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Split equality problem and multiple-sets split equality problem for quasi-nonexpansive multi-valued mappings

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Abstract

The multiple-sets split equality problem (MSSEP) requires finding a point $x \in \bigcap_{i=1}^{N} C_i$, $y \in \bigcap_{j=1}^{M} Q_j$, such that $Ax = By$, where N and M are positive integers, $\{C_1, C_2, \dots, C_N\}$ and $\{Q_1, Q_2, \dots, Q_M\}$ are closed convex subsets of Hilbert spaces H_1, H_2, respectively, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators. When $N = M = 1$, the MSSEP is called the split equality problem (SEP). If let $B = I$, then the MSSEP and SEP reduce to the well-known multiple-sets split feasibility problem (MSSFP) and split feasibility problem (SFP), respectively. Recently, some authors proposed many algorithms to solve the SEP and MSSEP. However, to implement these algorithms, one has to find the projection on the closed convex sets, which is not possible except in simple cases. One of the purposes of this paper is to study the SEP and MSSEP for a family of quasi-nonexpansive multi-valued mappings in the framework of infinite-dimensional Hilbert spaces, and propose an algorithm to solve the SEP and MSSEP without the need to compute the projection on the closed convex sets.

Keywords: split equality problem; iterative algorithms; converge strongly

1 Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space, C is a subset of H. Denote by $CB(H)$ the collection of all nonempty closed and bounded subsets of H and by $\mathrm{Fix}(T)$ the set of the fixed points of a mapping T. The *Hausdorff* metric $\tilde{H}$ on $CB(H)$ is defined by

$$\tilde{H}(C, D) := \max\Big\{\sup_{x\in C} d(x, D), \sup_{y\in D} d(y, C)\Big\}, \quad \forall C, D \in CB(H),$$

where $d(x, K) := \inf_{y\in K} d(x, y)$.

Definition 1.1 Let $R : H \to CB(H)$ be a multi-valued mapping. An element $p \in H$ is said to be a *fixed point of* R, if $p \in Rp$. The set of fixed points of R will be denoted by $\mathrm{Fix}(R)$. R is said to be

(i) nonexpansive, if $\tilde{H}(Rx, Ry) \le \|x - y\|$, $\forall x, y \in H$;
(ii) quasi-nonexpansive, if $\mathrm{Fix}(R) \ne \emptyset$ and $\tilde{H}(Rx, Ry) \le \|x - y\|$, $\forall x \in H$, $y \in \mathrm{Fix}(R)$.

Let $\{C_1, C_2, \dots, C_N\}$ and $\{Q_1, Q_2, \dots, Q_M\}$ be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator.

Recall that the *multiple-sets split feasibility problem* (MSSFP) is to find a point x satisfying the property:

$$x \in \bigcap_{i=1}^{N} C_i, \qquad Ax \in \bigcap_{j=1}^{M} Q_j,$$

if such a point exists. If $N = M = 1$, then the MSSFP reduce to the well-known *split feasibility problem* (SFP).

The SFP and MSSFP was first introduced by Censor and Elfving [1], and Censor *et al.* [2], respectively, which attracted many authors' attention due to the applications in signal processing [1] and intensity-modulated radiation therapy [2]. Various algorithms have been invented to solve it (see [1–8], *etc.*).

Recently, Moudafi [9] proposed a new *split equality problem* (SEP): Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$, $Q \subseteq H_2$ be two nonempty closed convex sets, and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Find $x \in C$, $y \in Q$ satisfying

$$Ax = By. \tag{1.1}$$

When $B = I$, SEP reduces to the well-known SFP.

Naturally, we propose the following *multiple-sets split equality problem* (MSSEP) requiring to find a point $x \in \bigcap_{i=1}^{N} C_i$, $y \in \bigcap_{j=1}^{M} Q_j$, such that

$$Ax = By, \tag{1.2}$$

where N and M are positive integers, $\{C_1, C_2, \ldots, C_N\}$ and $\{Q_1, Q_2, \ldots, Q_M\}$ are closed convex subsets of Hilbert spaces H_1, H_2, respectively, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators.

In the paper [9], Moudafi give the alternating CQ-algorithm and relaxed alternating CQ-algorithm iterative algorithm for solving the split equality problem.

Let $S = C \times Q$ in $H = H_1 \times H_2$, define $G : H \to H_3$ by $G = [A, -B]$, then $G^*G : H \to H$ has the matrix form

$$G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

The original problem can now be reformulated as finding $w = (x, y) \in S$ with $Gw = 0$, or, more generally, minimizing the function $\|Gw\|$ over $w \in S$. Therefore solving SEP (1.1) is equivalent to solving the following minimization problem:

$$\min_{w \in S} f(w) = \frac{1}{2}\|Gw\|^2.$$

In the paper [10], we use the well-known Tychonov regularization to get some algorithms that converge strongly to the minimum-norm solution of the SEP.

Note that to implement these algorithms, one has to find the projection on the closed convex sets, which is not possible except in simple cases.

The purpose of this paper is to introduce and study the following split equality problem for quasi-nonexpansive multi-valued mappings in infinitely dimensional Hilbert spaces, *i.e.*, to find $w = (x, y) \in C$ such that

$$Ax = By, \tag{1.3}$$

where H_1, H_2, H_3 are real Hilbert spaces, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, $R_i : H_i \to CB(H_i)$, $i = 1, 2$ are two quasi-nonexpansive multi-valued mappings, $C = \mathrm{Fix}(R_1)$, $Q = \mathrm{Fix}(R_2)$. In the rest of this paper, we still use Γ to denote the set of solutions of SEP (1.3), and assume consistency of SEP so that Γ is closed, convex, and nonempty, *i.e.*, $\Gamma = \{(x, y) \in H_1 \times H_2, Ax = By, x \in C, y \in Q\} \neq \emptyset$. The multiple-sets split equality problem (MSSEP) for a family quasi-nonexpansive multi-valued mappings in infinitely dimensional Hilbert spaces, *i.e.*, to find $w = (x, y) \in C$ such that

$$Ax = By, \tag{1.4}$$

where $R_i^j : H_i \to CB(H_i)$, $i = 1, 2$, $j = 1, 2, \dots, N$ is a family of quasi-nonexpansive multi-valued mappings, $C = \bigcap_{j=1}^N \mathrm{Fix}(R_1^j)$, $Q = \bigcap_{j=1}^N \mathrm{Fix}(R_2^j)$. In the rest of this paper, we use $\bar{\Gamma}$ to denote the set of solutions of MSSEP (1.4), and assume consistency of MSSEP so that $\bar{\Gamma}$ is closed, convex, and nonempty, *i.e.*, $\bar{\Gamma} = \{(x, y) \in H_1 \times H_2, Ax = By, x \in C, y \in Q\} \neq \emptyset$.

In this paper, we study the SEP and MSSEP for a family of quasi-nonexpansive multi-valued mappings in the framework of infinite-dimensional Hilbert spaces, and propose an algorithm to solve the SEP and MSSEP not requiring to compute the projection on the closed convex sets.

We now collect some definitions and elementary facts which will be used in the proofs of our main results.

Definition 1.2 Let H be a Banach space.

(1) A multi-valued mapping $R : H \to CB(H)$ is said to be *demi-closed at the origin* if, for any sequence $\{x_n\} \subseteq H$ with x_n converges weakly to x and $d(x_n, Rx_n) \to 0$, we have $x \in Rx$.

(2) A multi-valued mapping $R : H \to CB(H)$ is said to be *semi-compact* if, for any bounded sequence $\{x_n\} \subseteq H$ with $d(x_n, Rx_n) \to 0$, there exists a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges strongly to a point $x \in H$.

Lemma 1.3 [11, 12] *Let X be a Banach space, C a closed convex subset of X, and $T : C \to C$ a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 1.4 [13] *Let H be a Hilbert space and $\{w_n\}$ a sequence in H such that there exists a nonempty set $S \subseteq H$ satisfying the following*:

(i) *for every $w \in S$, $\lim_{n\to\infty} \|w_n - w\|$ exists*;

(ii) *any weak-cluster point of the sequence $\{w_n\}$ belongs to S.*

Then there exists $\tilde{w} \in s$ such that $\{w_n\}$ weakly converges to $\tilde{w}$.

Lemma 1.5 [10] *Let $T = I - \gamma G^*G$, where $0 < \gamma < \lambda = 2/\rho(G^*G)$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G on H, $S = C \times Q$. Then we have the following*:

(1) $\|T\| \leq 1$ (*i.e., T is nonexpansive*) *and averaged*;
(2) $\mathrm{Fix}(T) = \{(x,y) \in H, Ax = By\}$, $\mathrm{Fix}(P_S T) = \mathrm{Fix}(P_S) \cap \mathrm{Fix}(T) = \Gamma$.

2 Iterative algorithm for SEP

In this section, we establish an iterative algorithm that converges strongly to a solution of SEP (1.3).

Algorithm 2.1 For an arbitrary initial point $w_0 = (x_0, y_0)$, the sequence $\{w_n = (x_n, y_n)\}$ is generated by the iteration:

$$w_{n+1} = \alpha_n \bigl(I - \gamma G^* G\bigr) w_n + (1 - \alpha_n) v_n, \quad v_n \in R\bigl(w_n - \gamma G^* G w_n\bigr), \tag{2.1}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ and $0 < \gamma < \lambda = 2/\rho(G^*G)$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G on H, $R : H_1 \times H_2 \to H_1 \times H_2$ by

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix},$$

and R_1, R_2 are quasi-nonexpansive multi-valued mappings on H_1, H_2, respectively.

To prove its convergence we need the following lemma.

Lemma 2.2 *Any sequence $\{w_n\}$ generated by Algorithm* (2.1) *is Féjer-monotone with respect to Γ, namely for every $w \in \Gamma$,*

$$\|w_{n+1} - w\| \leq \|w_n - w\|, \quad \forall n \geq 1,$$

provided that $\alpha_n > 0$ is a sequence in $(0,1)$ *and* $0 < \gamma < \lambda = 2/\rho(G^*G)$.

Proof Let $u_n = (I - \gamma G^*G) w_n$ and taking $w \in \Gamma$, by Lemma 1.5, $w \in \mathrm{Fix}(P_S) \cap \mathrm{Fix}(I - \gamma G^*G)$, $Gw = 0$ and we have

$$\begin{aligned} \|w_{n+1} - w\|^2 &= \bigl\|\alpha_n u_n + (1 - \alpha_n) v_n - w\bigr\|^2 \\ &\leq \alpha_n \|u_n - w\|^2 + (1 - \alpha_n)\|v_n - w\|^2 - \alpha_n(1 - \alpha_n)\|u_n - v_n\|^2 \\ &\leq (1 - \alpha_n)\|u_n - w\|^2 + \alpha_n \tilde{H}(Ru_n - Rw)^2 - \alpha_n(1 - \alpha_n)\|u_n - v_n\|^2 \\ &\leq (1 - \alpha_n)\|u_n - w\|^2 + \alpha_n\|u_n - w\|^2 - \alpha_n(1 - \alpha_n)\|u_n - v_n\|^2 \\ &= \|u_n - w\|^2 - \alpha_n(1 - \alpha_n)\|u_n - v_n\|^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \|u_n - w\|^2 &= \bigl\|\bigl(I - \gamma G^*G\bigr) w_n - w\bigr\|^2 \\ &= \|w_n - w\|^2 + \bigl\|\gamma G^* G w_n\bigr\|^2 - 2\bigl\langle w_n - w, \gamma G^* G w_n\bigr\rangle \\ &= \|w_n - w\|^2 + \gamma^2 \bigl\langle G w_n, G G^* G w_n\bigr\rangle - 2\gamma \langle G w_n - Gw, G w_n\rangle \\ &\leq \|w_n - w\|^2 + \gamma^2 \lambda \|G w_n\|^2 - 2\gamma \langle G w_n - 0, G w_n\rangle \\ &= \|w_n - w\|^2 - \gamma(2 - \lambda\gamma)\|G w_n\|^2. \end{aligned}$$

Hence, we have

$$\|w_{n+1} - w\|^2 \le \|w_n - w\|^2 - \alpha_n(1 - \alpha_n)\|u_n - v_n\|^2 - \gamma(2 - \lambda\gamma)\|Gw_n\|^2. \tag{2.2}$$

It follows that $\|w_{n+1} - w\| \le \|w_n - w\|$, $\forall w \in \Gamma$, $n \ge 1$. □

Theorem 2.3 *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ *and* R_1, R_2 *are demi-closed at the origin, then the sequence* $\{w_n\}$ *generated by Algorithm* (2.1) *converges weakly to a solution of SEP* (1.3). *In addition, if* R_1, R_2 *are semi-compact, then* $\{w_n\}$ *converges strongly to a solution of SEP* (1.3).

Proof For any solution of SEP w, according to Lemma 2.2, we see that the sequence $\|w_n - w\|$ is monotonically decreasing and thus converges to some positive real. Since $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$ and $0 < \gamma < \lambda$, by (2.2), we can obtain

$$\|u_n - v_n\| \to 0, \qquad \|Gw_n\| \to 0, \quad \text{when } n \to \infty.$$

Since $v_n \in Ru_n$, we can get $d(u_n, Ru_n) \le \|u_n - v_n\| \to 0$.

From the Féjer-monotonicity of $\{w_n\}$ it follows that the sequence is bounded. Denoting by $\tilde{w}$ a weak-cluster point of $\{w_n\}$ let $\nu = 0, 1, 2, \ldots$ be the sequence of indices, such that w_{n_ν} converges weakly to $\tilde{w}$. Then, by Lemma 1.3, we obtain $G\tilde{w} = 0$, and it follows that $\tilde{w} \in \operatorname{Fix}(I - \gamma G^*G)$.

Since R_1, R_2 are demi-closed at the origin, it is easy to check that R is demi-closed at the origin. Now, by setting $u_n = (I - \gamma G^*G)w_n$, it follows that u_{n_ν} converges weakly to $\tilde{w}$. Since $d(u_n, Ru_n) \to 0$, and R is demi-closed at the origin, we obtain $\tilde{w} \in \operatorname{Fix} R = C \times Q$, *i.e.*, $P_S(\tilde{w}) = \tilde{w}$. That is to say, $\tilde{w} \in \operatorname{Fix}(P_S)$.

Hence $\tilde{w} \in \operatorname{Fix}(P_S) \cap \operatorname{Fix}(I - \gamma G^*G)$. By Lemma 1.5, we find that $\tilde{w}$ is a solution of SEP (1.3).

The weak convergence of the whole sequence $\{w_n\}$ holds true since all conditions of the well-known Opial lemma (Lemma 1.4) are fulfilled with $S = \Gamma$.

Moreover, if R_1, R_2 are semi-compact, it is easy to prove that R is semi-compact, and since $d(u_n, Ru_n) \to 0$, we get the result that there exists a subsequence of $\{u_{n_i}\} \subseteq \{u_n\}$ such that u_{n_i} converges strongly to w^*. Since u_{n_ν} converges weakly to $\tilde{w}$, we have $w^* = \tilde{w}$ and so u_{n_i} converges strongly to $\tilde{w} \in \Gamma$. From the Féjer-monotonicity of $\{w_n\}$ and $\|w_{n+1} - u_n\| = (1 - \alpha_n)\|u_n - v_n\| \to 0$, we can find that $\|w_n - \tilde{w}\| \to 0$, *i.e.*, $\{w_n\}$ converges strongly to a solution of the SEP (1.3). □

3 Iterative algorithm for MSSEP

In this section, we establish an iterative algorithm that converges strongly to a solution of the following MSSEP (1.4) for a family quasi-nonexpansive multi-valued mappings in infinitely dimensional Hilbert spaces.

Let $C_j = \operatorname{Fix} R_1^j$, $Q_j = \operatorname{Fix} R_2^j$ and $S_j = C_j \times Q_j$, $j = 1, 2, \ldots, N$, $S = \bigcap_{j=1}^N S_j$. The original problem can now be reformulated as finding $w = (x, y) \in S$ with $Gw = 0$, or, more generally, minimizing the function $\|Gw\|$ over $w \in S$.

Algorithm 3.1 For an arbitrary initial point $w_0 = (x_0, y_0)$, sequence $\{w_n = (x_n, y_n)\}$ is generated by the iteration:

$$w_{n+1} = \alpha_n\big(I - \gamma G^*G\big)w_n + (1 - \alpha_n)v_n, \quad v_n \in R_{i(n)}\big(w_n - \gamma G^*Gw_n\big), \tag{3.1}$$

where $i(n) = n(\mathrm{mod}\, N) + 1$, $\alpha_n > 0$ is a sequence in $(0,1)$ and $0 < \gamma < \lambda = 2/\rho(G^*G)$, $R_{i(n)} : H_1 \times H_2 \to H_1 \times H_2$ by

$$R_{i(n)} = \begin{bmatrix} R_1^{i(n)} & 0 \\ 0 & R_2^{i(n)} \end{bmatrix},$$

and $R_1^{i(n)}$, $R_2^{i(n)}$ are a family of quasi-nonexpansive multi-valued mappings on H_1, H_2, respectively.

The proof of the following lemma is similar to Lemma 1.5, and we omit its proof.

Lemma 3.2 *Let* $T = I - \gamma G^*G$, *where* $0 < \gamma < \lambda = 2/\rho(G^*G)$. *Then we have* $\mathrm{Fix}(T) = \{(x,y) \in H, Ax = By\}$, $\mathrm{Fix}(P_{\bigcap S_j}T) = \mathrm{Fix}(P_{\bigcap S_j}) \cap \mathrm{Fix}(T) = \bar{\Gamma}$ *and* $\bigcap \mathrm{Fix}(P_{S_j}T) = \bigcap[\mathrm{Fix}(P_{S_j}) \cap \mathrm{Fix}(T)] = \bar{\Gamma}$.

To prove its convergence we also need the following lemma.

Lemma 3.3 *Any sequence* $\{w_n\}$ *generated by Algorithm* (3.1) *is the Féjer-monotone with respect to* $\bar{\Gamma}$, *namely for every* $w \in \bar{\Gamma}$,

$$\|w_{n+1} - w\| \le \|w_n - w\|, \quad \forall n \ge 1,$$

provided that $\alpha_n > 0$ *is a sequence in* $(0,1)$ *and* $0 < \gamma < \lambda = 2/\rho(G^*G)$.

Proof Let $u_n = (I - \gamma G^*G)w_n$ and taking $w \in \bar{\Gamma}$, by Lemma 3.2, $w \in \mathrm{Fix}(P_{S_j}) \cap \mathrm{Fix}(I - \gamma G^*G)$, $\forall N \ge i \ge 1$, $Gw = 0$ and we have

$$\begin{aligned} \|w_{n+1} - w\|^2 &= \big\|\alpha_n u_n + (1-\alpha_n)v_n - w\big\|^2 \\ &\le \alpha_n\|u_n - w\|^2 + (1-\alpha_n)\|v_n - w\|^2 - \alpha_n(1-\alpha_n)\|u_n - v_n\|^2 \\ &\le \alpha_n\|u_n - w\|^2 + (1-\alpha_n)\tilde{H}(R_{i(n)}u_n - R_{i(n)}w)^2 - \alpha_n(1-\alpha_n)\|u_n - v_n\|^2 \\ &\le \alpha_n\|u_n - w\|^2 + (1-\alpha_n)\|u_n - w\|^2 - \alpha_n(1-\alpha_n)\|u_n - v_n\|^2 \\ &= \|u_n - w\|^2 - \alpha_n(1-\alpha_n)\|u_n - v_n\|^2. \end{aligned}$$

On the other hand, in the same way as in the proof of Lemma 2.2, we have

$$\|u_n - w\|^2 \le \|w_n - w\|^2 - \gamma(2 - \lambda\gamma)\|Gw_n\|^2.$$

Hence, we have

$$\|w_{n+1} - w\|^2 \le \|w_n - w\|^2 - \alpha_n(1-\alpha_n)\|u_n - v_n\|^2 - \gamma(2 - \lambda\gamma)\|Gw_n\|^2. \tag{3.2}$$

It follows that $\|w_{n+1} - w\| \le \|w_n - w\|$, $\forall w \in \bar{\Gamma}$, $n \ge 1$. □

Theorem 3.4 *If* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *then the sequence* $\{w_n\}$ *generated by Algorithm* (3.1) *converges weakly to a solution of MSSEP* (1.4). *In addition, if there exists* $1 \le j \le N$ *such that* R_1^j, R_2^j *are semi-compact, then* $\{w_n\}$ *converges strongly to a solution of MSSEP* (1.4).

Proof From (3.2), and the fact that $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and $0 < \gamma < \lambda = 2/\rho(G^*G)$, we obtain

$$\sum_{n=0}^{\infty} \|u_n - v_n\|^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|Gw_n\|^2 < \infty.$$

Therefore,

$$\lim_{n\to\infty} \|u_n - v_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Gw_n\| = 0.$$

Since $v_n \in R_{i(n)}u_n$, we get $d(u_n, R_{i(n)}u_n) \leq \|u_n - v_n\| \to 0$.

It follows from the Féjer-monotonicity of $\{w_n\}$ that the sequence is bounded. Let $\tilde{w}$ be a weak-cluster point of $\{w_n\}$. Take a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that w_{n_k} converges weakly to $\tilde{w}$. Then, by Lemma 1.3, we obtain $G\tilde{w} = 0$, and it follows that $\tilde{w} \in \operatorname{Fix}(I - \gamma G^*G)$.

Now, by setting $u_n = (I - \gamma G^*G)w_n$, it follows that u_{n_k} converges weakly to $\tilde{w}$.

Since

$$\begin{aligned} \|w_{n+1} - w_n\|^2 &= \big\|\alpha_n u_n + (1-\alpha_n)v_n - w_n\big\|^2 \\ &= \big\|(1-\alpha_n)(v_n - u_n) + u_n - w_n\big\|^2 \\ &\leq 2(1-\alpha_n)^2\big\|(v_n - u_n)\big\|^2 + 2\big\|\gamma G^*Gw_n\big\|^2 \\ &= 2(1-\alpha_n)^2\big\|(v_n - u_n)\big\|^2 + 2\gamma^2\big\langle Gw_n, GG^*Gw_n\big\rangle \\ &\leq 2(1-\alpha_n)^2\big\|(v_n - u_n)\big\|^2 + 2\gamma^2\lambda\|Gw_n\|^2, \end{aligned}$$

we have

$$\sum_{n=0}^{\infty} \|w_{n+1} - w_n\|^2 < \infty.$$

On the other hand,

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &= \big\|w_{n+1} - w_n + \gamma G^*G(w_{n+1} - w_n)\big\|^2 \\ &\leq 2\big(\|w_{n+1} - w_n\|^2 + \big\|\gamma G^*G(w_{n+1} - w_n)\big\|^2\big) \\ &\leq 2\big(\|w_{n+1} - w_n\|^2 + \gamma^2\lambda\big\|(w_{n+1} - w_n)\big\|^2\big), \end{aligned}$$

that is,

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 < \infty.$$

Thus, $\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0$ and $\lim_{n\to\infty} \|u_{n+j} - u_n\| = 0$ for all $j = 1, 2, \ldots, N$.

It follows that, for any $j = 1, 2, \ldots, N$,

$$\begin{aligned} d(u_n, R_{i(n+j)}u_n) \leq{}& \|u_n - u_{n+j}\| + d(u_{n+j}, R_{i(n+j)}u_{n+j}) \\ &+ \tilde{H}(R_{i(n+j)}u_{n+j}, R_{i(n+j)}u_n) \end{aligned}$$

$$\begin{aligned} &\leq 2\|u_n - u_{n+j}\| + d(u_{n+j}, R_{i(n+j)}u_{n+j}) \\ &\to 0. \end{aligned}$$

Hence, $\lim_{n\to\infty} d(u_n, R_j u_n) = 0$ for all $j = 1, 2, \ldots, N$. Since R_1^j, R_2^j are demi-closed at the origin, it is easy to check that R_j is demi-closed at the origin, and it follows that $\tilde{w} \in \bigcap_{j=1}^N \operatorname{Fix} R_j = C \times Q$, *i.e.*, $P_S(\tilde{w}) = \tilde{w}$. That is to say $\tilde{w} \in \operatorname{Fix}(P_S)$. Hence $\tilde{w} \in \operatorname{Fix}(P_S) \cap \operatorname{Fix}(I - \gamma G^* G)$. By Lemma 3.2, we get that $\tilde{w}$ is a solution of the MSSEP (1.4).

The weak convergence of the whole sequence $\{w_n\}$ holds true since all conditions of the well-known Opial lemma (Lemma 1.4) are fulfilled with $S = \bar{\Gamma}$.

Moreover, if R_1^j, R_2^j are semi-compact, it is easy to prove that R_j is semi-compact, since $d(u_n, R_j u_n) \to 0$, we find that there exists a subsequence of $\{u_{n_i}\} \subseteq \{u_n\}$ such that u_{n_i} converges strongly to w^*. Since u_{n_ν} converges weakly to $\tilde{w}$, we have $w^* = \tilde{w}$ and so u_{n_i} converges strongly to $\tilde{w} \in \Gamma$. From the Féjer-monotonicity of $\{w_n\}$ and $\|w_{n+1} - u_n\| = (1 - \alpha_n)\|u_n - v_n\| \to 0$, we can see that $\|w_n - \tilde{w}\| \to 0$, *i.e.*, $\{w_n\}$ converges strongly to a solution of the MSSEP (1.4). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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