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Algorithms with variant anchors for pseudocontractive mappings

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Abstract
The purpose of this paper is to find the fixed points of pseudocontractive mappings by using the iterative technique. Two algorithms with variant anchors have been introduced. Strong convergence results are given. Especially, we can find the minimum-norm fixed point of pseudocontractive mappings.

MSC: 47H05; 47H10; 47H17

Keywords: algorithms; pseudocontractive mapping; fixed point; strong convergence

1 Introduction

In this paper, we assume that H is a real Hilbert space and $C \subset H$ is a nonempty closed convex subset. Recall that a mapping $T : C \to C$ is said to be Lipschitzian if

$$\left\| Tu - Tu^\dagger \right\| \leq \kappa \left\| u - u^\dagger \right\|, \quad \forall u, u^\dagger \in C,$$

where $\kappa > 0$ is a constant, which is in general called the Lipschitz constant. If $\kappa = 1$, T is called nonexpansive.

A mapping $T : C \to C$ is said to be pseudocontractive if

$$\left\langle Tu - Tu^\dagger, u - u^\dagger \right\rangle \leq \left\| u - u^\dagger \right\|^2, \quad \forall u, u^\dagger \in C.$$

We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T.

In the literature, there are a large number references associated with the fixed point algorithms for the pseudocontractive mappings. See, for instance, [1–24]. (The interest of pseudocontractions lies in their connection with monotone operators; namely, T is a pseudocontraction if and only if the complement $I - T$ is a monotone operator.)

Now there exists an example which shows that Mann iteration does not converge for the pseudocontractive mappings [2]. At present, it is still an interesting topic to construct algorithms for finding the fixed points of the pseudocontractive mappings.

On the other hand, there are perturbations always occurring in the iterative processes because the manipulations are inaccurate. Recently, in order to find the fixed points of the nonexpansive mappings, Yao and Shahzad [25] introduced the following algorithms with perturbations and obtained the strong convergence results.

Algorithm 1.1 Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. For given $x_0 \in C$, define a sequence $\{x_m\}$ in the following manner:

$$x_m = \mathrm{proj}_C\big[\alpha_m u_m + (1-\alpha_m)Tx_m\big], \quad m \geq 0, \tag{1.1}$$

where $\{\alpha_m\}$ is a sequence in $[0,1]$ and the sequence $\{u_m\} \subset H$ is a small perturbation for the m-step iteration satisfying $\|u_m\| \to 0$ as $m \to \infty$.

Theorem 1.2 *Suppose* $\mathrm{Fix}(T) \neq \emptyset$. *Then, as* $\alpha_m \to 0$, *the sequence* $\{x_m\}$ *generated by the implicit method* (1.1) *converges to* $\tilde{x} \in \mathrm{Fix}(T)$, *which is the minimum-norm fixed point of* T.

Algorithm 1.3 Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping. For given $x_0 \in C$, define a sequence $\{x_n\}$ in the following manner:

$$x_{n+1} = (1-\beta_n)x_n + \beta_n \mathrm{proj}_C\big[\alpha_n u_n + (1-\alpha_n)Tx_n\big], \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ and the sequence $\{u_n\} \subset H$ is a perturbation for the n-step iteration.

Theorem 1.4 *Suppose that* $\mathrm{Fix}(T) \neq \emptyset$. *Assume the following conditions are satisfied*:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(iii) $\sum_{n=0}^{\infty} \alpha_n \|u_n\| < \infty$.
Then the sequence $\{x_n\}$ *generated by the explicit iterative method* (1.2) *converges to* $\tilde{x} \in \mathrm{Fix}(T)$, *which is the minimum-norm fixed point of* T.

Note that the idea of the iterative algorithms with perturbations has been extended to the other topics, see, for example, [26].

Motivated by the above ideas and the results in the literature, in the present paper, we present two algorithms with variant anchors for finding the fixed points of the pseudocontractive mappings in Hilbert spaces. Strong convergence results are given. As special cases, we can find the minimum-norm fixed point of the pseudocontractive mappings.

2 Preliminaries

Recall that the metric projection $\mathrm{proj}_C : H \to C$ is defined by

$$\mathrm{proj}_C x := \arg\min_{\forall y \in C} \|x - y\|, \quad x \in H.$$

It is obvious that proj_C satisfies

$$\|x - \mathrm{proj}_C x\| \leq \|x - y\|, \quad \forall y \in C,$$

and is characterized by

$$\mathrm{proj}_C x \in C, \quad \langle x - \mathrm{proj}_C x, y - \mathrm{proj}_C x\rangle \leq 0, \quad \forall y \in C.$$

The following two lemmas will be useful for our main results.

Lemma 2.1 ([24]) *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping. Then* $\mathrm{Fix}(T)$ *is a closed convex subset of C and the mapping $I - T$ is demiclosed at* 0, *i.e., whenever $\{x_n\} \subset C$ is such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$.*

Lemma 2.2 ([27]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ *and $\{\delta_n\}$ is a sequence in R such that*
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} |\delta_n\gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In the sequel, we assume that C is a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ is a κ-Lipschitzian pseudocontractive mapping with nonempty fixed points set $\mathrm{Fix}(T)$.

The first result is on the convergence of the path for the pseudocontractive mappings. Now, we define our path as follows.

For fixed $\zeta, t \in (0,1)$ and $u_t \in H$, we define a mapping $G_t : C \to C$ by

$$G_t x = (1 - \zeta)\,\mathrm{proj}_C\bigl[tu_t + (1 - t)x\bigr] + \zeta\, Tx, \quad \forall x \in C,$$

where $\mathrm{proj}_C : H \to C$ is the metric projection from H on C.

Next, we show that the mapping G_t is strongly pseudocontractive. Indeed, for $x, y \in C$, we have

$$\begin{aligned}
\langle G_t x - G_t y, x - y\rangle &= (1 - \zeta)\bigl\langle \mathrm{proj}_C\bigl[tu_t + (1 - t)x\bigr] - \mathrm{proj}_C\bigl[tu_t + (1 - t)y\bigr], x - y\bigr\rangle \\
&\quad + \zeta\,\langle Tx - Ty, x - y\rangle \\
&\le (1 - \zeta)\bigl\|\mathrm{proj}_C\bigl[tu_t + (1 - t)x\bigr] - \mathrm{proj}_C\bigl[tu_t + (1 - t)y\bigr]\bigr\|\,\|x - y\| \\
&\quad + \zeta\,\|x - y\|^2 \\
&\le (1 - \zeta)(1 - t)\|x - y\|^2 + \zeta\,\|x - y\|^2 \\
&= \bigl[1 - (1 - \zeta)t\bigr]\|x - y\|^2.
\end{aligned}$$

Since $\zeta, t \in (0,1)$, $1 - (1 - \zeta)t \in (0,1)$. Hence, G_t is a strongly pseudocontractive mapping. By [2], G_t has a unique fixed point $x_t \in C$. That is, x_t satisfies

$$x_t = (1 - \zeta)\,\mathrm{proj}_C\bigl[tu_t + (1 - t)x_t\bigr] + \zeta\, Tx_t, \quad \forall t \in (0,1). \tag{3.1}$$

Remark 3.1 $u_t \in H$ can be seen as a perturbation.

Next, we prove the convergence of the path (3.1).

Theorem 3.2 *If* $\lim_{t\to 0} u_t = u \in H$, *then the path* $\{x_t\}$ *defined by* (3.1) *converges strongly to* $\mathrm{proj}_{\mathrm{Fix}(T)}(u)$.

Proof Let $p \in \mathrm{Fix}(T)$. We get from (3.1) that

$$\begin{aligned}\|x_t - p\|^2 &= (1-\zeta)\bigl\langle \mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - p, x_t - p\bigr\rangle + \zeta\langle Tx_t - p, x_t - p\rangle \\ &\le (1-\zeta)\bigl\|\mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - p\bigr\|\,\|x_t - p\| + \zeta\|x_t - p\|^2 \\ &\le (1-\zeta)\bigl\|t(u_t - p) + (1-t)(x_t - p)\bigr\|\,\|x_t - p\| + \zeta\|x_t - p\|^2 \\ &\le (1-\zeta)\bigl[(1-t)\|x_t - p\| + t\|u_t - p\|\bigr]\|x_t - p\| + \zeta\|x_t - p\|^2.\end{aligned}$$

It follows that

$$\|x_t - p\| \le \|u_t - p\|.$$

Since $\lim_{t\to 0} u_t = u \in H$, there exists a constant $M > 0$ such that $\sup_{t\in(0,1)}\|u_t - u\| \le M$. So,

$$\|x_t - p\| \le \|u_t - p\| \le \|u_t - u\| + \|u - p\| \le M + \|u - p\|.$$

Thus, $\{x_t\}$ is bounded.

By (3.1), we have

$$\begin{aligned}\|x_t - Tx_t\| &= \bigl\|(1-\zeta)\,\mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] + \zeta Tx_t - Tx_t\bigr\| \\ &\le (1-\zeta)\bigl\|\mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - Tx_t\bigr\| \\ &\le (1-\zeta)\bigl[\|x_t - Tx_t\| + t\|u_t - x_t\|\bigr].\end{aligned}$$

Therefore,

$$\|x_t - Tx_t\| \le \frac{(1-\zeta)t}{\zeta}\|u_t - x_t\| \le \frac{(1-\zeta)t}{\zeta}\bigl(\|u_t - u\| + \|x_t - u\|\bigr) \to 0 \quad (\text{as } t \to 0). \tag{3.2}$$

Let $\{t_n\} \subset (0,1)$ be a sequence satisfying $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$. By (3.2), we get

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.3}$$

By (3.1), we obtain

$$\begin{aligned}\|x_t - p\|^2 &= (1-\zeta)\bigl\langle \mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - p, x_t - p\bigr\rangle + \zeta\langle Tx_t - p, x_t - p\rangle \\ &\le (1-\zeta)\bigl\|\mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - p\bigr\|\,\|x_t - p\| + \zeta\|x_t - p\|^2 \\ &\le \frac{1-\zeta}{2}\bigl(\bigl\|\mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - p\bigr\|^2 + \|x_t - p\|^2\bigr) + \zeta\|x_t - p\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|x_t - p\|^2 &\le \bigl\|\mathrm{proj}_C\bigl[tu_t + (1-t)x_t\bigr] - p\bigr\|^2 \\ &\le \bigl\|x_t - p + t(u_t - x_t)\bigr\|^2\end{aligned}$$

$$\begin{aligned} &= \|x_t - p\|^2 + 2t\langle u_t - x_t, x_t - p\rangle + t^2\|u_t - x_t\|^2 \\ &= \|x_t - p\|^2 - 2t\langle x_t - p, x_t - p\rangle + 2t\langle u_t - p, x_t - p\rangle + t^2\|u_t - x_t\|^2 \\ &= (1-2t)\|x_t - p\|^2 + 2t\langle u_t - p, x_t - p\rangle + t^2\|u_t - x_t\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_t - p\|^2 &\le \langle u_t - p, x_t - p\rangle + \frac{t}{2}\|u_t - x_t\|^2 \\ &\le \langle u_t - p, x_t - p\rangle + tM_1. \end{aligned} \tag{3.4}$$

Here $M_1 > 0$ is a constant such that $\sup_{t\in(0,1)} \frac{\|u_t - x_t\|^2}{2} \le M_1$. In particular, we obtain

$$\|x_n - p\|^2 \le \langle u_n - p, x_n - p\rangle + t_n M_1, \quad \forall p \in \mathrm{Fix}(T). \tag{3.5}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ satisfying $x_{n_i} \to x^* \in C$ weakly. By (3.3), we get

$$\lim_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0. \tag{3.6}$$

Applying Lemma 2.1 to (3.6) to deduce $x^* \in \mathrm{Fix}(T)$.

By (3.5), we derive

$$\left\|x_{n_i} - x^*\right\|^2 \le \left\langle u_{n_i} - x^*, x_{n_i} - x^*\right\rangle + t_{n_i} M_1. \tag{3.7}$$

Since $u_{n_i} - x^* \to u - x^*$ and $t_{n_i} \to 0$, we deduce that $x_{n_i} \to x^*$ by (3.7). By (3.5), we have

$$\left\|x^* - p\right\|^2 \le \left\langle u - p, x^* - p\right\rangle, \quad \forall p \in \mathrm{Fix}(T). \tag{3.8}$$

Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ satisfying $x_{n_j} \to x^\dagger$ weakly. Similarly, we can prove that $x_{n_j} \to x^\dagger \in \mathrm{Fix}(T)$, which satisfies

$$\left\|x^\dagger - p\right\|^2 \le \left\langle u - p, x^\dagger - p\right\rangle, \quad \forall p \in \mathrm{Fix}(T). \tag{3.9}$$

In (3.8), we pick up $p = x^\dagger$ to get

$$\left\|x^* - x^\dagger\right\|^2 \le \left\langle u - x^\dagger, x^* - x^\dagger\right\rangle. \tag{3.10}$$

In (3.9), we pick up $p = x^*$ to get

$$\left\|x^\dagger - x^*\right\|^2 \le \left\langle u - x^*, x^\dagger - x^*\right\rangle. \tag{3.11}$$

Adding (3.10) and (3.11), we deduce

$$\left\|x^\dagger - x^*\right\|^2 \le 0.$$

Thus, $x^* = x^\dagger$. This indicates that the weak limit set of $\{x_n\}$ is singleton and the path $\{x_t\}$ converges strongly to $x^* = \mathrm{proj}_{\mathrm{Fix}(T)}(u)$ by (3.8). This completes the proof. □

Corollary 3.3 *The path* $\{x_t\}$ *defined by*

$$x_t = (1-\zeta)\operatorname{proj}_C\big[(1-t)x_t\big] + \zeta Tx_t, \quad \forall t \in (0,1),$$

converges strongly to $\operatorname{proj}_{\mathrm{Fix}(T)}(0)$, *which is the minimum-norm fixed point of* T.

Now, we introduce another algorithm, which is an explicit manner.

Algorithm 3.4 Let $\{\varsigma_n\}$ and $\{\zeta_n\}$ be two real number sequences in $(0,1)$. Let $\{u_n\} \subset H$ be a sequence. For $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$x_{n+1} = (1-\zeta_n)\operatorname{proj}_C\big[\varsigma_n u_n + (1-\varsigma_n)x_n\big] + \zeta_n Tx_n, \quad n \geq 0. \tag{3.12}$$

Theorem 3.5 *Assume the following conditions are satisfied*:

(C1) $\lim_{n\to\infty}\varsigma_n = \lim_{n\to\infty}\frac{\varsigma_n}{\zeta_n} = \lim_{n\to\infty}\frac{\zeta_n^2}{\varsigma_n} = 0$;

(C2) $\lim_{n\to\infty}u_n = u \in H$.

Then we have

(1) *the sequence* $\{x_n\}$ *is bounded*;

(2) *the sequence* $\{x_n\}$ *is asymptotically regular, that is,* $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Further, if $\sum_{n=0}^{\infty}\varsigma_n = \infty$ *and* $\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\zeta_n} = 0$, *then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{proj}_{\mathrm{Fix}(T)}(u)$.

Proof By the condition (C1), we can find a sufficiently large positive integer m such that

$$1 - \frac{1}{1/2-\zeta_m}(\kappa+1)(\kappa+2)\left(\varsigma_m + 2\zeta_m + \frac{\zeta_m^2}{\varsigma_m}\right) > 0. \tag{3.13}$$

Let $p \in \mathrm{Fix}(T)$. For fixed m, we pick up a constant $M_2 > 0$ such that

$$\max\big\{\|x_0-p\|, \|x_1-p\|, \ldots, \|x_{m-1}-p\|, 4\|x_m-p\| + 4\|u_m-p\|\big\} \leq M_2. \tag{3.14}$$

Next, we show that $\|x_{m+1}-p\| \leq M_2$. Set $y_n = \operatorname{proj}_C[\varsigma_n u_n + (1-\varsigma_n)x_n]$ for all $n \geq 0$. Thus, we have $x_{n+1} = (1-\zeta_n)y_n + \zeta_n Tx_n$ for all $n \geq 0$.

Since $I-T$ is monotone, we have

$$\big\langle (I-T)x_{m+1}, x_{m+1}-p\big\rangle = \big\langle (I-T)x_{m+1} - (I-T)p, x_{m+1}-p\big\rangle \geq 0.$$

By (3.12), we obtain

$$\begin{aligned}
\|x_{m+1}-p\|^2 &= (1-\zeta_m)\langle y_m - p, x_{m+1}-p\rangle + \zeta_m\langle Tx_m - p, x_{m+1}-p\rangle \\
&= (1-\zeta_m)\big\langle y_m - \varsigma_m u_m - (1-\varsigma_m)x_m, x_{m+1}-p\big\rangle \\
&\quad + (1-\zeta_m)\big\langle \varsigma_m u_m + (1-\varsigma_m)x_m - p, x_{m+1}-p\big\rangle \\
&\quad + \zeta_m\langle Tx_m - p, x_{m+1}-p\rangle \\
&= (1-\zeta_m)\big\langle y_m - \varsigma_m u_m - (1-\varsigma_m)x_m, x_{m+1}-p\big\rangle \\
&\quad + (1-\zeta_m)\langle x_m - p, x_{m+1}-p\rangle + (1-\zeta_m)\varsigma_m\langle u_m - x_m, x_{m+1}-p\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \zeta_m \langle Tx_m - p, x_{m+1} - p\rangle \\
&= (1-\zeta_m)\big\langle y_m - \varsigma_m u_m - (1-\varsigma_m)x_m, x_{m+1} - p\big\rangle \\
&\quad + \langle x_m - p, x_{m+1} - p\rangle - (1-\zeta_m)\varsigma_m \langle x_{m+1} - p, x_{m+1} - p\rangle \\
&\quad - (1-\zeta_m)\varsigma_m \langle x_m - x_{m+1}, x_{m+1} - p\rangle - (1-\zeta_m)\varsigma_m \langle p - u_m, x_{m+1} - p\rangle \\
&\quad + \zeta_m \langle Tx_m - Tx_{m+1}, x_{m+1} - p\rangle + \zeta_m \langle x_{m+1} - x_m, x_{m+1} - p\rangle \\
&\quad - \zeta_m \langle x_{m+1} - Tx_{m+1}, x_{m+1} - p\rangle.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\big\|y_m - \varsigma_m u_m - (1-\varsigma_m)x_m\big\| &\le \|y_m - x_m\| + \varsigma_m \|x_m - u_m\| \\
&= \big\|\mathrm{proj}_C\big[\varsigma_m u_m + (1-\varsigma_m)x_m\big] - x_m\big\| + \varsigma_m \|x_m - u_m\| \\
&\le 2\varsigma_m \|x_m - u_m\|.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\|x_{m+1} - p\|^2 &\le (1-\zeta_m)\big\|y_m - \varsigma_m u_m - (1-\varsigma_m)x_m\big\| \|x_{m+1} - p\| \\
&\quad + \|x_m - p\| \|x_{m+1} - p\| - (1-\zeta_m)\varsigma_m \|x_{m+1} - p\|^2 \\
&\quad + (1-\zeta_m)\varsigma_m\big(\|x_{m+1} - x_m\| + \|u_m - p\|\big)\|x_{m+1} - p\| \\
&\quad + \zeta_m\big(\|Tx_m - Tx_{m+1}\| + \|x_{m+1} - x_m\|\big)\|x_{m+1} - p\| \\
&\le 2(1-\zeta_m)\varsigma_m \|x_m - u_m\| \|x_{m+1} - p\| + \|x_m - p\| \|x_{m+1} - p\| \\
&\quad + (1-\zeta_m)\varsigma_m\big(\|x_{m+1} - x_m\| + \|u_m - p\|\big)\|x_{m+1} - p\| \\
&\quad - (1-\zeta_m)\varsigma_m \|x_{m+1} - p\|^2 + \zeta_m(\kappa + 1)\|x_{m+1} - x_m\| \|x_{m+1} - p\| \\
&\le \|x_m - p\| \|x_{m+1} - p\| + 2(1-\zeta_m)\varsigma_m\big(\|x_m - p\| + \|u_m - p\|\big)\|x_{m+1} - p\| \\
&\quad - (1-\zeta_m)\varsigma_m \|x_{m+1} - p\|^2 + (\varsigma_m + \zeta_m)(\kappa + 1)\|x_{m+1} - x_m\| \|x_{m+1} - p\|.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\big[1 + (1-\zeta_m)\varsigma_m\big]\|x_{m+1} - p\| &\le \|x_m - p\| + 2\varsigma_m\big(\|x_m - p\| + \|u_m - p\|\big) \\
&\quad + (\kappa + 1)(\varsigma_m + \zeta_m)\|x_{m+1} - x_m\|. \qquad (3.15)
\end{aligned}
$$

By (3.12), we have

$$
\begin{aligned}
\|x_{m+1} - x_m\| &\le (1-\zeta_m)\big\|\mathrm{proj}_C\big[\varsigma_m u_m + (1-\varsigma_m)x_m\big] - x_m\big\| + \zeta_m \|Tx_m - x_m\| \\
&\le (1-\zeta_m)\varsigma_m\big(\|x_m - p\| + \|u_m - p\|\big) + \zeta_m\big(\|Tx_m - p\| + \|p - x_m\|\big) \\
&\le \varsigma_m\big(\|x_m - p\| + \|u_m - p\|\big) + \zeta_m(\kappa + 1)\|x_m - p\| \\
&\le (\kappa + 1)(\varsigma_m + \zeta_m)\|x_m - p\| + \varsigma_m \|u_m - p\| \\
&\le (\kappa + 2)(\varsigma_m + \zeta_m)M_2. \qquad (3.16)
\end{aligned}
$$

From condition (C1), we deduce $\varsigma_m \to 0$ and $\zeta_m \to 0$ as $m \to \infty$. Therefore, we get

$$\lim_{m\to\infty} \|x_{m+1} - x_m\| = 0.$$

That is, the sequence $\{x_m\}$ is asymptotically regular.

By (3.15) and (3.16), we have

$$\begin{aligned}
&\bigl[1 + (1-\zeta_m)\varsigma_m\bigr]\|x_{m+1} - p\| \\
&\quad\le \|x_m - p\| + \varsigma_m\bigl(2\|x_m - p\| + 2\|u_m - p\|\bigr) + (\kappa+1)(\kappa+2)(\varsigma_m + \zeta_m)^2 M_2 \\
&\quad\le \left(1 + \frac{1}{2}\varsigma_m\right)M_2 + (\kappa+1)(\kappa+2)(\varsigma_m + \zeta_m)^2 M_2.
\end{aligned}$$

This together with (3.13) and (3.14) imply that

$$\begin{aligned}
\|x_{m+1} - p\| &\le \left[1 - \frac{(1/2 - \zeta_m)\varsigma_m - (\kappa+1)(\kappa+2)(\varsigma_m + \zeta_m)^2}{1 + (1-\zeta_m)\varsigma_m}\right]M_2 \\
&= \left\{1 - \frac{(1/2 - \zeta_m)\varsigma_m[1 - \frac{1}{1/2-\zeta_m}(\kappa+1)(\kappa+2)(\varsigma_m + 2\zeta_m + (\zeta_m^2/\varsigma_m))]}{1 + (1-\zeta_m)\varsigma_m}\right\}M_2 \\
&\le M_2.
\end{aligned}$$

By induction, we get

$$\|x_n - p\| \le M_2, \quad \forall n \ge 0.$$

So $\{x_n\}$ is bounded.

By (3.12), we have

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-\zeta_n)\bigl\|\mathrm{proj}_C\bigl[\varsigma_n u_n + (1-\varsigma_n)x_n\bigr] - Tx_n\bigr\| \\
&\le \|x_n - x_{n+1}\| + (1-\zeta_n)\|x_n - Tx_n\| + \varsigma_n\|x_n - u_n\|.
\end{aligned}$$

It follows that

$$\|x_n - Tx_n\| \le \frac{1}{\zeta_n}\|x_n - x_{n+1}\| + \frac{\varsigma_n}{\zeta_n}\|x_n - u_n\|.$$

By the condition $\lim_{n\to\infty}\frac{\varsigma_n}{\zeta_n} = 0$ and the assumption $\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\zeta_n} = 0$, we deduce

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.17}$$

Let the net $\{z_t\}$ be defined by $z_t = (1-\zeta)\,\mathrm{proj}_C[tu_t + (1-t)z_t] + \zeta Tz_t$. By Theorem 3.2, we know that z_t converges strongly to $\mathrm{proj}_{\mathrm{Fix}(T)}(u)$. Next, we prove

$$\limsup_{n\to\infty}\bigl\langle \mathrm{proj}_{\mathrm{Fix}(T)}(u) - u_n, \mathrm{proj}_{\mathrm{Fix}(T)}(u) - y_n\bigr\rangle \le 0.$$

By the definition of $\{z_t\}$, we have

$$z_t - x_n = (1-\zeta)\big(\mathrm{proj}_C\big[tu_t + (1-t)z_t\big] - x_n\big) + \zeta(Tz_t - Tx_n) + \zeta(Tx_n - x_n).$$

It follows that

$$\begin{aligned}\|z_t - x_n\|^2 &= (1-\zeta)\big\langle \mathrm{proj}_C\big[tu_t + (1-t)z_t\big] - x_n, z_t - x_n\big\rangle + \zeta\langle Tz_t - Tx_n, z_t - x_n\rangle \\ &\quad + \zeta\langle Tx_n - x_n, z_t - x_n\rangle \\ &= (1-\zeta)\big\langle \mathrm{proj}_C\big[tu_t + (1-t)z_t\big] - tu_t - (1-t)z_t, z_t - x_n\big\rangle \\ &\quad + (1-\zeta)\big\langle tu_t + (1-t)z_t - x_n, z_t - x_n\big\rangle + \zeta\langle Tz_t - Tx_n, z_t - x_n\rangle \\ &\quad + \zeta\langle Tx_n - x_n, z_t - x_n\rangle.\end{aligned}$$

Since $x_n \in C$, by the characteristic inequality of metric projection, we have

$$\big\langle \mathrm{proj}_C\big[tu_t + (1-t)z_t\big] - tu_t - (1-t)z_t, z_t - x_n\big\rangle \le 0.$$

Then

$$\begin{aligned}\|z_t - x_n\|^2 &\le (1-\zeta)\big\langle tu_t + (1-t)z_t - x_n, z_t - x_n\big\rangle + \zeta\|z_t - x_n\|^2 \\ &\quad + \zeta\|Tx_n - x_n\|\|z_t - x_n\| \\ &= (1-\zeta)\|z_t - x_n\|^2 - (1-\zeta)t\langle z_t - u_t, z_t - x_n\rangle + \zeta\|z_t - x_n\|^2 \\ &\quad + \zeta\|Tx_n - x_n\|\|z_t - x_n\|,\end{aligned}$$

which implies that

$$\langle z_t - u_t, z_t - x_n\rangle \le \frac{\zeta}{(1-\zeta)t}\|Tx_n - x_n\|\|z_t - x_n\|.$$

By (3.17), we deduce

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u_t, z_t - x_n\rangle \le 0. \tag{3.18}$$

Note the fact that the two limits $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ are interchangeable. This together with $z_t \to \mathrm{proj}_{\mathrm{Fix}(T)}(u)$, $u_t \to u$ and (3.18) implies that

$$\limsup_{n\to\infty}\big\langle \mathrm{proj}_{\mathrm{Fix}(T)}(u) - u, \mathrm{proj}_{\mathrm{Fix}(T)}(u) - x_n\big\rangle \le 0.$$

Note that $\|y_n - x_n\| \to 0$ and $u_n - u \to 0$. We derive

$$\limsup_{n\to\infty}\big\langle \mathrm{proj}_{\mathrm{Fix}(T)}(u) - u_n, \mathrm{proj}_{\mathrm{Fix}(T)}(u) - y_n\big\rangle \le 0.$$

Finally, we prove that $x_n \to \operatorname{proj}_{\operatorname{Fix}(T)}(u)$. Note that

$$\begin{aligned}
&\langle Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), x_{n+1} - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\quad = \langle Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle + \langle Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), x_{n+1} - x_n\rangle \\
&\quad \le \|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 + \|Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|\|x_{n+1} - x_n\|
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&\|y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 \\
&\quad = \langle y_n - \varsigma_n u_n - (1-\varsigma_n)x_n, y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\qquad + \langle \varsigma_n u_n + (1-\varsigma_n)x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\quad \le \langle \varsigma_n u_n + (1-\varsigma_n)x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\quad = (1-\varsigma_n)\langle x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\qquad - \varsigma_n\langle \operatorname{proj}_{\operatorname{Fix}(T)}(u) - u_n, y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\quad \le \frac{(1-\varsigma_n)}{2}\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 + \frac{1}{2}\|y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 \\
&\qquad - \varsigma_n\langle \operatorname{proj}_{\operatorname{Fix}(T)}(u) - u_n, y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
\|y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 &\le (1-\varsigma_n)\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 \\
&\quad - 2\varsigma_n\langle \operatorname{proj}_{\operatorname{Fix}(T)}(u) - u_n, y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle.
\end{aligned} \tag{3.20}$$

By (3.12), (3.16), and (3.20), we get

$$\begin{aligned}
&\|x_{n+1} - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 \\
&\quad = \|(1-\zeta_n)(y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)) + \zeta_n(Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u))\|^2 \\
&\quad \le \|(1-\zeta_n)(y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u))\|^2 \\
&\qquad + 2\zeta_n\langle Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u), x_{n+1} - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\quad \le (1-\zeta_n)^2(1-\varsigma_n)\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 + 2\zeta_n\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 \\
&\qquad - 2\varsigma_n(1-\zeta_n)^2\langle \operatorname{proj}_{\operatorname{Fix}(T)}(u) - u_n, y_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\rangle \\
&\qquad + 2\zeta_n\|Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|\|x_{n+1} - x_n\| \\
&\quad \le [1 - (1-2\zeta_n)\varsigma_n]\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 + \zeta_n^2\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 \\
&\qquad + 2\varsigma_n(1-\zeta_n)^2\langle \operatorname{proj}_{\operatorname{Fix}(T)}(u) - u_n, \operatorname{proj}_{\operatorname{Fix}(T)}(u) - y_n\rangle \\
&\qquad + 2\zeta_n\|Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|(\kappa + 2)(\varsigma_n + \zeta_n)M_2 \\
&\quad = (1-\gamma_n)\|x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\|^2 + \gamma_n\delta_n,
\end{aligned} \tag{3.21}$$

where $\gamma_n = (1-2\zeta_n)\varsigma_n$ and

$$\begin{aligned}\delta_n = {}& \frac{2(1-\zeta_n)^2}{1-2\zeta_n}\langle \operatorname{proj}_{\operatorname{Fix}(T)}(u) - u_n, \operatorname{proj}_{\operatorname{Fix}(T)}(u) - y_n\rangle + \frac{\zeta_n^2}{(1-2\zeta_n)\varsigma_n}\left\| x_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\right\|^2 \\ & + \frac{2\zeta_n}{1-2\zeta_n}\left\| Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\right\|(\kappa+2)M_2 \\ & + \frac{2\zeta_n^2}{(1-2\zeta_n)\varsigma_n}\left\| Tx_n - \operatorname{proj}_{\operatorname{Fix}(T)}(u)\right\|(\kappa+2)M_2.\end{aligned}$$

It is clear that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. We can therefore apply Lemma 2.2 to (3.21) and conclude that $x_n \to \operatorname{proj}_{\operatorname{Fix}(T)}(u)$ as $n\to\infty$. This completes the proof. □

Corollary 3.6 *Let* $\{\varsigma_n\}$ *and* $\{\zeta_n\}$ *be two real number sequences in* $(0,1)$. *For* $x_0 \in C$ *arbitrarily, let the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = (1-\zeta_n)\operatorname{proj}_C\bigl[(1-\varsigma_n)x_n\bigr] + \zeta_n Tx_n, \quad n \ge 0. \tag{3.22}$$

Assume $\lim_{n\to\infty}\varsigma_n = \lim_{n\to\infty}\frac{\varsigma_n}{\zeta_n} = \lim_{n\to\infty}\frac{\zeta_n^2}{\varsigma_n} = 0$. *Then we have*

(1) *the sequence* $\{x_n\}$ *is bounded*;

(2) *the sequence* $\{x_n\}$ *is asymptotically regular, that is,* $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$.

Further, if $\sum_{n=0}^{\infty}\varsigma_n = \infty$ *and* $\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\zeta_n} = 0$, *then the sequence* $\{x_n\}$ *converges strongly to* $\operatorname{proj}_{\operatorname{Fix}(T)}(0)$, *which is the minimum-norm fixed point of* T.

Proof Letting $u_n = u = 0$ in (3.12), we obtain (3.22). Consequently, by Theorem 3.5, we find that the sequence $\{x_n\}$ generated by (3.22) converges strongly to $\operatorname{proj}_{\operatorname{Fix}(T)}(0)$, which is the minimum-norm fixed point of T. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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