Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The k-quasi-$*$-class $\mathcal{A}$ contractions have property PF

Ilmi Hoxha* and Naim L Braha

*Correspondence:
ilmihoxha011@gmail.com
Department of Mathematics and Computer Sciences, University of Prishtina, Avenue 'Mother Theresa' 5, Prishtinë, 10000, Kosovo

Abstract
First, we will see that if T is a contraction of the k-quasi-$*$-class $\mathcal{A}$ operator, then the nonnegative operator $D = T^{*k}(|T^2| - |T^*|^2)T^k$ is a contraction whose power sequence $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P and $TT^{*k}P = 0$. Second, it will be proved that if T is a contraction of the k-quasi-$*$-class $\mathcal{A}$ operator, then either T has a non-trivial invariant subspace or T is a proper contraction. Finally it will be proved that if T belongs to the k-quasi-$*$-class $\mathcal{A}$ and is a contraction, then T has a *Wold-type decomposition* and T has the *PF property*.
MSC: 47A10; 47B37; 15A18

Keywords: k-quasi-$*$-class $\mathcal{A}$; contractions; proper contractions; Wold-type decomposition; PF property; supercyclic operator; hypercyclic operator

1 Introduction

Throughout this paper, let H and K be infinite dimensional separable complex Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$. We denote by $L(H,K)$ the set of all bounded operators from H into K. To simplify, we put $L(H) := L(H,H)$. For $T \in L(H)$, we denote by $\ker T$ the null space and by $T(H)$ the range of T. The closure of a set M will be denoted by $\overline{M}$. We shall denote the set of all complex numbers by $\mathbb{C}$ and the set of all nonnegative integers by $\mathbb{N}$.

For an operator $T \in L(H)$, as usual, by T^* we mean the adjoint of T and $|T| = (T^*T)^{\frac{1}{2}}$. An operator T is said to be hyponormal, if $|T|^2 \geq |T^*|^2$. An operator T is said to be paranormal, if

$$\|T^2x\| \geq \|Tx\|^2$$

for any unit vector x in H [1]. Further, T is said to be $*$-paranormal, if

$$\|T^2x\| \geq \|T^*x\|^2$$

for any unit vector x in H [2]. T is said to be a k-paranormal operator if $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\|x\|^k$ for all $x \in H$, and T is said to be a k-$*$-paranormal operator if $\|T^*x\|^{k+1} \leq \|T^{k+1}x\|\|x\|^k$, for all $x \in H$.

Furuta *et al.* [3] introduced a very interesting class of bounded linear Hilbert space operators: class $\mathcal{A}$ defined by

$$|T^2| \geq |T|^2,$$

and they showed that the class $\mathcal{A}$ is a subclass of paranormal operators and contains hyponormal operators. Jeon and Kim [4] introduced the quasi-class $\mathcal{A}$. An operator T is said to be a quasi-class $\mathcal{A}$, if

$$T^*|T^2|T \geq T^*|T|^2T.$$

We denote the set of quasi-class $\mathcal{A}$ by $\mathcal{QA}$. An operator T is said to be a k-quasi-class $\mathcal{A}$, if

$$T^{*k}|T^2|T^k \geq T^{*k}|T|^2T^k.$$

We denote the set of quasi-class $\mathcal{A}$ by $\mathcal{QA}(k)$.

Duggal *et al.* [5], introduced $*$-class $\mathcal{A}$ operator. An operator T is said to be a $*$-class $\mathcal{A}$ operator, if

$$|T^2| \geq |T^*|^2.$$

A $*$-class $\mathcal{A}$ is a generalization of a hyponormal operator [5, Theorem 1.2], and $*$-class $\mathcal{A}$ is a subclass of the class of $*$-paranormal operators [5, Theorem 1.3]. We denote the set of $*$-class $\mathcal{A}$ by $\mathcal{A}^*$. Shen *et al.* in [6] introduced the quasi-$*$-class $\mathcal{A}$ operator: an operator T is said to be a quasi-$*$-class $\mathcal{A}$ operator, if

$$T^*|T^2|T \geq T^*|T^*|^2T.$$

We denote the set of quasi-$*$-class $\mathcal{A}$ by $\mathcal{QA}^*$. Mecheri [7] introduced the k-quasi-$*$-class $\mathcal{A}$ operator.

Definition 1.1 An operator $T \in L(H)$ is said to be a k-quasi-$*$-class $\mathcal{A}$ operator, if

$$T^{*k}(|T^2| - |T^*|^2)T^k \geq O$$

for a nonnegative integer k.

We denote the set of the k-quasi-$*$-class $\mathcal{A}$ by $\mathcal{QA}^*(k)$.

Example 1.2 Let T be an operator defined by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $|T^2| - |T^*|^2 \ngeq O$ and so T is not a class $\mathcal{A}^*$. However, $T^{*k}(|T^2| - |T^*|^2)T^k = O$ for every positive number k, which implies that T is a k-quasi-class $\mathcal{A}^*$ operator.

A contraction is an operator T such that $\|Tx\| \leq \|x\|$ for all $x \in H$. A proper contraction is an operator T such that $\|Tx\| < \|x\|$ for every nonzero $x \in H$ [8]. A strict contraction is an operator such that $\|T\| < 1$ (*i.e.*, $\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} < 1$). Obviously, every strict contraction is a proper contraction and every proper contraction is a contraction. An operator T is said to be completely non-unitary (c.n.u.) if T restricted to every reducing subspace of H is non-unitary.

An operator T on H is uniformly stable, if the power sequence $\{T^n\}_{n=1}^{\infty}$ converges uniformly to the null operator (*i.e.*, $\|T^n\| \to O$). An operator T on H is strongly stable, if the power sequence $\{T^n\}_{n=1}^{\infty}$ converges strongly to the null operator (*i.e.*, $\|T^n x\| \to 0$, for every $x \in H$).

A contraction T is of class $C_{0\cdot}$ if T is strongly stable (*i.e.*, $\|T^n x\| \to 0$ and $\|Tx\| \leq \|x\|$ for every $x \in H$). If T^* is a strongly stable contraction, then T is of class $C_{\cdot 0}$. T is said to be of class $C_{1\cdot}$ if $\lim_{n\to\infty} \|T^n x\| > 0$ (equivalently, if $T^n x \nrightarrow 0$ for every nonzero x in H). T is said to be of class $C_{\cdot 1}$ if $\lim_{n\to\infty} \|T^{*n} x\| > 0$ (equivalently, if $T^{*n} x \nrightarrow 0$ for every nonzero x in H). We define the class $C_{\alpha\beta}$ for $\alpha, \beta = 0, 1$ by $C_{\alpha\beta} = C_{\alpha\cdot} \cap C_{\cdot\beta}$. These are the Nagy-Foiaş classes of contractions [9, p.72]. All combinations are possible leading to classes C_{00}, C_{01}, C_{10}, and C_{11}. In particular, T and T^* are both strongly stable contractions if and only if T is a C_{00} contraction. Uniformly stable contractions are of class C_{00}.

Lemma 1.3 [10, Holder-McCarthy inequality] *Let T be a positive operator. Then the following inequalities hold for all $x \in H$*:
(1) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $0 < r < 1$;
(2) $\langle T^r x, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $r \geq 1$.

Lemma 1.4 [7, Lemma 2.1] *Let T be a k-quasi-$*$-class $\mathcal{A}$ operator, where T^k does not have a dense range, and let T have the following representation*:

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on } H = \overline{T^k(H)} \oplus \ker T^{*k}.$$

Then A is class $\mathcal{A}^*$ on $\overline{T^k(H)}$, $C^k = O$, and $\sigma(T) = \sigma(A) \cup \{0\}$.

2 Main results

Theorem 2.1 *If T is a contraction of the k-quasi-$*$-class $\mathcal{A}$ operator, then the nonnegative operator*

$$D = T^{*k}\left(\left|T^2\right| - \left|T^*\right|^2\right)T^k$$

is a contraction whose power sequence $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P and $T^* T^k P = O$.

Proof Suppose that T is a contraction of the k-quasi-$*$-class $\mathcal{A}$ operator. Then

$$D = T^{*k}\left(\left|T^2\right| - \left|T^*\right|^2\right)T^k \geq O.$$

Let $R = D^{\frac{1}{2}}$ be the unique nonnegative square root of D, then for every x in H and any nonnegative integer n, we have

$$\begin{aligned}\left\langle D^{n+1}x, x\right\rangle &= \left\|R^{n+1}x\right\|^2 \\ &= \left\langle DR^n x, R^n x\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \langle T^{*k}|T^2|T^kR^nx, R^nx\rangle - \langle T^{*k}|T^*|^2T^kR^nx, R^nx\rangle \\
&\le \||T^2|^{\frac{1}{2}}T^kR^nx\|^2 - \|T^*T^kR^nx\|^2 \\
&\le \|R^nx\|^2 - \|T^*T^kR^nx\|^2 \\
&\le \|R^nx\|^2 \\
&= \langle D^nx, x\rangle.
\end{aligned}
$$

Thus R (and so D) is a contraction (set $n = 0$), and $\{D^n\}_{n=1}^{\infty}$ is a decreasing sequence of nonnegative contractions. Then $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P. Moreover,

$$\sum_{n=0}^{m}\|T^*T^kR^nx\|^2 \le \sum_{n=0}^{m}(\|R^nx\|^2 - \|R^{n+1}x\|^2) = \|x\|^2 - \|R^{m+1}x\|^2 \le \|x\|^2$$

for all nonnegative integers m and for every $x \in H$. Therefore $\|T^*T^kR^nx\| \to 0$ as $n \to \infty$. Then we have

$$T^*T^kPx = T^*T^k\lim_{n\to\infty}D^nx = \lim_{n\to\infty}T^*T^kR^{2n}x = 0$$

for every $x \in H$. So that $T^*T^kP = O$. □

A subspace M of space H is said to be non-trivial invariant (alternatively, T-invariant) under T if $\{0\} \neq M \neq H$ and $T(M) \subseteq M$. A closed subspace $M \subseteq H$ is said to be a non-trivial hyperinvariant subspace for T if $\{0\} \neq M \neq H$ and is invariant under every operator $S \in L(H)$, which fulfills $TS = ST$.

Recently Duggal *et al.* [11] showed that if T is a class $\mathcal{A}$ contraction, then either T has a non-trivial invariant subspace or T is a proper contraction and the nonnegative operator $D = |T^2| - |T|^2$ is strongly stable. Duggal *et al.* [12] extended these results to contractions in $\mathcal{QA}$. Jeon and Kim [13] extended these results to contractions $\mathcal{QA}(k)$. Gao and Li [14] have proved that if a contraction $T \in \mathcal{A}^*$ has a no non-trivial invariant subspace, then (a) T is a proper contraction and (b) the nonnegative operator $D = |T^2| - |T^*|^2$ is a strongly stable contraction. In this paper we extend these results to contractions in the k-quasi-$*$-class $\mathcal{A}$ for $k > 0$.

Theorem 2.2 *Let T be a contraction of the k-quasi-$*$-class $\mathcal{A}$ for $k > 0$. If T has a no non-trivial invariant subspace, then*:

(1) *T is a proper contraction*;
(2) *the nonnegative operator*

$$D = T^{*k}(|T^2| - |T^*|^2)T^k$$

is a strongly stable contraction.

Proof We may assume that T is a nonzero operator.

(1) If either $\ker T$ or $\overline{T^k(H)}$ is a non-trivial subspace (*i.e.*, $\ker T \neq \{0\}$ or $\overline{T^k(H)} \neq H$), then T has a non-trivial invariant subspace. Hence, if T has no non-trivial invariant subspace,

then T is injective and $\overline{T^k(H)} = H$. Furthermore, T is a class $\mathcal{A}^*$ operator. The proof now follows from [14, Theorem 2.2].

(2) Let T be a contraction of the k-quasi-$*$-class $\mathcal{A}$. By the above theorem, we see that D is a contraction, $\{D^n\}_{n=1}^{\infty}$ converges strongly to a projection P, and $T^*T^kP = O$. So, $PT^{*k}T = O$. Suppose T has no non-trivial invariant subspaces. Since $\ker P$ is a nonzero invariant subspace for T whenever $PT^{*k}T = O$ and $T \neq O$, it follows that $\ker P = H$. Hence $P = O$, and we see that $\{D^n\}_{n=1}^{\infty}$ converges strongly to the null operator O, so D is a strongly stable contraction. Since D is self-adjoint, $D \in C_{00}$. □

Corollary 2.3 *Let T be a contraction of the k-quasi-$*$-class $\mathcal{A}$. If T has no non-trivial invariant subspace, then both T and the nonnegative operators*

$$D = T^{*k}\left(\left|T^2\right| - \left|T^*\right|^2\right)T^k$$

are proper contractions.

Proof A self-adjoint operator T is a proper contraction if and only if T is a C_{00} contraction. □

Definition 2.4 If the contraction T is a direct sum of the unitary and $C_{\cdot 0}$ (c.n.u.) contractions, then we say that T has a *Wold-type decomposition*.

Definition 2.5 [15] An operator $T \in L(H)$ is said to have the Fuglede-Putnam commutativity property (*PF property* for short) if $T^*X = XJ$ for any $X \in L(K,H)$ and any isometry $J \in L(K)$ such that $TX = XJ^*$.

Lemma 2.6 [16, 17] *Let T be a contraction. The following conditions are equivalent*:

(1) *For any bounded sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}} \subset H$ such that $Tx_{n+1} = x_n$ the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is constant*;
(2) *T has a* Wold-type decomposition;
(3) *T has the* PF property.

Duggal and Cubrusly in [16] have proved: Each k-paranormal contraction operator has a *Wold-type decomposition*. Pagacz in [17] has proved the same and also proved that each k-$*$-paranormal operator has a *Wold-type decomposition*. In this paper, we extend to contractions in $\mathcal{QA}^*(k)$.

Theorem 2.7 *Let T be a contraction of the k-quasi-$*$-class $\mathcal{A}$. Then T has a* Wold-type decomposition.

Proof Since T is a contraction operator, the decreasing sequence $\{T^nT^{*n}\}_{n=1}^{\infty}$ converges strongly to a nonnegative contraction. We denote by

$$S = \left(\lim_{n\to\infty} T^nT^{*n}\right)^{\frac{1}{2}}.$$

The operators T and S are related by $T^*S^2T = S^2$, $O \leq S \leq I$ and S is self-adjoint operator. By [18] there exists an isometry $V : \overline{S(H)} \to \overline{S(H)}$ such that $VS = ST^*$, and thus $SV^* = TS$, and $\|SV^m x\| \to \|x\|$ for every $x \in \overline{S(H)}$. The isometry V can be extended to an isometry on H, which we still denote by V.

For an $x \in \overline{S(H)}$, we can define $x_n = SV^n x$ for $n \in \mathbb{N} \cup \{0\}$. Then for all nonnegative integers m we have

$$T^m x_{n+m} = T^m SV^{m+n} x = SV^{*m} V^{m+n} x = SV^n x = x_n,$$

and for all $m \le n$ we have

$$T^m x_n = x_{n-m}.$$

Since T is a k-quasi-$*$-class $\mathcal{A}$ operator and the non-trivial $x \in \overline{S(H)}$ we have

$$\begin{aligned}
\|x_n\|^4 &= \left\|T^k x_{n+k}\right\|^4 \\
&= \left\langle T^* T T^{k-1} x_{n+k}, T^{k-1} x_{n+k}\right\rangle^2 \\
&\le \left\|T^* T^k x_{n+k}\right\|^2 \left\|T^{k-1} x_{n+k}\right\|^2 \\
&= \left\langle T^{*k} \left|T^*\right|^2 T^k x_{n+k}, x_{n+k}\right\rangle^2 \|x_{n+1}\|^2 \\
&\le \left\langle T^{*k} \left|T^2\right| T^k x_{n+k}, x_{n+k}\right\rangle^2 \|x_{n+1}\|^2 \\
&\le \left\langle \left|T^2\right|^2 T^k x_{n+k}, T^k x_{n+k}\right\rangle^{\frac{1}{2}} \left\|T^k x_{n+k}\right\|^{2(1-\frac{1}{2})} \|x_{n+1}\|^2 \\
&= \left\|T^{k+2} x_{n+k}\right\| \left\|T^k x_{n+k}\right\| \|x_{n+1}\|^2 \\
&= \|x_{n-2}\| \|x_n\| \|x_{n+1}\|^2.
\end{aligned}$$

Then

$$\|x_n\|^3 \le \|x_{n-2}\| \|x_{n+1}\|^2;$$

hence

$$\|x_n\| \le \|x_{n-2}\|^{\frac{1}{3}} \|x_{n+1}\|^{\frac{2}{3}} \le \frac{1}{3}\left(\|x_{n-2}\| + 2\|x_{n+1}\|\right).$$

Thus

$$2\left(\|x_{n+1}\| - \|x_n\|\right) \ge \|x_n\| - \|x_{n-2}\| = \left(\|x_n\| - \|x_{n-1}\|\right) + \left(\|x_{n-1}\| - \|x_{n-2}\|\right).$$

Put

$$b_n = \|x_n\| - \|x_{n-1}\|,$$

and we have

$$2b_{n+1} \ge b_n + b_{n-1}. \tag{1}$$

Since $x_n = Tx_{n+1}$, we have

$$\|x_n\| = \|Tx_{n+1}\| \le \|x_{n+1}\| \quad \text{for every } n \in \mathbb{N},$$

then the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is increasing. From

$$SV^n = SV^*V^{n+1} = TSV^{n+1}$$

we have

$$\|x_n\| = \|SV^n x\| = \|TSV^{n+1}x\| \leq \|SV^{n+1}x\| \leq \|x\|$$

for every $x \in \overline{S(H)}$ and $n \in \mathbb{N} \cup \{0\}$. Then $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is bounded. From this we have $b_n \geq 0$ and $b_n \to 0$ as $n \to \infty$.

It remains to check that all b_n equal zero. Suppose that there exists an integer $i \geq 1$ such that $b_i > 0$. Using the inequality (1) we get $b_{i+1} > 0$ and $b_{i+2} > 0$, so there exists $\epsilon > 0$ such that $b_{i+1} > \epsilon$ and $b_{i+2} > \epsilon$. From that, and using again the inequality (1), we can show by induction that $b_n > \epsilon$ for all $n > i$, thus arriving at a contradiction. So $b_n = 0$ for all $n \in \mathbb{N}$ and thus $\|x_{n-1}\| = \|x_n\|$ for all $n \geq 1$. Thus the sequence $\{\|x_n\|\}_{n\in\mathbb{N}\cup\{0\}}$ is constant.

From Lemma 2.6, T has a *Wold-type decomposition*. □

For $T \in L(H)$ and $x \in H$, $\{T^n x\}_{n=0}^{\infty}$ is called the orbit of x under T, and is denoted by $\mathcal{O}(x,T)$. When the linear span of the orbit $\mathcal{O}(x,T)$ is norm dense in H, x is called a cyclic vector for T and T is said to be a cyclic operator. If $\mathcal{O}(x,T)$ is norm dense in H, then x is called a hypercyclic vector for T. An operator $T \in L(H)$ is called hypercyclic if there is at least one hypercyclic vector for T. We say that an operator $T \in L(H)$ is supercyclic if there exists a vector $x \in H$ such that $\mathbb{C}\mathcal{O}(x,T) = \{\lambda T^n x : \lambda \in \mathbb{C}, n = 0,1,2,\ldots\}$ is norm dense in H.

Theorem 2.8 *Let $T \in L(H)$ be a quasi-$*$-class $\mathcal{A}$ such that $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. If the inverse of T is a quasi-$*$-class $\mathcal{A}$, then T is not a supercyclic operator.*

Proof Let $T \in L(H)$ be a quasi-$*$-class $\mathcal{A}$. Since $\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, T is an invertible operator. From [7] T is normaloid, thus $\|T\| = r(T) = 1$. Since $T^{-1} \in \mathcal{Q}(\mathcal{A}^*)$, $\|T^{-1}\| = 1$. Consequently, T is unitary. Since no unitary operator on an infinite dimensional Hilbert space can be supercyclic, we see that T is not a supercyclic operator. □

Remark 2.9 The condition that the inverse of the operator T belongs to quasi-$*$-class $\mathcal{A}$ cannot be removed from Theorem 2.8, because there are invertible operators from the quasi-$*$-class $\mathcal{A}$, such that their inverse does not belong to the quasi-$*$-class $\mathcal{A}$. This is shown in the following example.

Given a bounded sequence of complex numbers $\{\alpha_n : n \in \mathbb{Z}\}$ (called weights), let T be the bilateral weighted shift on an infinite dimensional Hilbert space operator $H = l_2$, with the canonical orthonormal basis $\{e_n : n \in \mathbb{Z}\}$, defined by $Te_n = \alpha_n e_{n+1}$ for all $n \in \mathbb{Z}$.

Lemma 2.10 *Let T be a bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T is a quasi-$*$-class $\mathcal{A}$ operator if and only if*

$$|\alpha_n|^2 \leq |\alpha_{n+1}||\alpha_{n+2}|$$

for all $n \in \mathbb{Z}$.

Lemma 2.11 *Let T be a non-singular bilateral weighted shift operator with weights $\{\alpha_n : n \in \mathbb{Z}\}$. Then T^{-1} is a quasi-$*$-class $\mathcal{A}$ operator if and only if*

$$|\alpha_{n-1}|^2 \geq |\alpha_{n-2}||\alpha_{n-3}|$$

for all $n \in \mathbb{Z}$.

Example 2.12 Let us denote by T the bilateral weighted shift operator, with weighted sequence $\{\alpha_n : n \in \mathbb{Z}\}$, given by the relation

$$\alpha_n = \begin{cases} 1 & \text{if } n \leq 1, \\ 2 & \text{if } n = 2, \\ 1 & \text{if } n = 3, \\ 4 & \text{if } n = 4, \\ 1 & \text{if } n = 5, \\ 16 & \text{if } n \geq 6. \end{cases}$$

From Lemma 2.10 it follows that T is a quasi-$*$-class $\mathcal{A}$ operator. Since $\{\alpha_n : n \in \mathbb{Z}\}$ is a bounded sequence of positive numbers with $\inf\{\alpha_n : n \in \mathbb{Z}\} > 0$, T is an invertible operator [19, Proposition II.6.8]. But T^{-1} is not a quasi-$*$-class $\mathcal{A}$ operator, which follows from Lemma 2.11, for $n = 4$.

Theorem 2.13 *Let $T \in L(H)$ be a quasi-$*$-class $\mathcal{A}$ operator and $\mathbb{D} = \{z : |z| < 1\}$. If T^* is a hypercyclic operator and for every hyperinvariant $M \subseteq H$ of T, the inverse of $T|_M$, whenever it exists, is a normaloid operator, then $\sigma(T|_M) \cap \mathbb{D} \neq \emptyset$ and $\sigma(T|_M) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$.*

Proof Assume that T^* is a hypercyclic operator. Then there exists a vector $x \in H$ such that $\overline{\{(T^*)^n x\}_{n=0}^{\infty}} = H$. Let $S = T|_M$ for some closed T-invariant subspace and let P be the orthogonal projection of H onto M. Since $(S^*)^n Px = P(T^*)^n x$ for each $n \in \mathbb{N} \cup \{0\}$ we have

$$\overline{\{(S^*)^n (Px)\}_{n=0}^{\infty}} = P\overline{\{(T^*)^n x\}_{n=0}^{\infty}} = P(H) = M,$$

thus S^* is hypercyclic.

From [20, Corollary 3] we have $\|S^*\| > 1$. Since S is a quasi-$*$-class $\mathcal{A}$, S is normaloid, thus $r(T|_M) = \|S\| = \|S^*\| > 1$. Therefore $\sigma(T|_M) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$.

Suppose that $\sigma(T|_M) \subset (\mathbb{C} \setminus \overline{\mathbb{D}})$. Then $\sigma(S^{-1}) \subset \overline{\mathbb{D}}$, and since S^{-1} is normaloid, $\|S^{-1}\| = r(S^{-1}) \leq 1$. Since S^* is hypercyclic, from [20, Theorem 6] $(S^*)^{-1}$ is hypercyclic, so $\|(S^*)^{-1}\| > 1$. Thus $\|S^{-1}\| = \|(S^*)^{-1}\| > 1$. This is a contradiction, therefore $\sigma(T|_M) \cap \mathbb{D} \neq \emptyset$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Received: 21 June 2014 Accepted: 22 October 2014 Published: 31 October 2014

References

1. Furuta, T: On the class of paranormal operators. Proc. Jpn. Acad. **43**, 594-598 (1967)
2. Arora, SC, Thukral, JK: On a class of operators. Glas. Mat. **21(41)**(2), 381-386 (1986)
3. Furuta, T, Ito, M, Yamazaki, T: A subclass of paranormal operators including class of log-hyponormal and several classes. Sci. Math. **1**(3), 389-403 (1998)
4. Jeon, IH, Kim, IH: On operators satisfying $T^*|T^2|T \geq T^*|T|^2T$. Linear Algebra Appl. **418**, 854-862 (2006)
5. Duggal, BP, Jeon, IH, Kim, IH: On $*$-paranormal contractions and properties for $*$-class $\mathcal{A}$ operators. Linear Algebra Appl. **436**(5), 954-962 (2012)
6. Shen, JL, Zuo, F, Yang, CS: On operators satisfying $T^*|T^2|T \geq T^*|T^*|^2T$. Acta Math. Sin. Engl. Ser. **26**(11), 2109-2116 (2010)
7. Mecheri, S: Isolated points of spectrum of k-quasi-$*$-class $\mathcal{A}$ operators. Stud. Math. **208**, 87-96 (2012)
8. Kubrusly, CS, Levan, N: Proper contractions and invariant subspace. Int. J. Math. Sci. **28**, 223-230 (2001)
9. Sz-Nagy, B, Foiaş, C: Harmonic Analysis of Operators on Hilbert Space. North-Holland, Amsterdam (1970)
10. McCarthy, CA: $\mathbf{C}_p$. Isr. J. Math. **5**, 249-271 (1967)
11. Duggal, BP, Jeon, IH, Kubrusly, CS: Contractions satisfying the absolute value property $|A|^2 \leq |A^2|$. Integral Equ. Oper. Theory **49**, 141-148 (2004)
12. Duggal, BP, Jeon, IH, Kim, IH: On quasi-class $\mathcal{A}$ contractions. Linear Algebra Appl. **436**, 3562-3567 (2012)
13. Jeon, IH, Kim, IH: On k-quasi-class $\mathcal{A}$ contractions. Korean J. Math. **22**(1), 85-89 (2014)
14. Gao, F, Li, X: On $*$-class $\mathcal{A}$ contractions. J. Inequal. Appl. **2013**, 239 (2013). doi:10.1186/1029-242X-2013-239
15. Duggal, BP: On characterising contractions with C_{10} pure part. Integral Equ. Oper. Theory **27**, 314-323 (1997)
16. Duggal, BP, Cubrusly, CS: Paranormal contractions have property PF. Far East J. Math. Sci. **14**, 237-249 (2004)
17. Pagacz, P: On Wold-type decomposition. Linear Algebra Appl. **436**, 3065-3071 (2012)
18. Durszt, E: Contractions as restricted shifts. Acta Sci. Math. **48**, 129-134 (1985)
19. Conway, JB: The Theory of Subnormal Operators. Mathematical Surveys and Monographs, vol. 36. Am. Math. Soc., Providence (1991)
20. Miller, VG: Remarks on finitely hypercyclic and finitely supercyclic operators. Integral Equ. Oper. Theory **29**, 110-115 (1997)

doi:10.1186/1029-242X-2014-433
Cite this article as: Hoxha and Braha: **The k-quasi-$*$-class $\mathcal{A}$ contractions have property PF.** *Journal of Inequalities and Applications* 2014 **2014**:433.