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Three solutions for equations involving nonhomogeneous operators of p-Laplace type in $\mathbb{R}^N$

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Abstract

In this paper, we are concerned with the following elliptic equation

$$-\operatorname{div}(\varphi(x,\nabla u)) = \lambda f(x,u) \quad \text{in } \mathbb{R}^N,$$

where the function $\varphi(x,v)$ is of type $|v|^{p-2}v$ and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies a Carathéodory condition. We establish the existence of at least three weak solutions for the problem above which is based on an abstract three critical points theory due to Ricceri. Moreover, we determine precisely the intervals of λ's for which the given problem possesses either only the trivial solution or at least two nontrivial solutions.

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1 Introduction

In this paper, we establish the existence of at least three solutions for equations of the p-Laplace type

$$(\mathrm{P}_\lambda) \quad -\operatorname{div}\big(\varphi(x,\nabla u)\big) = \lambda f(x,u) \quad \text{in } \mathbb{R}^N,$$

where the function $\varphi(x,v)$ is of type $|v|^{p-2}v$ and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies a Carathéodory condition. A Ricceri-type three critical points theorem has been extensively studied by many researchers (see [1–5] and the references therein), but the results on the localization of the interval for the existence of three solutions are rare. The authors in [3, 4] investigated the existence of multiple solutions for quasilinear nonhomogeneous problems with Dirichlet boundary conditions by applying an abstract three critical points theorem which is the extension of the famous result of Ricceri [6, 7].

Ricceri's theorems in [6–8] gave no further information on the size and location of an interval of values $\lambda\in\mathbb{R}$ for the existence of at least three critical points. However, further information concerning these points was given in [9]. Also the authors in [3] investigated the localization of the interval for the existence of three solutions for the Dirichlet problem involving the p-Laplace type operators which was motivated by the work of Arcoya and Carmona [2]. It is well known that the first eigenvalue of the p-Laplacian plays a decisive role in obtaining these results in [3, 9]. Hence, by using the positivity of the principal eigenvalue of the p-Laplacian in $\mathbb{R}^N$, which was given in [10–12], we localize a three critical points interval for the problem above as in [3, 9]. Especially, the main aim of this paper is to determine precisely the intervals of λ's for which problem (P_λ) admits only the trivial solution and for which problem (P_λ) has at least two nontrivial solutions, following the basic idea in [3]. To do this, we consider some of the basic properties for the integral operator corresponding to problem (P_λ) in the setting of weighted Sobolev spaces.

To this end, we recall in what follows some definitions of the basic function space which will be treated in the next sections. For a deeper treatment on these spaces, we refer to [12, 13].

Let $\langle\cdot,\cdot\rangle$ be the Euclidean scalar product on $\mathbb{R}^N$ or the usual pairing of X^* and X, where X^* denotes the dual space of X. Let $1 < p < N$ and set $p^* := Np/(N-p)$. Let ω be a weight function defined by

$$\omega(x) = \frac{1}{(1+|x|)^p} \quad \text{for } x \in \mathbb{R}^N.$$

Assume that

(A) a belongs to $L^\infty(\mathbb{R}^N)$ and there is a positive constant a_0 such that

$$a(x) \geq a_0 \quad \text{for almost all } x \in \mathbb{R}^N.$$

Let X be the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_X = \left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx + \int_{\mathbb{R}^N} \omega(x)|u|^p\,dx\right)^{\frac{1}{p}}.$$

From Hardy's inequality and assumption (A), it follows that

$$\int_{\mathbb{R}^N} \omega(x)|u|^p\,dx \leq \frac{1}{a_0}\left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx,$$

which implies that on X, the norm $\|\cdot\|_X$ is equivalent to the other norm $\|\cdot\|_a$ given by

$$\|u\|_a = \left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Note that there exist positive constants c_* and c^* such that

$$c_*\|u\|_X \leq \|u\|_a \leq c^*\|u\|_X \tag{1.1}$$

for all $u \in X$. The following *Sobolev inequality* will be used in the sequel:

$$\left(\int_{\mathbb{R}^N} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leq c_0\left(\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx\right)^{\frac{1}{p}}$$

for some positive constant c_0 (see [12]).

This paper is organized as follows. We first present some properties of the corresponding integral operators. Then we give and prove our main results in Theorem 2.12 and Theorem 2.14.

2 Main results

Definition 2.1 We say that $u \in X$ is a weak solution of problem (P_λ) if

$$\int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx = \lambda \int_{\mathbb{R}^N} f(x, u) v \, dx$$

for all $v \in X$.

We assume that $\varphi(x, v) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous derivative with respect to v of the mapping $\Phi_0 : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $\Phi_0 = \Phi_0(x, v)$, that is, $\varphi(x, v) = \frac{d}{dv}\Phi_0(x, v)$. Suppose that φ and Φ_0 satisfy the following assumptions:

(J1) The following equalities

$$\Phi_0(x, 0) = 0 \quad \text{and} \quad \Phi_0(x, v) = \Phi_0(x, -v)$$

hold for all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J2) $\varphi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following conditions: $\varphi(\cdot, v)$ is measurable for all $v \in \mathbb{R}^N$ and $\varphi(x, \cdot)$ is continuous for almost all $x \in \mathbb{R}^N$.

(J3) There are a function $\sigma_0 \in L^{p'}(\mathbb{R}^N)$ and a positive constant d such that

$$\left|\varphi(x, v)\right| \le \sigma_0(x) + d|v|^{p-1}$$

for almost all $x \in \mathbb{R}^N$ and for all $v \in \mathbb{R}^N$.

(J4) $\Phi_0(x, \cdot)$ is strictly convex in $\mathbb{R}^N$ for all $x \in \mathbb{R}^N$.

(J5) The following relations

$$c_1 a(x)|v|^p \le \varphi(x, v) \cdot v \quad \text{and} \quad c_1 a(x)|v|^p \le p\Phi_0(x, v)$$

hold for all $x \in \mathbb{R}^N$ and $v \in \mathbb{R}^N$, where c_1 is a positive constant.

Let us define the functional $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_{\mathbb{R}^N} \Phi_0(x, \nabla u) \, dx$$

for any $u \in X$. Under assumptions (J1)-(J3) and (J5), it follows from [14, Lemma 3.2] that the functional Φ is well defined on X, $\Phi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\left\langle \Phi'(u), v \right\rangle = \int_{\mathbb{R}^N} \varphi(x, \nabla u) \cdot \nabla v \, dx \tag{2.1}$$

for any $u \in X$.

Next, taking inspiration from the argument given in [3], we will show that the operator Φ' is a mapping of type (S_+) which plays an important role in obtaining our main results.

Lemma 2.2 *Assume that* (A) *and* (J1)-(J5) *hold. Then the functional* $\Phi : X \to \mathbb{R}$ *is convex and weakly lower semicontinuous on* X. *Moreover, the operator* Φ' *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *in* X *as* $n \to \infty$ *and* $\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \le 0$, *then* $u_n \to u$ *in* X *as* $n \to \infty$.

Proof From assumption (J4), the operator Φ is strictly convex and thus Φ' is strictly monotone (see [15, Proposition 25.10]), namely

$$\langle \Phi'(u) - \Phi'(v), u - v \rangle > 0 \tag{2.2}$$

for $u \neq v$. The convexity of Φ_0 also implies that Φ is weakly lower semicontinuous in X, that is, $u_n \rightharpoonup u$ implies

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n). \tag{2.3}$$

Now we claim that the operator Φ' is a mapping of type (S_+). Let $\{u_n\}$ be a sequence in X such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and

$$\limsup_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leq 0. \tag{2.4}$$

From relations (2.2) and (2.4), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \bigl(\varphi(x, \nabla u_n) - \varphi(x, \nabla u)\bigr) \cdot (\nabla u_n - \nabla u)\, dx = \lim_{n\to\infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle = 0,$$

that is, the sequence $\{(\varphi(x, \nabla u_n) - \varphi(x, \nabla u)) \cdot (\nabla u_n - \nabla u)\}$ converges to 0 in $L^1(\mathbb{R}^N)$ as $n \to \infty$. Hence the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ such that

$$\lim_{k\to\infty} \bigl(\varphi(x, \nabla u_{n_k}(x)) - \varphi(x, \nabla u(x))\bigr) \cdot \bigl(\nabla u_{n_k(x)} - \nabla u(x)\bigr) = 0 \tag{2.5}$$

for almost all $x \in \mathbb{R}^N$. Thus there exists $M > 0$ such that

$$\begin{aligned} \varphi(x, \nabla u_{n_k}(x)) \cdot \nabla u_{n_k}(x) \leq{}& M + \bigl|\varphi(x, \nabla u_{n_k}(x))\bigr|\bigl|\nabla u(x)\bigr| \\ &+ \bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u_{n_k}(x)\bigr| + \bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u(x)\bigr| \end{aligned}$$

for almost all $x \in \mathbb{R}^N$. It follows from conditions (A), (J3) and (J5) that

$$\begin{aligned} c_1 a_0 \bigl|\nabla u_{n_k}(x)\bigr|^p &\leq c_1 a(x) \bigl|\nabla u_{n_k}(x)\bigr|^p \leq \varphi(x, \nabla u_{n_k}(x)) \cdot \nabla u_{n_k}(x) \\ &\leq M + \bigl|\varphi(x, \nabla u_{n_k}(x))\bigr|\bigl|\nabla u(x)\bigr| \\ &\quad + \bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u_{n_k}(x)\bigr| + \bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u(x)\bigr| \\ &\leq M + \bigl(\sigma_0(x) + d\bigl|\nabla u_{n_k}(x)\bigr|^{p-1}\bigr)\bigl|\nabla u(x)\bigr| \\ &\quad + \bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u_{n_k}(x)\bigr| + \bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u(x)\bigr| \end{aligned} \tag{2.6}$$

for almost all $x \in \mathbb{R}^N$. By using Young's inequality, we deduce that

$$d\bigl|\nabla u_{n_k}(x)\bigr|^{p-1}\bigl|\nabla u(x)\bigr| \leq \frac{c_1 a_0}{3}\bigl|\nabla u_{n_k}(x)\bigr|^p + \left(\frac{3d^{p'}}{c_1 a_0}\right)^{p-1}\bigl|\nabla u(x)\bigr|^p,$$

and

$$\bigl|\varphi(x, \nabla u(x))\bigr|\bigl|\nabla u_{n_k}(x)\bigr| \leq \left(\frac{3}{c_1 a_0}\right)^{\frac{1}{p-1}}\bigl|\varphi(x, \nabla u(x))\bigr|^{p'} + \frac{c_1 a_0}{3}\bigl|\nabla u_{n_k}(x)\bigr|^p$$

for almost all $x \in \mathbb{R}^N$. These together with relation (2.6) imply that

$$\frac{c_1 a_0}{3}\left|\nabla u_{n_k}(x)\right|^p \le M + \sigma_0(x)\left|\nabla u(x)\right| + \left(\frac{3d^{p'}}{c_1 a_0}\right)^{p-1}\left|\nabla u(x)\right|^p + \left(\frac{3}{c_1 a_0}\right)^{\frac{1}{p-1}}\left|\varphi\left(x, \nabla u(x)\right)\right|^{p'} + \left|\varphi\left(x, \nabla u(x)\right)\right|\left|\nabla u(x)\right|$$

for almost all $x \in \mathbb{R}^N$. Since c_1 and a_0 are positive constants, the above inequality implies that the sequence $\{|\nabla u_{n_k}(x)|\}$ is bounded, and so $\{\nabla u_{n_k}(x)\}$ is bounded in $\mathbb{R}^N$ for almost all $x \in \mathbb{R}^N$. By passing to a subsequence, we can suppose that $\nabla u_{n_k}(x) \to \xi$ as $k \to \infty$ for some $\xi \in \mathbb{R}^N$ and for almost all $x \in \mathbb{R}^N$. Then we have $\varphi(x, \nabla u_{n_k}(x)) \to \varphi(x, \xi)$ as $k \to \infty$ for almost all $x \in \mathbb{R}^N$. It follows from (2.5) that

$$\begin{aligned} 0 &= \lim_{k\to\infty}\left(\varphi\left(x, \nabla u_{n_k}(x)\right) - \varphi\left(x, \nabla u(x)\right)\right) \cdot \left(\nabla u_{n_k}(x) - \nabla u(x)\right) \\ &= \left(\varphi(x, \xi) - \varphi\left(x, \nabla u(x)\right)\right) \cdot \left(\xi - \nabla u(x)\right) \end{aligned}$$

for almost all $x \in \mathbb{R}^N$. Since φ is strictly monotone by (J4), this means $\xi = \nabla u(x)$, that is, $\nabla u_{n_k}(x) \to \nabla u(x)$ as $k \to \infty$ for almost all $x \in \mathbb{R}^N$. The arguments above hold for any subsequence $\{u_{n_k}\}$ of the sequence $\{u_n\}$. Hence we obtain $\nabla u_n(x) \to \nabla u(x)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$. Then it implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\varphi(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)\,dx = 0. \tag{2.7}$$

Since the functional Φ is convex, it is obvious that

$$\Phi(u) + \int_{\mathbb{R}^N}\varphi(x, \nabla u_n) \cdot (\nabla u_n - \nabla u)\,dx \ge \Phi(u_n),$$

and so we get $\Phi(u) \ge \limsup_{n\to\infty}\Phi(u_n)$. Therefore, it is derived from (2.3) that

$$\Phi(u) = \lim_{n\to\infty}\Phi(u_n). \tag{2.8}$$

Consider the sequence $\{g_n\}$ in $L^1(\mathbb{R}^N)$ defined pointwise by

$$g_n(x) = \frac{1}{2}\left(\Phi_0(x, \nabla u_n) + \Phi_0(x, \nabla u)\right) - \Phi_0\left(x, \frac{1}{2}(\nabla u_n - \nabla u)\right).$$

Then $g_n \ge 0$ for all $n \in \mathbb{N}$ by (J1) and (J4). Since $\Phi_0(x, \cdot)$ is continuous for almost all $x \in \mathbb{R}^N$, we obtain that $g_n \to \Phi_0(x, \nabla u)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^N$. Therefore, by the Fatou lemma and relation (2.8), we have

$$\Phi(u) \le \liminf_{n\to\infty}\int_{\mathbb{R}^N} g_n(x)\,dx = \Phi(u) - \limsup_{n\to\infty}\int_{\mathbb{R}^N}\Phi_0\left(x, \frac{1}{2}(\nabla u_n - \nabla u)\right)dx.$$

Hence

$$\limsup_{n\to\infty}\int_{\mathbb{R}^N}\Phi_0\left(x, \frac{1}{2}(\nabla u_n - \nabla u)\right)dx \le 0,$$

that is,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\Phi_0\left(x,\frac{1}{2}(\nabla u_n-\nabla u)\right)dx=0,$$

in other words, $\lim_{n\to\infty}\|u_n-u\|_a=0$ by (J5). Since $\|u_n-u\|_X\le\frac{1}{c_*}\|u_n-u\|_a$ by (1.1), in conclusion, $\lim_{n\to\infty}\|u_n-u\|_X=0$, as claimed. □

Corollary 2.3 *Assume that* (A) *and* (J1)-(J5) *hold. Then the operator* $\Phi':X\to X^*$ *is bounded homeomorphism onto* X^*.

Proof It is immediate that the operator Φ' is strictly monotone, coercive, and hemicontinuous. Hence the Browder-Minty theorem implies that the inverse operator $(\Phi')^{-1}:X^*\to X$ exists and is bounded; see Theorem 26.A in [15]. Since the operator Φ' is a mapping of type (S_+) by Lemma 2.2, it is easy to prove that the inverse operator $(\Phi')^{-1}$ is continuous and is omitted here. □

Before dealing with our main results in this section, we need the following assumptions for f. Let us put $F(x,t)=\int_0^t f(x,s)\,ds$.

(F1) $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\cdot,t)$ is measurable for all $t\in\mathbb{R}$ and $f(x,\cdot)$ is continuous for almost all $x\in\mathbb{R}^N$.

(F2) f satisfies the following growth condition: for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$,

$$\left|f(x,t)\right|\le\sigma(x)+\rho(x)|t|^{\gamma-1},$$

where $\sigma\in L^{(p^*)'}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\gamma\in\mathbb{R}$ such that $\gamma<p$, $\rho\in L^s(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ with $(1/s)+(\gamma/p^*)=1$.

(F3) There exist a real number s_0 and a positive constant r_0 so small that

$$\int_{B_N(x_0,r_0)}F(x,s_0)\,dx>0,$$

and $F(x,t)\ge0$ for almost all $x\in B_N(x_0,r_0)\setminus B_N(x_0,\sigma r_0)$ with $\sigma\in(0,1)$ and for all $0\le t\le|s_0|$, where $B_N(x_0,r_0)=\{x\in\mathbb{R}^N:|x-x_0|\le r_0\}\subset\mathbb{R}^N$.

Then we define the functionals $\Psi,I_\lambda:X\to\mathbb{R}$ by

$$\Psi(u)=-\int_{\mathbb{R}^N}F(x,u)\,dx\quad\text{and}\quad I_\lambda(u)=\Phi(u)+\lambda\Psi(u)$$

for any $u\in X$. It is easy to check that $\Psi\in C^1(X,\mathbb{R})$ and its Fréchet derivative is

$$\left\langle\Psi'(u),v\right\rangle=-\int_{\mathbb{R}^N}f(x,u)v\,dx \tag{2.9}$$

for any $u,v\in X$.

Lemma 2.4 *Assume that* (A), *and* (F1)-(F2) *hold. Then* Ψ *and* Ψ' *are weakly-strongly continuous on* X.

Proof The analogous arguments as in Lemma 4.4 of [12] imply that functionals Ψ and Ψ' are weakly-strongly continuous on X. □

Lemma 2.5 *Assume that* (A), (J1)-(J3), (J5), *and* (F1)-(F2) *hold. Then we have*

$$\lim_{\|u\|_X \to \infty} \big(\Phi(u) + \lambda\Psi(u)\big) = +\infty$$

for all $\lambda \in \mathbb{R}$.

Proof If $\|u\|_X$ is large enough and $\lambda \in \mathbb{R}$, then it follows from (J5), (F2) and Hölder's inequality that

$$\begin{aligned}
\Phi(u) + \lambda\Psi(u) &= \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\,dx - \lambda \int_{\mathbb{R}^N} F(x,u)\,dx \\
&\geq \frac{c_1}{p} \int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx - |\lambda| \int_{\mathbb{R}^N} \big|\sigma(x)\big||u|\,dx - |\lambda| \int_{\mathbb{R}^N} \frac{1}{\gamma}\big|\rho(x)\big||u|^\gamma\,dx \\
&\geq \frac{c_1}{p}\|u\|_a^p - |\lambda|\|\sigma\|_{L^{(p^*)'}(\mathbb{R}^N)}\|u\|_{L^{p^*}(\mathbb{R}^N)} - \frac{|\lambda|}{\gamma}\|\rho\|_{L^s(\mathbb{R}^N)}\|u\|_{L^{p^*}(\mathbb{R}^N)}^\gamma \\
&\geq \frac{c_1 c_*^p}{p}\|u\|_X^p - |\lambda| C_1\|u\|_X - \frac{|\lambda| C_2}{\gamma}\|u\|_X^\gamma
\end{aligned}$$

for some positive constants C_1 and C_2. Since $p > \gamma$, we get that

$$\lim_{\|u\|_X \to \infty} \big(\Phi(u) + \lambda\Psi(u)\big) = +\infty$$

for all $\lambda \in \mathbb{R}$. □

Now we will localize the interval for which problem (P_λ) has at least three solutions as the application of three critical points theorems given in [9] and [2], respectively. To do this, we consider the following eigenvalue problem:

(E) $-\operatorname{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) = \lambda m(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N.$

Proposition 2.6 ([11, 12]) *Assume that* (A) *and* (J1)-(J5) *hold. Moreover, suppose that*

(M) $m(x) > 0$ *for all* $x \in \mathbb{R}^N$ *such that* $m \in L^\infty(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$, *and* $m \in L^{\kappa_1}(\mathbb{R}^N)$, *where*

$$\kappa_1 = \frac{p^*}{p^* - \kappa} \quad \textit{with } p < \kappa < p^*.$$

Denote the quantity

$$\lambda_1 = \inf_{u \in X\setminus\{0\}} \left(\frac{\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx}{\int_{\mathbb{R}^N} m(x)|u|^p\,dx} \right).$$

Then the eigenvalue problem (E) *has a pair* (λ_1, u_1) *of a principal eigenvalue* λ_1 *and an eigenfunction* u_1 *with* $\lambda_1 > 0$ *and* $0 < u_1 \in X \cap L^\infty(\mathbb{R}^N)$. *Moreover,* λ_1 *is simple and* $u_1(x)$ *decays uniformly as* $|x| \to \infty$.

Definition 2.7 Let X be a real Banach space. We call that W_X is the class of all functionals $\Phi : X \to \mathbb{R}$ satisfying the following property: if $\{u_n\}$ is a sequence such that $u_n \rightharpoonup u$ in X as $n \to \infty$ and $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u_n)$, then $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ and $u_{n_k} \to u$ in X as $k \to \infty$.

The following lemma is three critical points theory which was introduced by Ricceri [9].

Lemma 2.8 ([9]) *Let X be a separable and reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to W_X, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*. Let $\Psi : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = \Psi(u_0) = 0$. Finally, set*

$$\alpha = \max\left\{0, \limsup_{\|u\|_X\to\infty}\left(-\frac{\Psi(u)}{\Phi(u)}\right), \limsup_{u\to u_0}\left(-\frac{\Psi(u)}{\Phi(u)}\right)\right\}, \qquad \beta = \sup_{u\in\Phi^{-1}((0,+\infty))}\left(-\frac{\Psi(u)}{\Phi(u)}\right).$$

Assume that $\alpha < \beta$. Then, for each compact interval $[a,b] \subset (\frac{1}{\beta}, \frac{1}{\alpha})$ (with the conventions $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [a,b]$, the equation $\Phi'(u) + \lambda\Psi'(u) = 0$ has at least three solutions whose norms are less than R.

In order to apply the above lemma to (P_λ), we have to show that the functional Φ belongs to W_X. To do this, we need the following additional assumption:

(J6) The following relation holds for all $u, v \in \mathbb{R}^N$:

$$\frac{1}{2}\big(\Phi_0(x,u) + \Phi_0(x,v)\big) \ge \Phi_0\left(x, \frac{u+v}{2}\right) + \Phi_0\left(x, \frac{u-v}{2}\right).$$

To consider some examples that satisfy hypothesis (J6), we observe the following argument which is given in [16].

Remark 2.9 If $\phi(t)$ is a continuous, strictly increasing function for $t \ge 0$ with $\phi(0) = 0$ and

$$t \mapsto \phi(\sqrt{t}) \quad \text{is convex for all } t \in [0,\infty), \tag{2.10}$$

then the following estimate

$$\frac{1}{2}\big(\phi(|u|) + \phi(|v|)\big) \ge \phi\left(\left|\frac{u+v}{2}\right|\right) + \phi\left(\left|\frac{u-v}{2}\right|\right)$$

holds for all $u, v \in \mathbb{R}^N$.

Example 2.10 Let us consider

$$\varphi(x,v) = |v|^{p-2}v \quad \text{and} \quad \Phi_0(x,v) = \frac{1}{p}|v|^p$$

for all $v \in \mathbb{R}^N$. If $p \geq 2$, then we obtain a Clarkson-type inequality for the function Φ_0, *i.e.*,

$$\frac{1}{2}\left(|u|^p + |v|^p\right) \geq \left|\frac{u+v}{2}\right|^p + \left|\frac{u-v}{2}\right|^p$$

for all $u, v \in \mathbb{R}^N$. Therefore assumption (J6) holds.

Example 2.11 Let $p \geq 2$. Suppose that $w \in L^{2p'}(\mathbb{R}^N)$ and there exists a positive constant w_0 such that $w(x) \geq w_0$ for almost all $x \in \mathbb{R}^N$. Let us consider

$$\varphi(x,v) = \left(w(x) + |v|^2\right)^{\frac{p}{2}-1} v \quad \text{and} \quad \Phi_0(x,v) = \frac{1}{p}\left[\left(w(x) + |v|^2\right)^{\frac{p}{2}} - 1\right]$$

for all $v \in \mathbb{R}^N$. Set $\phi(t) = (1/p)[(w(x) + t^2)^{p/2} - 1]$ for $t \geq 0$. Then it is easy to calculate that ϕ satisfies all the assumptions of Remark 2.9 and therefore condition (J6) is verified.

Combining with Proposition 2.6 and Lemma 2.8, we derive the following consequence.

Theorem 2.12 *Assume that conditions* (A), (J1)-(J6), (F1)-(F3) *and* (M) *hold. Moreover, suppose that*

(F4) $\limsup_{|s|\to\infty} \frac{F(x,s)}{m(x)|s|^p} \leq 0$ *for* $x \in \mathbb{R}^N$ *uniformly.*

(F5) $\limsup_{s\to 0} \frac{F(x,s)}{m(x)|s|^p} \leq 0$ *for* $x \in \mathbb{R}^N$ *uniformly.*

(F6) *For all compact* $K \subset \mathbb{R}$, *there exists a function* $\psi_K \in L^1(\mathbb{R}^N)$ *such that*

$$F(x,s) \leq \psi_K(x)$$

for almost all $x \in \mathbb{R}^N$ *and for all* $s \in K$.

Assume also that the condition $\gamma < p$ *is removed and replaced by the more general condition* $\gamma < p^*$ *in assumption* (F2). *Set* $\xi = \sup_{u\in X\setminus\{0\}}(-\frac{\Psi(u)}{\Phi(u)})$. *Then, for each compact interval* $[a,b] \subset (\frac{1}{\xi}, +\infty)$, *there exists* $R > 0$ *with the following property: for every* $\lambda \in [a,b]$, *problem* (P_λ) *has at least three solutions whose norms are less than* R.

Proof It is obvious that the functional Φ is coercive, sequentially weakly lower semicontinuous of class C^1, bounded on each subset of X, and whose derivative is a homeomorphism by Corollary 2.3. Moreover, the functional $\Psi \in C^1(X,\mathbb{R})$ has a compact derivative due to Lemma 2.4.

First of all, let us claim that the functional Φ belongs to W_X. It follows from the same argument as in the proof of Theorem 3.1 in [17]. For the sake of convenience, we give the proof. Let $\{u_n\}$ be a sequence in X that converges weakly to u in X as $n \to \infty$ and $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$. By Lemma 2.2, Φ is sequentially weakly lower semicontinuous, namely $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n)$. Thus there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\lim_{n\to\infty} \Phi(u_n) = \Phi(u)$. Since $u_n \rightharpoonup u$ as $n \to \infty$, the sequence $\{(u_n + u)/2\}$ also converges weakly to u in X as $n \to \infty$, and we get

$$\Phi(u) \leq \liminf_{n\to\infty} \Phi\left(\frac{u_n + u}{2}\right). \tag{2.11}$$

If $\{u_n\}$ does not converge to u as n approaches infinity, the sequence $\{(u_n - u)/2\}$ also does not converge to 0 as $n \to \infty$. So we can choose $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$

such that $\|(u_{n_k}-u)/2\|_X \geq \varepsilon_0$ for all $k \in \mathbb{N}$. By assumption (J5) and (1.1), we deduce that

$$\begin{aligned}\Phi\left(\frac{u_{n_k}-u}{2}\right) &= \int_{\mathbb{R}^N} \Phi_0\left(x, \frac{\nabla u_{n_k}-\nabla u}{2}\right) dx \geq \frac{c_1}{p}\int_{\mathbb{R}^N} a(x)\left|\frac{\nabla u_{n_k}-\nabla u}{2}\right|^p dx \\ &= \frac{c_1}{p}\left\|\frac{u_{n_k}-u}{2}\right\|_a^p \geq \frac{c_1 c_*^p}{p}\left\|\frac{u_{n_k}-u}{2}\right\|_X^p \geq \frac{c_1 c_*^p}{p}\varepsilon_0^p\end{aligned}$$

for all $k \in \mathbb{N}$. From (J6), we know

$$\begin{aligned}&\frac{1}{2}\left(\int_{\mathbb{R}^N} \Phi_0(x, \nabla u_{n_k})\, dx + \int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx\right) \\ &\quad \geq \int_{\mathbb{R}^N} \Phi_0\left(x, \frac{\nabla u_{n_k}+\nabla u}{2}\right) dx + \int_{\mathbb{R}^N} \Phi_0\left(x, \frac{\nabla u_{n_k}-\nabla u}{2}\right) dx.\end{aligned}$$

Thus we deduce that the following relation

$$\frac{1}{2}\big(\Phi(u_{n_k})+\Phi(u)\big) \geq \Phi\left(\frac{u_{n_k}+u}{2}\right) + \Phi\left(\frac{u_{n_k}-u}{2}\right) \geq \Phi\left(\frac{u_{n_k}+u}{2}\right) + \frac{c_1 c_*^p}{p}\varepsilon_0^p \tag{2.12}$$

holds for all $k \in \mathbb{N}$. From (2.11) and (2.12), we have $\Phi(u) \geq \Phi(u) + (c_1 c_*^p/p)\varepsilon_0^p$ as $k \to \infty$, a contradiction. Therefore, we conclude that $u_n \to u$ as $n \to \infty$ and so $\Phi \in W_X$.

Observe now that $\Phi(u) > 0$ for every $u \in X \setminus \{0\}$. Then 0 is a strict local (even global) minimum with $\Phi(0) = \Psi(0) = 0$. By assumptions (F4) and (F6), for every $\varepsilon > 0$, we get

$$F(x,s) \leq \varepsilon m(x)|s|^p + \psi_\varepsilon(x)$$

for almost all $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$, where $\psi_\varepsilon \in L^1(\mathbb{R}^N)$. It implies that

$$\int_{\mathbb{R}^N} F(x,u)\, dx \leq \varepsilon \int_{\mathbb{R}^N} m(x)|u|^p\, dx + \int_{\mathbb{R}^N} \psi_\varepsilon(x)\, dx. \tag{2.13}$$

Notice that

$$\lambda_1 = \inf_{u \in X\setminus\{0\}}\left(\frac{\int_{\mathbb{R}^N} a(x)|\nabla u|^p\, dx}{\int_{\mathbb{R}^N} m(x)|u|^p\, dx}\right) > 0 \tag{2.14}$$

by Proposition 2.6. Then it follows from (2.13), (2.14) and (J5) that

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u)\, dx &\leq \frac{\varepsilon}{\lambda_1}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\, dx + \int_{\mathbb{R}^N} \psi_\varepsilon(x)\, dx \\ &\leq \frac{\varepsilon p}{\lambda_1 c_1}\int_{\mathbb{R}^N} \Phi_0(x, \nabla u)\, dx + \int_{\mathbb{R}^N} \psi_\varepsilon(x)\, dx \\ &\leq \frac{\varepsilon p}{\lambda_1 c_1}\Phi(u) + \int_{\mathbb{R}^N} \psi_\varepsilon(x)\, dx.\end{aligned}$$

Hence we have

$$\limsup_{\|u\|_X \to \infty} \frac{\int_{\mathbb{R}^N} F(x,u)\, dx}{\Phi(u)} \leq \varepsilon\left(\frac{p}{\lambda_1 c_1}\right).$$

Since ε is arbitrary, the following inequality holds:

$$\limsup_{\|u\|_X\to\infty}\left(-\frac{\Psi(u)}{\Phi(u)}\right)=\limsup_{\|u\|_X\to\infty}\frac{\int_{\mathbb{R}^N}F(x,u)\,dx}{\Phi(u)}\le 0. \tag{2.15}$$

On the other hand, by conditions (F4) and (F5), we have that for every $\varepsilon>0$, there exists $C_\varepsilon>0$ verifying that

$$F(x,s)\le \varepsilon m(x)|s|^p+C_\varepsilon m(x)|s|^\kappa \tag{2.16}$$

for almost all $x\in\mathbb{R}^N$ and for all $s\in\mathbb{R}$. From (2.14), (2.16) and (J5), we deduce

$$\begin{aligned}\int_{\mathbb{R}^N}F(x,u)\,dx&\le \varepsilon\int_{\mathbb{R}^N}m(x)|u|^p\,dx+C_\varepsilon\int_{\mathbb{R}^N}m(x)|u|^\kappa\,dx\\&\le\frac{\varepsilon}{\lambda_1}\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,dx+C_\varepsilon\|m\|_{L^{\kappa_1}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|u|^{p^*}\,dx\right)^{\frac{\kappa}{p^*}}\\&\le\frac{\varepsilon p}{\lambda_1 c_1}\int_{\mathbb{R}^N}\Phi_0(x,\nabla u)\,dx+C_\varepsilon\|m\|_{L^{\kappa_1}(\mathbb{R}^N)}\|u\|^\kappa_{L^{p^*}(\mathbb{R}^N)}\\&\le\frac{\varepsilon p}{\lambda_1 c_1}\Phi(u)+C_\varepsilon C_3\|u\|_X^\kappa\end{aligned}$$

for some positive constant C_3. Then it follows that

$$\frac{\int_{\mathbb{R}^N}F(x,u)\,dx}{\Phi(u)}\le\varepsilon\left(\frac{p}{\lambda_1c_1}\right)+C_\varepsilon C_3\frac{\|u\|_X^\kappa}{\Phi(u)}.$$

Hence we obtain

$$\limsup_{\|u\|_X\to 0}\left(-\frac{\Psi(u)}{\Phi(u)}\right)=\limsup_{\|u\|_X\to 0}\frac{\int_{\mathbb{R}^N}F(x,u)\,dx}{\Phi(u)}\le\varepsilon\left(\frac{p}{\lambda_1c_1}\right)$$

for all $\varepsilon>0$, which leads to

$$\limsup_{\|u\|_X\to 0}\left(-\frac{\Psi(u)}{\Phi(u)}\right)\le 0. \tag{2.17}$$

Taking now assumption (F3) into account, it follows from (2.15) and (2.17) that

$$\max\left\{0,\limsup_{\|u\|_X\to\infty}\left(-\frac{\Psi(u)}{\Phi(u)}\right),\limsup_{u\to 0}\left(-\frac{\Psi(u)}{\Phi(u)}\right)\right\}=0<\sup_{u\in\Phi^{-1}((0,+\infty))}\left(-\frac{\Psi(u)}{\Phi(u)}\right).$$

Therefore, all the conditions of Lemma 2.8 are fulfilled and thus the proof is completed. □

In the rest of this section, we determine precisely the intervals of λ's for which problem (P_λ) possesses either only the trivial solution or at least two nontrivial solutions. To do this, we assume that

(F7) $\limsup_{s\to 0}\frac{|f(x,s)|}{m(x)|s|^{\kappa-1}}<+\infty$ uniformly for almost all $x\in\mathbb{R}^N$.

Then we get that $\limsup_{s\to 0}\frac{|F(x,s)|}{m(x)|s|^{\kappa}} < +\infty$ uniformly for almost all $x \in \mathbb{R}^N$ by the L'Hôspital's rule. Let us consider that two functions

$$\chi_1(r) = \inf_{u\in\Psi^{-1}((-\infty,r))} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r}, \tag{2.18}$$

$$\chi_2(r) = \sup_{u\in\Psi^{-1}((r,+\infty))} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \tag{2.19}$$

for every $r \in (\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u))$. Also we consider the following crucial value:

$$C_f = \operatorname{ess}\sup_{s\neq 0, x\in\mathbb{R}^N} \frac{|f(x,s)|}{m(x)|s|^{p-1}}.$$

Then the same arguments in [3] imply that C_f is a positive constant. From this fact, we obtain

$$\operatorname{ess}\sup_{s\neq 0, x\in\mathbb{R}^N} \frac{|F(x,s)|}{m(x)|s|^{p}} = \frac{C_f}{p}. \tag{2.20}$$

The next lemma represents the differentiable form of the Arcoya and Carmona Theorem 3.4 in [2].

Lemma 2.13 *Let Φ and Ψ be two functionals on X such that Φ and Ψ are weakly lower semicontinuous and continuously Gâteaux differentiable in X, and Ψ is nonconstant. Let also $\Phi' : X \to X^*$ have the (S_+) property, and that Ψ' is a compact operator. Assume that there exists an interval $I \subset \mathbb{R}$ such that the one parameter family of functionals $I_\lambda = \Phi + \lambda\Psi$ is coercive in X for all $\lambda \in I$. Let us assume that there exists*

$$r \in \left(\inf_{u\in X} \Psi(u), \sup_{u\in X} \Psi(u)\right) \quad \textit{such that} \quad \chi_1(r) < \chi_2(r), \tag{2.21}$$

then the following properties hold.

(i) *The functional I_λ admits at least one critical point for every $\lambda \in I$.*
(ii) *If furthermore $(\chi_1(r), \chi_2(r)) \cap I \neq 0$, then*
  (a) *I_λ has at least three critical points for every $\lambda \in (\chi_1(r), \chi_2(r)) \cap I$.*
  (b) *$I_{\chi_1(r)}$ has at least two critical points provided that $\chi_1(r) \in I$.*
  (c) *$I_{\chi_2(r)}$ has at least two critical points provided that $\chi_2(r) \in I$.*

Theorem 2.14 *Assume that* (A), (J1)-(J5), (F1)-(F3) *and* (M) *hold. Then we have*

(i) *If $\lambda \in [0, \ell_*)$, where $\ell_* = c_1\lambda_1/C_f$, then problem* (P_λ) *has only the trivial solution, where λ_1 is the principal eigenvalue of problem* (E), *c_1 is a positive constant in* (J5), *and both of c_* and c^* are positive constants from* (1.1).
(ii) *If furthermore f satisfies condition* (F7), *then there exists a positive constant ℓ^* with $\ell^* \geq \ell_*$ such that problem* (P_λ) *has at least two nontrivial solutions for all $\lambda \in (\ell^*, +\infty)$.*

Proof By Lemma 2.2, the functional $\Phi : X \to \mathbb{R}$ is a sequentially weakly lower semicontinuous C^1-functional and the operator Φ' is a mapping of type (S_+). It follows from

Lemma 2.4 that the functional Ψ is also sequentially weakly lower semicontinuous C^1-functional and the operator $\Psi' : X \to X^*$ is compact. Due to Lemma 2.5, we have

$$\lim_{\|u\|_X\to\infty}\bigl(\Phi(u)+\lambda\Psi(u)\bigr)=+\infty$$

for all $u \in X$ and for all $\lambda \in \mathbb{R}$.

First we claim the assertion (i). Let $u \in X$ be a nontrivial weak solution of problem (P_λ), that is,

$$\int_{\mathbb{R}^N}\varphi(x,\nabla u)\cdot\nabla v\,dx=\lambda\int_{\mathbb{R}^N}f(x,u)v\,dx$$

for all $v \in X$. If we put $v = u$, then it follows from (J5) that

$$\begin{aligned}
c_1\lambda_1\|u\|_a^p &\le \lambda_1\int_{\mathbb{R}^N}\varphi(x,\nabla u)\cdot\nabla u\,dx=\lambda_1\lambda\int_{\mathbb{R}^N}f(x,u)u\,dx\\
&=\lambda_1\lambda\int_{\mathbb{R}^N}\frac{f(x,u)}{m(x)|u|^{p-1}}m(x)|u|^p\,dx\le\lambda_1\lambda C_f\int_{\mathbb{R}^N}m(x)|u|^p\,dx\\
&\le\lambda C_f\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,dx=\lambda C_f\|u\|_a^p.
\end{aligned}$$

Thus if u is a nontrivial weak solution of problem (P_λ), then necessarily $\lambda \ge \ell_* = c_1\lambda_1/C_f$, as required.

Next let us prove assertion (ii). Let $s_0 \ne 0$ be from (F3). For $\sigma \in (0,1)$, define

$$u_\sigma(x)=\begin{cases}0 & \text{if } x\in\mathbb{R}^N\setminus B_N(x_0,r_0),\\ |s_0| & \text{if } x\in B_N(x_0,\sigma r_0),\\ \frac{|s_0|}{r_0(1-\sigma)}(r_0-|x-x_0|) & \text{if } x\in B_N(x_0,r_0)\setminus B_N(x_0,\sigma r_0).\end{cases} \tag{2.22}$$

Then it is obvious that $0 \le u_\sigma(x) \le |s_0|$ for all $x \in \mathbb{R}^N$ and $u_\sigma \in X$. From condition (F3),

$$\begin{aligned}
-\Psi(u_\sigma)&=\int_{B_N(x_0,\sigma r_0)}F\bigl(x,|s_0|\bigr)\,dx\\
&\quad+\int_{B_N(x_0,r_0)\setminus B_N(x_0,\sigma r_0)}F\biggl(x,\frac{|s_0|}{r_0(1-\sigma)}\bigl(r_0-|x-x_0|\bigr)\biggr)\,dx\\
&>0.
\end{aligned}$$

It follows that the crucial number

$$\ell^*=\chi_1(0)=\inf_{u\in\Psi^{-1}((-\infty,0))}-\frac{\Phi(u)}{\Psi(u)}$$

is well defined. Let u be in X with $u \not\equiv 0$. Using (J5) and (2.20), we have

$$\begin{aligned}
\frac{\Phi(u)}{|\Psi(u)|}&=\frac{\int_{\mathbb{R}^N}\Phi_0(x,\nabla u)\,dx}{\int_{\mathbb{R}^N}F(x,u)\,dx}\ge\frac{\frac{c_1}{p}\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,dx}{\int_{\mathbb{R}^N}\frac{|F(x,u)|}{m(x)|u|^p}m(x)|u|^p\,dx}\\
&\ge\frac{\frac{c_1}{p}\int_{\mathbb{R}^N}a(x)|\nabla u|^p\,dx}{\frac{C_f}{p}\int_{\mathbb{R}^N}m(x)|u|^p\,dx}\ge\frac{c_1\lambda_1}{C_f}=\ell_*.
\end{aligned}$$

Hence we get $\ell^* \geq \ell_*$. To employ Lemma 2.13, we have to verify assumption (2.21). For all $u \in \Psi^{-1}((-\infty, 0))$, we have that

$$\chi_1(r) = \inf_{u\in\Psi^{-1}((-\infty,r))} \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \leq \frac{\inf_{v\in\Psi^{-1}(r)} \Phi(v) - \Phi(u)}{\Psi(u) - r} \leq \frac{\Phi(u)}{r - \Psi(u)}$$

for all $r \in (\Psi(u), 0)$, and hence

$$\limsup_{r\to 0-} \chi_1(r) \leq -\frac{\Phi(u)}{\Psi(u)}$$

for all $u \in \Psi^{-1}((-\infty, 0))$. Then it implies that

$$\limsup_{r\to 0-} \chi_1(r) \leq \chi_1(0) = \ell^*.$$

By assumption (F7), there exists a positive real number $M_* > 0$ such that

$$\bigl|F(x,s)\bigr| \leq M_* m(x)|s|^\kappa \tag{2.23}$$

for almost all $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$. Indeed, denote

$$M_0 = \limsup_{s\to 0} \frac{|F(x,s)|}{m(x)|s|^\kappa}.$$

Then there exists $\delta > 0$ such that $|F(x,s)| \leq (M_0 + 1)m(x)|s|^\kappa$ for almost all $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$ with $|s| < \delta$. Let s be fixed with $|s| \geq \delta$. According to (2.20),

$$\bigl|F(x,s)\bigr| \leq \frac{C_f}{p}|s|^{p-\kappa} m(x)|s|^\kappa \leq \frac{C_f \delta^{p-\kappa}}{p} m(x)|s|^\kappa$$

for almost all $x \in \mathbb{R}^N$. Put $M_* = \max\{M_0 + 1, C_f \delta^{p-\kappa}/p\}$. Then relation (2.23) holds.

Hence we deduce that

$$\bigl|\Psi(u)\bigr| \leq \int_{\mathbb{R}^N} M_* m(x)|u|^\kappa \,dx \leq C_4 \|m\|_{L^{\kappa_1}(\mathbb{R}^N)} \|u\|_X^\kappa$$

for some positive constant C_4. If $r < 0$ and $v \in \Psi^{-1}(r)$, then we obtain by (J5) that

$$r = \Psi(v) \geq -C_4 \|m\|_{L^{\kappa_1}(\mathbb{R}^N)} \|v\|_X^\kappa \geq -C_4 \|m\|_{L^{\kappa_1}(\mathbb{R}^N)} \left(\frac{p}{c_1 c_*^p} \Phi(v)\right)^{\frac{\kappa}{p}}.$$

Since $u = 0 \in \Psi^{-1}((r, \infty))$, by using (2.19), we have

$$\chi_2(r) \geq \frac{1}{|r|} \inf_{v\in\Psi^{-1}(r)} \Phi(v) \geq \frac{|r|^{\frac{p}{\kappa}-1}}{C_4^{\frac{p}{\kappa}} \|m\|_{L^{\kappa_1}(\mathbb{R}^N)}^{\frac{p}{\kappa}}} \frac{c_1 c_*^p}{p},$$

and so $\lim_{r\to 0-} \chi_2(r) = \infty$ since $\kappa > p$. Therefore, we conclude

$$\limsup_{r\to 0-} \chi_1(r) \leq \chi_1(0) = \ell^* < \lim_{r\to 0-} \chi_2(r) = +\infty.$$

It means that there exists a negative sequence $\{r_n\}$ such that $r_n \to 0$ as $n \to \infty$, so that $\chi_1(r_n) < \ell^* + 1/n < n < \chi_2(r_n)$ for all integers n with $n \geq n^* = 2 + [\ell^*]$. By Lemma 2.5, we put $I = \mathbb{R}$. Since $u \equiv 0$ is a critical point of I_λ, according to the part (a) of (ii) in Lemma 2.13, problem (P_λ) admits at least two nontrivial solutions for all

$$\lambda \in (\ell^*, +\infty) = \bigcup_{n=n^*}^{\infty} \left[\ell^* + \frac{1}{n}, n\right] \subset \bigcup_{n=n^*}^{\infty} \big(\chi_1(r_n), \chi_2(r_n)\big),$$

as claimed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References
1. Aouaoui, S: On some degenerate quasilinear equations involving variable exponents. Nonlinear Anal. **75**, 1843-1858 (2012)
2. Arcoya, D, Carmona, J: A nondifferentiable extension of a theorem of Pucci and Serrin and applications. J. Differ. Equ. **235**, 683-700 (2007)
3. Colasuonno, F, Pucci, P, Varga, C: Multiple solutions for an eigenvalue problem involving p-Laplacian type operators. Nonlinear Anal. **75**, 4496-4512 (2012)
4. Kristály, A, Lisei, H, Varga, C: Multiple solutions for p-Laplacian type equations. Nonlinear Anal. **68**, 1375-1382 (2008)
5. Liu, J, Shi, X: Existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x), q(x))$-Laplacian. Nonlinear Anal. **71**, 550-557 (2009)
6. Ricceri, B: Existence of three solutions for a class of elliptic eigenvalue problem. Math. Comput. Model. **32**, 1485-1494 (2000)
7. Ricceri, B: On three critical points theorem. Arch. Math. (Basel) **75**, 220-226 (2000)
8. Ricceri, B: Three critical points theorem revisited. Nonlinear Anal. **70**, 3084-3089 (2009)
9. Ricceri, B: A further three critical points theorem. Nonlinear Anal. **71**, 4151-4157 (2009)
10. Allegretto, W, Huang, YX: Eigenvalues of the indefinite-weight p-Laplacian in weighted spaces. Funkc. Ekvacioj **38**, 233-242 (1995)
11. Drábek, P: Nonlinear eigenvalue problem for the p-Laplacian in $\mathbb{R}^N$. Math. Nachr. **173**, 131-139 (1995)
12. Drábek, P, Kufner, A, Nicolosi, F: Quasilinear Elliptic Equations with Degenerations and Singularities. de Gruyter, Berlin (1997)
13. Kim, I-S: A global bifurcation result for generalized Laplace operators in $\mathbb{R}^N$. J. Nonlinear Convex Anal. **13**, 667-680 (2012)
14. Kim, IH, Kim, Y-H: Mountain pass type solutions and positivity of the infimum eigenvalue for quasilinear elliptic equations with variable exponents (submitted)
15. Zeidler, E: Nonlinear Functional Analysis and Its Applications II/B. Springer, New York (1990)
16. Lamperti, J: On the isometries of certain function-spaces. Pac. J. Math. **8**, 459-466 (1958)
17. Cammaroto, F, Vilasi, L: Multiple solutions for a nonhomogeneous Dirichlet problem in Orlicz-Sobolev spaces. Appl. Math. Comput. **218**, 11518-11527 (2012)

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