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Convergence theorems for continuous pseudocontractive mappings in Banach spaces

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Abstract
In this paper, we prove some strong and weak convergence theorems for continuous pseudocontractive mapping and a weak convergence theorem for nonexpansive mapping in real uniformly convex Banach spaces. As an application of the strong convergence theorem, we give an interesting example.

Keywords: uniformly convex Banach space; pseudocontractive mapping; nonexpansive mapping; implicit iteration process; convergence theorem

1 Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}$$

for all $x \in E$, where $\langle\cdot,\cdot\rangle$ denotes duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j.

Let C be a nonempty closed convex subset of a real Banach space E. A mapping $T : C \to C$ is said to be *pseudocontractive* [1] if, for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2. \tag{1.1}$$

It is well known [2] that (1.1) is equivalent to the following:

$$\|x - y\| \le \|x - y + s[(I - T)x - (I - T)y]\| \tag{1.2}$$

for all $s > 0$ and $x, y \in C$.

A mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$.

Remark 1.1 It is easy to see that, if T is nonexpansive, then T is continuous pseudocontractive, but the converse is not true in general.

Example 1.1 Let $E = (-\infty, \infty)$ with the usual norm $|\cdot|$. Then $E^* = E$, $\langle x, f\rangle = xf$ for all $x \in E$ and $f \in E^*$ and the normalized duality mapping $J : E \to 2^{E^*}$ is as follows:

$$J(x) = \{f \in E^* : \langle x, f\rangle = |x||f|, |f| = |x|\} = \{x\}$$

for all $x \in E$. Let $C = E$ and define a mapping $T : C \to C$ by $Tx = -2x$ for all $x \in C$. It is easy to prove that T is continuous pseudocontractive, but not nonexpansive.

In 1974, Deimling [3] proved the following fixed point theorem.

Theorem 1.1 *Let E be a real Banach space, C be a nonempty closed convex subset of E, and $T : C \to C$ be a continuous strongly pseudocontractive mapping. Then T has a unique fixed point in C.*

Let E be a real Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous pseudocontractive mapping. For all $u \in C$ and $t \in (0, 1)$, define the mapping $S_t : C \to C$ by

$$S_t x = tu + (1 - t)Tx$$

for all $x \in C$. It is easy to prove that S_t is a continuous strongly pseudocontractive mapping. By Theorem 1.1, there exists a unique fixed point $x_t \in C$ of S_t such that

$$x_t = tu + (1 - t)Tx_t$$

for all $t \in (0, 1)$.

Let $T : C \to C$ be a continuous pseudocontractive mapping and define an implicit iteration process $\{x_n\}$ by

$$\begin{cases} x_0 \in C, \\ x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n \end{cases} \tag{1.3}$$

for each $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ with some conditions.

In 2008, Zhou [4] studied the implicit iteration process (1.3) and proved a weak convergence theorem for the strict pseudocontraction in a real reflexive Banach space which satisfies Opial's condition.

The purpose of this paper is to discuss the implicit iteration process (1.3) and to prove some strong and weak convergence theorems for a continuous pseudocontractive mapping and a weak convergence theorem for a nonexpansive mapping in real uniformly convex Banach spaces. As an application of the strong convergence theorem, we give an interesting example.

In order to prove the main results, we need the following:

A Banach space E is said to satisfy *Opial's condition* [5] if, for any sequence $\{x_n\}$ of E, $x_n \to x$ weakly as $n \to \infty$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. A Banach space E is said to have the *Kadec-Klee property* [6] if, for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$, it follows that $x_n \to x$ strongly.

Lemma 1.1 ([7]) *Let E be a uniformly convex Banach space with the modulus of uniform convexity δ_E. Then $\delta_E : [0,2] \to [0,1]$ is continuous, increasing, $\delta_E(0) = 0$, $\delta_E(t) > 0$ for $t > 0$ and, further,*

$$\big\|cu + (1-c)v\big\| \leq 1 - 2\min\{c, 1-c\}\delta_E\big(\|u - v\|\big)$$

whenever $0 \leq c \leq 1$ and $\|u\|, \|v\| \leq 1$.

Lemma 1.2 ([8]) *Let X be a uniformly convex Banach space and C be a convex subset of X. Then there exists a strictly increasing continuous convex function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ such that, for each $S : C \to C$ with Lipschitz constant L,*

$$\big\|\alpha Sx + (1-\alpha)Sy - S\big[\alpha x + (1-\alpha)y\big]\big\| \leq L\gamma^{-1}\left(\|x - y\| - \frac{1}{L}\|Sx - Sy\|\right)$$

for all $x, y \in C$ and $0 < \alpha < 1$.

Lemma 1.3 ([8]) *Let X be a uniformly convex Banach space such that its dual space X^* has the Kadec-Klee property. Suppose that $\{x_n\}$ is a bounded sequence and $f_1, f_2 \in W_w(\{x_n\})$ (where $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of a bounded sequence $\{x_n\}$ in X) such that*

$$\lim_{n\to\infty} \big\|\alpha x_n + (1-\alpha)f_1 - f_2\big\|$$

exists for all $\alpha \in [0,1]$. Then $f_1 = f_2$.

Lemma 1.4 ([4]) *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous pseudocontractive mapping. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I - T)x_n\}$ converges strongly to 0, then $(I - T)q = 0$.*

2 Convergence for continuous pseudocontractive mappings

Lemma 2.1 *Let E be a real uniformly convex Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous pseudocontractive mapping with $F(T) = \{x \in C : Tx = x\} \neq \emptyset$. Let $\{x_n\}$ be defined by* (1.3), *where $\alpha_n \in (0,1)$ and $\limsup_{n\to\infty} \alpha_n < 1$. Then*

(1) *$\|x_n - p\| \leq \|x_{n-1} - p\|$ for all $n \geq 1$ and all $p \in F(T)$;*
(2) *$\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$;*
(3) *$\lim_{n\to\infty} d(x_n, F(T))$ exists;*
(4) *$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof For all $n \geq 1$ and $p \in F(T)$, since T is pseudocontractive, it follows from (1.3) that

$$\begin{aligned}
\|x_n - p\|^2 &= \langle x_n - p, j(x_n - p)\rangle \\
&= \alpha_n\langle x_{n-1} - p, j(x_n - p)\rangle + (1-\alpha_n)\langle Tx_n - p, j(x_n - p)\rangle \\
&\leq \alpha_n\|x_{n-1} - p\|\|x_n - p\| + (1-\alpha_n)\|x_n - p\|^2,
\end{aligned}$$

which implies that $\|x_n - p\| \le \|x_{n-1} - p\|$ and $d(x_n, F(T)) \le d(x_{n-1}, F(T))$. Therefore, $\lim_{n\to\infty} \|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, F(T))$ exist. Thus (1), (2), and (3) are proved.

Now, we prove (4). By using (1.2), (1.3), and Lemma 1.1, it follows that, for all $p \in F(T)$,

$$\begin{aligned}
\|x_n - p\| &\le \left\| x_n - p + \frac{1-\alpha_n}{2\alpha_n}(x_n - Tx_n) \right\| \\
&= \left\| x_n - p + \frac{1-\alpha_n}{2}(x_{n-1} - Tx_n) \right\| \\
&= \left\| x_n - p + \frac{x_{n-1} - x_n}{2} \right\| \\
&= \left\| \frac{x_{n-1} + x_n}{2} - p \right\| \\
&= \|x_{n-1} - p\| \left\| \frac{1}{2}\frac{x_{n-1} - p}{\|x_{n-1} - p\|} + \frac{1}{2}\frac{x_n - p}{\|x_{n-1} - p\|} \right\| \\
&\le \|x_{n-1} - p\| \left[1 - \delta_E\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - p\|} \right) \right].
\end{aligned}$$

This implies that

$$\|x_{n-1} - p\| \delta_E\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - p\|} \right) \le \|x_{n-1} - p\| - \|x_n - p\|.$$

If $\|x_n - p\| \nrightarrow 0$, then we have $x_n - x_{n-1} \to 0$ as $n \to \infty$ by the properties of δ_E. It follows from $\|x_{n-1} - Tx_n\| = \frac{1}{1-\alpha_n}\|x_n - x_{n-1}\|$ and $\limsup_{n\to\infty} \alpha_n < 1$ that $x_{n-1} - Tx_n \to 0$ as $n \to \infty$ and so

$$\|x_n - Tx_n\| = \alpha_n \|x_{n-1} - Tx_n\| \le \|x_{n-1} - Tx_n\| \to 0$$

as $n \to \infty$.

If $\|x_n - p\| \to 0$, then, since T is continuous, it follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \|p - Tp\| = 0.$$

This completes the proof. □

Theorem 2.1 *Under the assumptions of Lemma* 2.1, $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof The necessity is obvious. So, we will prove the sufficiency. Assume that

$$\liminf_{n\to\infty} d\bigl(x_n, F(T)\bigr) = 0.$$

By Lemma 2.1, limit $\lim_{n\to\infty} d(x_n, F(T))$ exists and so $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Now, we show that $\{x_n\}$ is a Cauchy sequence in C. In fact, it follows from Lemma 2.1 that $\|x_m - p\| \le \|x_n - p\|$ for all positive integers m, n with $m > n \ge 1$ and $p \in F(T)$. So,

$$\|x_m - x_n\| \le \|x_n - p\| + \|x_m - p\| \le 2\|x_n - p\|.$$

Taking the infimum over all $p \in F(T)$, we have

$$\|x_m - x_n\| \leq 2d\big(x_n, F(T)\big).$$

It follows from $\lim_{n\to\infty} d(x_n, F(T)) = 0$ that $\{x_n\}$ is a Cauchy sequence. C is a closed subset of E and so $\{x_n\}$ converges strongly to some $q \in C$. Further, by the continuity of T, it is easy to prove that $F(T)$ is closed and it follows from $\lim_{n\to\infty} d(x_n, F(T)) = 0$ that $q \in F(T)$. This completes the proof. □

Corollary 2.1 *Under the assumptions of Lemma* 2.1, $\{x_n\}$ *converges strongly to a fixed point* p *of* T *if and only if there exists a subsequence* $\{x_{n_k}\}$ *of* $\{x_n\}$ *such that* $\{x_{n_k}\}$ *converges strongly to* p.

Proof Since

$$\liminf_{n\to\infty} d\big(x_n, F(T)\big) \leq \liminf_{k\to\infty} d\big(x_{n_k}, F(T)\big) \leq \lim_{k\to\infty} \|x_{n_k} - p\|,$$

it follows from Theorem 2.1 that Corollary 2.1 holds. This completes the proof. □

Theorem 2.2 *Under the assumptions of Lemma* 2.1, *if there exists a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(r) > 0$ *for all* $r \in (0,\infty)$ *such that*

$$f\big(d\big(x, F(T)\big)\big) \leq \|x - Tx\|$$

for all $x \in C$, *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof Since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ by Lemma 2.1 and so $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Further, by using Lemma 2.1 $\lim_{n\to\infty} d(x_n, F(T))$ exists, and we assume $\lim_{n\to\infty} d(x_n, F(T)) = r$.

If $r > 0$, there exists a positive integer N such that $d(x_n, F(T)) > \frac{r}{2}$ for all $n > N$. Thus we have

$$\lim_{n\to\infty} f\big(d\big(x_n, F(T)\big)\big) \geq f\left(\frac{r}{2}\right) > 0,$$

which is a contradiction. Therefore, $r = 0$. It follows from Theorem 2.1 that Theorem 2.2 holds. This completes the proof. □

Definition 2.1 ([9]) Let E be a real normed linear space, C be a nonempty subset of E, and $T : C \to E$ be a mapping. The pair (T, C) is said to satisfy the *condition* (A) if, for any bounded closed subset G of C, $\{z = x - Tx : x \in G\}$ is a closed subset of E.

Now, we prove strong convergence and weak convergence theorems for a continuous pseudocontractive mapping in real uniformly convex Banach spaces.

Theorem 2.3 *Under the assumptions of Lemma* 2.1, *if the pair* (T, C) *satisfies the condition* (A), *then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof The sequence $\{x_n\}$ is bounded in C by Lemma 2.1. Letting $G = \overline{\{x_n\}}$, where $\overline{A}$ denotes the closure of A, G is a bounded closed subset of C and so $M = \{z = x - Tx : x \in G\}$ is a closed subset of E since the pair (T, C) satisfies the condition (A). It follows from $\{x_n - Tx_n\} \subset M$ and $x_n - Tx_n \to 0$ as $n \to \infty$ by Lemma 2.1(3) that the zero vector $0 \in M$ and so there exists $q \in G$ such that $q = Tq$. This shows that q is a fixed point of T and so there exists a positive integer n_0 such that $x_{n_0} = q$ or there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to q$ as $n \to \infty$.

If $x_{n_0} = q$, then it follows from Lemma 2.1(1) that $x_n = q$ for all $n \geq n_0$ and so $x_n \to q$ as $n \to \infty$.

If $x_{n_k} \to q$, then, since $\lim_{n\to\infty} \|x_n - q\|$ exists by Lemma 2.1(2), $x_n \to q$ as $n \to \infty$. This completes the proof. □

As an application of Theorem 2.1, we give the following.

Example 2.1 Let $E = (-\infty, \infty)$ with the usual norm $|\cdot|$. Then $E^* = E$, $\langle x, f\rangle = xf$ for all $x \in E$ and $f \in E^*$ and $J(x) = \{x\}$ for all $x \in E$. Let $C = [0, \infty)$. Define a mapping $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{4}, & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}x, & \frac{1}{2} < x < \infty. \end{cases}$$

Then T is continuous pseudocontractive with $F(T) = \{\frac{1}{4}\}$ and the pair (T, C) satisfies the condition (A). In fact, for all $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, \infty)$, we have $4x < 2y + 1$ and so

$$0 < \frac{1}{2}y - \frac{1}{4} < y - x. \tag{2.1}$$

For all $x \in (\frac{1}{2}, \infty)$ and $y \in [0, \frac{1}{2}]$, we obtain $4y < 2x + 1$ and so

$$0 < \frac{1}{2}x - \frac{1}{4} < x - y. \tag{2.2}$$

Thus, for all $x, y \in C$, taking $j(x - y) = x - y \in J(x - y)$, it follows from (2.1) and (2.2) that

$$\begin{aligned} &\bigl\langle Tx - Ty, j(x - y)\bigr\rangle \\ &= \begin{cases} \langle \frac{1}{4} - \frac{1}{4}, x - y\rangle = 0 \leq |x - y|^2, & x, y \in [0, \frac{1}{2}], \\ \langle \frac{1}{4} - \frac{1}{2}y, x - y\rangle = (\frac{1}{2}y - \frac{1}{4})(y - x) < |x - y|^2, & x \in [0, \frac{1}{2}], y \in (\frac{1}{2}, \infty), \\ \langle \frac{1}{2}x - \frac{1}{4}, x - y\rangle = (\frac{1}{2}x - \frac{1}{4})(x - y) < |x - y|^2, & x \in (\frac{1}{2}, \infty), y \in [0, \frac{1}{2}], \\ \langle \frac{1}{2}x - \frac{1}{2}y, x - y\rangle = \frac{1}{2}(x - y)^2 \leq |x - y|^2, & x, y \in (\frac{1}{2}, \infty). \end{cases} \end{aligned}$$

This shows that T is continuous pseudocontractive.

Now, we prove that the pair (T, C) satisfies the condition (A). For any bounded closed subset G of C, we denote $M = \{z = x - Tx : x \in G\}$. Then M is closed. Indeed, for any $z_n \in M$ with $z_n \to z$, there exists $x_n \in G$ such that $z_n = x_n - Tx_n$. We consider the following cases.

Case 1. There exists a positive integer n_0 such that $x_n \in [0, \frac{1}{2}]$ for all $n \geq n_0$.
Case 2. There exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \in (\frac{1}{2}, \infty)$ for all $k \geq 1$.

If Case 1 holds, then $z_n = x_n - Tx_n = x_n - \frac{1}{4}$ for all $n \geq n_0$ and so $x_n = z_n + \frac{1}{4} \to z + \frac{1}{4} \in [0, \frac{1}{2}]$ as $n \to \infty$. Since G is closed, it follows that $z + \frac{1}{4} \in G$ and so $z = (z + \frac{1}{4}) - T(z + \frac{1}{4}) \in M$.

If Case 2 holds, then $z_{n_k} = x_{n_k} - Tx_{n_k} = \frac{1}{2}x_{n_k}$ for all $k \geq 1$ and so $x_{n_k} = 2z_{n_k} \to 2z \in G$. If $2z = \frac{1}{2}$, we have $z = \frac{1}{4} = \frac{1}{2} - T\frac{1}{2} = 2z - T(2z) \in M$. Otherwise, we have $2z \in (\frac{1}{2}, \infty)$ and so $z = 2z - T(2z) \in M$.

By Theorem 1.1, it is easy to prove that, for any $x_0 \in C$, there exists a unique $x_n \in C$ such that

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n$$

for all $n \geq 1$, where $\alpha_n = \frac{2n}{3n+1} \in (0,1)$ for all $n \geq 1$ and

$$\limsup_{n\to\infty} \alpha_n = \lim_{n\to\infty} \alpha_n = \frac{2}{3} < 1.$$

Thus it follows from Theorem 2.3 that the sequence $\{x_n\}$ converges to $\frac{1}{4}$.

Theorem 2.4 *Under the assumptions of Lemma* 2.1, *if E satisfies Opial's condition*, *then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof Since $\{x_n\}$ is bounded by Lemma 2.1 and E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $p \in C$. By Lemma 2.1, we have $\lim_{k\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$. It follows from Lemma 1.4 that $p \in F(T)$.

Now, we prove that $\{x_n\}$ converges weakly to p. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to a point $p^* \in C$. Then $p = p^*$. In fact, if $p \neq p^*$, then it follows from Opial's condition that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \limsup_{k\to\infty} \|x_{n_k} - p\| < \limsup_{k\to\infty} \|x_{n_k} - p^*\| = \lim_{n\to\infty} \|x_n - p^*\| \\ &= \limsup_{j\to\infty} \|x_{n_j} - p^*\| < \limsup_{j\to\infty} \|x_{n_j} - p\| = \lim_{n\to\infty} \|x_n - p\|,\end{aligned}$$

which is a contradiction. So $p = p^*$. Therefore, $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

Remark 2.1 By Remark 1.1, clearly, Theorems 2.1, 2.2, 2.3 and 2.4 still hold for nonexpansive mappings.

3 Weak convergence for nonexpansive mappings

In this section, we prove a weak convergence theorem for a nonexpansive mapping in real uniformly convex Banach spaces.

Lemma 3.1 *Let E be a real uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $\{x_n\}$ be the sequence defined by* (1.3), *where $\alpha_n \in (0,1)$ and* $\limsup_{n\to\infty} \alpha_n < 1$. *Then*, *for all $p_1, p_2 \in F(T)$, the limit* $\lim_{n\to\infty} \|tx_n + (1-t)p_1 - p_2\|$ *exists for all $t \in [0,1]$*.

Proof Letting $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$ for all $t \in [0,1]$, $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty} \|x_n - p_2\|$ exists by Remark 1.1 and Lemma 2.1. Thus it remains to prove Lemma 3.1 for any $t \in (0,1)$. For all $n \geq 1$ and $x \in C$, we define a mapping $A_{x,n-1} : C \to C$ by

$$A_{x,n-1}y = \alpha_n x + (1 - \alpha_n)Ty$$

for all $y \in C$. Then we have

$$\|A_{x,n-1}u - A_{x,n-1}v\| = (1 - \alpha_n)\|Tu - Tv\| \leq (1 - \alpha_n)\|u - v\|$$

for all $u, v \in C$. It follows from $0 < 1 - \alpha_n < 1$ that $A_{x,n-1}$ is contractive and so it has a unique fixed point in C, which is denoted by $G_{n-1}x$. Define a mapping $G_n : C \to C$ by

$$G_n = \alpha_{n+1}I + (1 - \alpha_{n+1})TG_n, \tag{3.1}$$

where I is a identity mapping. It follows from (3.1) that

$$\|G_n x - G_n y\| \leq \alpha_{n+1}\|x - y\| + (1 - \alpha_{n+1})\|G_n x - G_n y\|$$

for all $x, y \in C$ and so

$$\|G_n x - G_n y\| \leq \|x - y\|. \tag{3.2}$$

Using (3.1) and (1.3), we obtain

$$\|G_n x_n - x_{n+1}\| \leq (1 - \alpha_{n+1})\|G_n x_n - x_{n+1}\| \tag{3.3}$$

and

$$\|G_n p - p\| \leq (1 - \alpha_{n+1})\|G_n p - p\| \tag{3.4}$$

for all $p \in F(T)$. It follows from (3.3), (3.4) and $0 < 1 - \alpha_{n+1} < 1$ that $G_n x_n = x_{n+1}$ and $G_n p = p$. For each $m \geq 1$, let

$$S_{n,m} = G_{n+m-1}G_{n+m-2}\cdots G_n$$

and

$$b_{n,m} = \left\| S_{n,m}\big(tx_n + (1-t)p_1\big) - \big(tS_{n,m}x_n + (1-t)S_{n,m}p_1\big)\right\|. \tag{3.5}$$

By (3.2), we have

$$\|S_{n,m}x - S_{n,m}y\| \leq \|x - y\|$$

for all $x, y \in C$. This shows that $S_{n,m}$ is nonexpansive, $S_{n,m}x_n = x_{n+m}$ and $S_{n,m}p = p$ for all $p \in F(T)$. By Lemma 1.2, we obtain

$$b_{n,m} \leq \gamma^{-1}\big(\|x_n - p_1\| - \|x_{n+m} - p_1\|\big). \tag{3.6}$$

It is easy to prove that

$$\begin{aligned} a_{n+m}(t) &\leq \|tx_n + (1-t)p_1 - p_2\| + b_{n,m} \leq a_n(t) + b_{n,m} \\ &\leq a_n(t) + \gamma^{-1}(\|x_n - p_1\| - \|x_{n+m} - p_1\|). \end{aligned}$$

For fixed n and letting $m \to \infty$, we have

$$\limsup_{m\to\infty} a_m(t) \leq a_n(t) + \gamma^{-1}\Bigl(\|x_n - p_1\| - \lim_{m\to\infty}\|x_m - p_1\|\Bigr).$$

Again, letting $n \to \infty$, we obtain

$$\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t) + \gamma^{-1}(0) = \liminf_{n\to\infty} a_n(t).$$

This shows that

$$\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$$

exists for all $t \in (0,1)$. This completes the proof. □

Theorem 3.1 *Under the assumptions of Lemma* 3.1, *if the dual space E^* of E has the Kadec-Klee property, then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof Using the same method as in the proof of Theorem 2.4, we can prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, which converges weakly to a point $p \in F(T)$.

Now, we prove that $\{x_n\}$ converges weakly to p. Suppose that there exists a subsequence $\{x_{m_j}\}$ of $\{x_n\}$ such that $\{x_{m_j}\}$ converges weakly to a point $p^* \in C$. Then $p = p^*$. In fact, it follows from Lemma 3.1 that the limit

$$\lim_{n\to\infty}\|tx_n + (1-t)p - p^*\|$$

exists for all $t \in [0,1]$. Again, since $p, p^* \in W_w(\{x_n\})$, we have $p^* = p$ by Lemma 1.3. This shows that $\{x_n\}$ converges weakly to p. This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this paper and they read and approved the final manuscript.

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