac{\|C_\varphi D^m(z^i)\|_{\mathcal{LB}}}{\|z^i\|_{\mathcal{LB}}}. \tag{4.4}$$

By (ii) of Lemma 4.1 and Cauchy's integral formula, we have

$$\begin{aligned}
&\limsup_{n \to \infty} I_{21} \\
&\quad = \limsup_{n \to \infty} \sup_{\|f\|_{\mathcal{LB}} \leq 1} \sup_{k \leq i < N-1} \sup_{z \in \mathbb{D}_i} \left| \big((I - L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\right| \left|\varphi'(z)\right| \big(1 - |z|\big) \log \frac{e}{1 - |z|} \\
&\quad \leq \|\varphi\|_{\mathcal{LB}} \limsup_{n \to \infty} \sup_{\|f\|_{\mathcal{LB}} \leq 1} \sup_{|\varphi(z)| < r_{m,N-1}} \left| \big((I - L_n)f\big)^{(m+1)}\big(\varphi(z)\big)\right| \\
&\quad = 0,
\end{aligned}$$

which together with (4.4) implies that

$$I_2 \preceq \left(\frac{e^m}{(m+1)^{m+1}} + \varepsilon \right) \sup_{N \leq i} \frac{\|C_\varphi D^m(z^i)\|_{\mathcal{LB}}}{\|z^i\|_{\mathcal{LB}}}. \tag{4.5}$$

From (4.5) we obtain

$$\left\| C_\varphi D^m \right\|_e^{\mathcal{LB} \to \mathcal{LB}} \leq I_1 + I_2 \preceq \left(\frac{e^m}{(m+1)^{m+1}} + \varepsilon \right) \sup_{N \leq i} \frac{\|C_\varphi D^m(z^i)\|_{\mathcal{LB}}}{\|z^i\|_{\mathcal{LB}}}.$$

By the arbitrariness of ε, we get

$$\left\| C_\varphi D^m \right\|_e^{\mathcal{LB} \to \mathcal{LB}} \preceq \frac{e^m}{(m+1)^{m+1}} \limsup_{n \to \infty} \frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}.$$

The proof is complete. □

From Theorem 4.3, we obtain the following result.

Corollary 4.4 *Let $m \in \mathbb{N}$ and φ be an analytic self-map of $\mathbb{D}$ such that $C_\varphi D^m$ is bounded on $\mathcal{LB}$. Then $C_\varphi D^m$ is compact on $\mathcal{LB}$ if and only if*

$$\limsup_{n\to\infty} \frac{\|C_\varphi D^m(z^n)\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}} = 0.$$

Especially, when $m = 0$, from the proof of Theorem 4.3, we get the exact formula for essential norm of composition operator on $\mathcal{LB}$.

Corollary 4.5 *Let φ be an analytic self-map of $\mathbb{D}$. Suppose that C_φ is bounded on $\mathcal{LB}$; then*

$$\|C_\varphi\|_e^{\mathcal{LB}\to\mathcal{LB}} = \limsup_{n\to\infty} \frac{\|\varphi^n\|_{\mathcal{LB}}}{\|z^n\|_{\mathcal{LB}}}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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