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Note on certain inequalities for Neuman means

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Abstract
In this paper, we give the explicit formulas for the Neuman means N_{AH}, N_{HA}, N_{AC}, and N_{CA}, and present the best possible upper and lower bounds for these means in terms of the combinations of harmonic mean H, arithmetic mean A, and contraharmonic mean C.

MSC: 26E60

Keywords: Schwab-Borchardt mean; Neuman mean; harmonic mean; arithmetic mean; contraharmonic mean

1 Introduction

Let $a, b, c \geq 0$ with $ab + ac + bc \neq 0$. Then the symmetric integral $R_F(a, b, c)$ [1] of the first kind is defined as

$$R_F(a,b,c) = \frac{1}{2}\int_0^\infty \bigl[(t+a)(t+b)(t+c)\bigr]^{-1/2}\,dt.$$

The degenerate case of R_F, denoted by R_C, plays an important role in the theory of special functions [1, 2], which is given by

$$R_C(a,b) = R_F(a,b,b).$$

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $\mathrm{SB}(a, b)$ [3–5] of a and b is given by

$$\mathrm{SB}(a,b) = \begin{cases} \frac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \frac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases}$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

Carlson [6] (see also [7, (3.21)]) proved that

$$\mathrm{SB}(a,b) = \bigl[R_C\bigl(a^2, b^2\bigr)\bigr]^{-1}.$$

Recently, the Schwab-Borchardt mean has been the subject of intensive research. In particular, many remarkable inequalities for the Schwab-Borchardt mean and its generated means can be found in the literature [3–5, 8–11].

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Let $a > b > 0$, $v = (a-b)/(a+b) \in (0,1)$, $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0,\log(2+\sqrt{3}))$, and $s \in (0,\pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$, $H(a,b) = 2ab/(a+b)$, $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$, $Q(a,b) = \sqrt{(a^2+b^2)/2}$, and $C(a,b) = (a^2+b^2)/(a+b)$ be, respectively, the harmonic, geometric, arithmetic, quadratic, and contraharmonic means of a and b, $S_{AH}(a,b) = \mathrm{SB}[A(a,b),H(a,b)]$, $S_{HA}(a,b) = \mathrm{SB}[H(a,b),A(a,b)]$, $S_{AC}(a,b) = \mathrm{SB}[A(a,b),C(a,b)]$, $S_{CA}(a,b) = \mathrm{SB}[C(a,b),A(a,b)]$. Then Neuman [10] gave the explicit formulas

$$S_{AH}(a,b) = A(a,b)\frac{\tanh(p)}{p}, \qquad S_{HA}(a,b) = A(a,b)\frac{\sin q}{q}, \tag{1.1}$$

$$S_{CA}(a,b) = A(a,b)\frac{\sinh(r)}{r}, \qquad S_{AC}(a,b) = A(a,b)\frac{\tan s}{s}. \tag{1.2}$$

Very recently, Neuman [12] found a new mean $N(a,b)$ derived from the Schwab-Borchardt mean as follows:

$$N(a,b) = \frac{1}{2}\left[a + \frac{b^2}{\mathrm{SB}(a,b)}\right]. \tag{1.3}$$

Let $N_{AH}(a,b) = N[A(a,b),H(a,b)]$, $N_{HA}(a,b) = N[H(a,b),A(a,b)]$, $N_{AG}(a,b) = N[A(a,b),G(a,b)]$, $N_{GA}(a,b) = N[G(a,b),A(a,b)]$, $N_{AC}(a,b) = N[A(a,b),C(a,b)]$, $N_{CA}(a,b) = N[C(a,b),A(a,b)]$, $N_{AQ}(a,b) = N[A(a,b),Q(a,b)]$, and $N_{QA}(a,b) = N[Q(a,b),A(a,b)]$ be the Neuman means. Then Neuman [12] proved that

$$G(a,b) < N_{AG}(a,b) < N_{GA}(a,b) < A(a,b) < N_{QA}(a,b) < N_{AQ}(a,b) < Q(a,b)$$

for all $a,b > 0$ with $a \neq b$, and the double inequalities

$$\alpha_1 A(a,b) + (1-\alpha_1)G(a,b) < N_{GA}(a,b) < \beta_1 A(a,b) + (1-\beta_1)G(a,b),$$

$$\alpha_2 Q(a,b) + (1-\alpha_2)A(a,b) < N_{AQ}(a,b) < \beta_2 Q(a,b) + (1-\beta_2)A(a,b),$$

$$\alpha_3 A(a,b) + (1-\alpha_3)G(a,b) < N_{AG}(a,b) < \beta_3 A(a,b) + (1-\beta_3)G(a,b),$$

$$\alpha_4 Q(a,b) + (1-\alpha_4)A(a,b) < N_{QA}(a,b) < \beta_4 Q(a,b) + (1-\beta_4)A(a,b)$$

hold for all $a,b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/3$, $\beta_1 \geq \pi/4$, $\alpha_2 \leq 2/3$, $\beta_2 \geq (\pi-2)/[4(\sqrt{2}-1)] = 0.689\ldots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq 1/2$, $\alpha_4 \leq 1/3$, and $\beta_4 \geq [\log(1+\sqrt{2}) + \sqrt{2} - 2]/[2(\sqrt{2}-1)] = 0.356\ldots$.

Zhang *et al.* [13] presented the best possible parameters $\alpha_1, \alpha_2, \beta_1, \beta_2 \in [0,1/2]$ and $\alpha_3, \alpha_4, \beta_3, \beta_4 \in [1/2,1]$ such that the double inequalities

$$G\bigl(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a\bigr) < N_{AG}(a,b) < G\bigl(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a\bigr),$$

$$G\bigl(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a\bigr) < N_{GA}(a,b) < G\bigl(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a\bigr),$$

$$Q\bigl(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a\bigr) < N_{QA}(a,b) < Q\bigl(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a\bigr),$$

$$Q\bigl(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a\bigr) < N_{AQ}(a,b) < Q\bigl(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a\bigr)$$

hold for all $a,b > 0$ with $a \neq b$.

In [14], the authors found the greatest values α_1, α_2, α_3, α_4, α_5, α_6, α_7, α_8, and the least values β_1, β_2, β_3, β_4, β_5, β_6, β_7, β_8 such that the double inequalities

$$A^{\alpha_1}(a,b)G^{1-\alpha_1}(a,b) < N_{GA}(a,b) < A^{\beta_1}(a,b)G^{1-\beta_1}(a,b),$$
$$\frac{\alpha_2}{G(a,b)} + \frac{1-\alpha_2}{A(a,b)} < \frac{1}{N_{GA}(a,b)} < \frac{\beta_2}{G(a,b)} + \frac{1-\beta_2}{A(a,b)},$$
$$A^{\alpha_3}(a,b)G^{1-\alpha_3}(a,b) < N_{AG}(a,b) < A^{\beta_3}(a,b)G^{1-\beta_3}(a,b),$$
$$\frac{\alpha_4}{G(a,b)} + \frac{1-\alpha_4}{A(a,b)} < \frac{1}{N_{AG}(a,b)} < \frac{\beta_4}{G(a,b)} + \frac{1-\beta_4}{A(a,b)},$$
$$Q^{\alpha_5}(a,b)A^{1-\alpha_5}(a,b) < N_{AQ}(a,b) < Q^{\beta_5}(a,b)A^{1-\beta_5}(a,b),$$
$$\frac{\alpha_6}{A(a,b)} + \frac{1-\alpha_6}{Q(a,b)} < \frac{1}{N_{AQ}(a,b)} < \frac{\beta_6}{A(a,b)} + \frac{1-\beta_6}{Q(a,b)},$$
$$Q^{\alpha_7}(a,b)A^{1-\alpha_7}(a,b) < N_{QA}(a,b) < Q^{\beta_7}(a,b)A^{1-\beta_7}(a,b),$$
$$\frac{\alpha_8}{A(a,b)} + \frac{1-\alpha_8}{Q(a,b)} < \frac{1}{N_{QA}(a,b)} < \frac{\beta_8}{A(a,b)} + \frac{1-\beta_8}{Q(a,b)}$$

hold for all $a, b > 0$ with $a \neq b$.

The main purpose of this paper is to give the explicit formulas for the Neuman means N_{AH}, N_{HA}, N_{AC}, and N_{CA}, and to present the best possible upper and lower bounds for these means in terms of the combinations of harmonic, arithmetic, and contraharmonic means. Our main results are Theorems 1.1-1.3.

Theorem 1.1 *Let* $a > b > 0$, $v = (a-b)/(a+b) \in (0,1)$, $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0,\log(2+\sqrt{3}))$, *and* $s \in (0,\pi/3)$ *be the parameters such that* $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$. *Then we have*

$$N_{AH}(a,b) = \frac{1}{2}A(a,b)\left[1 + \frac{2p}{\sinh(2p)}\right], \tag{1.4}$$

$$N_{HA}(a,b) = \frac{1}{2}A(a,b)\left[\cos(q) + \frac{q}{\sin(q)}\right], \tag{1.5}$$

$$N_{CA}(a,b) = \frac{1}{2}A(a,b)\left[\cosh(r) + \frac{r}{\sinh(r)}\right], \tag{1.6}$$

$$N_{AC}(a,b) = \frac{1}{2}A(a,b)\left[1 + \frac{2s}{\sin(2s)}\right], \tag{1.7}$$

and

$$\begin{aligned} H(a,b) &< N_{AH}(a,b) < N_{HA}(a,b) < A(a,b) \\ &< N_{CA}(a,b) < N_{AC}(a,b) < C(a,b). \end{aligned} \tag{1.8}$$

Theorem 1.2 *The double inequalities*

$$\alpha_1 A(a,b) + (1-\alpha_1)H(a,b) < N_{AH}(a,b) < \beta_1 A(a,b) + (1-\beta_1)H(a,b), \tag{1.9}$$

$$\alpha_2 A(a,b) + (1-\alpha_2)H(a,b) < N_{HA}(a,b) < \beta_2 A(a,b) + (1-\beta_2)H(a,b), \tag{1.10}$$

$$\alpha_3 C(a,b) + (1-\alpha_3)A(a,b) < N_{CA}(a,b) < \beta_3 C(a,b) + (1-\beta_3)A(a,b), \tag{1.11}$$

$$\alpha_4 C(a,b) + (1-\alpha_4)A(a,b) < N_{AC}(a,b) < \beta_4 C(a,b) + (1-\beta_4)A(a,b) \tag{1.12}$$

hold for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leq 1/3$, $\beta_1 \geq 1/2$, $\alpha_2 \leq 2/3$, $\beta_2 \geq \pi/4 = 0.7853\ldots$, $\alpha_3 \leq 1/3$, $\beta_3 \geq \sqrt{3}\log(2+\sqrt{3})/6 = 0.3801\ldots$, $\alpha_4 \leq 2/3$, *and* $\beta_4 \geq (4\sqrt{3}\pi - 9)/18 = 0.7901\ldots$.

Theorem 1.3 *The double inequalities*

$$\frac{\alpha_5}{H(a,b)} + \frac{1-\alpha_5}{A(a,b)} < \frac{1}{N_{AH}(a,b)} < \frac{\beta_5}{H(a,b)} + \frac{1-\beta_5}{A(a,b)}, \tag{1.13}$$

$$\frac{\alpha_6}{H(a,b)} + \frac{1-\alpha_6}{A(a,b)} < \frac{1}{N_{HA}(a,b)} < \frac{\beta_6}{H(a,b)} + \frac{1-\beta_6}{A(a,b)}, \tag{1.14}$$

$$\frac{\alpha_7}{A(a,b)} + \frac{1-\alpha_7}{C(a,b)} < \frac{1}{N_{CA}(a,b)} < \frac{\beta_7}{A(a,b)} + \frac{1-\beta_7}{C(a,b)}, \tag{1.15}$$

$$\frac{\alpha_8}{A(a,b)} + \frac{1-\alpha_8}{C(a,b)} < \frac{1}{N_{AC}(a,b)} < \frac{\beta_8}{A(a,b)} + \frac{1-\beta_8}{C(a,b)}, \tag{1.16}$$

hold for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha_5 \leq 0$, $\beta_5 \geq 2/3$, $\alpha_6 \leq 0$, $\beta_6 \geq 1/3$, $\alpha_7 \leq [2\sqrt{3} - \log(2+\sqrt{3})]/[2\sqrt{3} + \log(2+\sqrt{3})] = 0.4490\ldots$, $\beta_7 \geq 2/3$, $\alpha_8 \leq (9\sqrt{3} - 4\pi)/(3\sqrt{3} + 4\pi) = 0.1701\ldots$, *and* $\beta_8 \geq 1/3$.

2 Lemmas

In order to prove our main results we need several lemmas, which we present in this section.

Lemma 2.1 (See [15, Theorem 1.25]) *For* $-\infty < a < b < \infty$, *let* $f, g : [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$, *and be differentiable on* (a,b), *let* $g'(x) \neq 0$ *on* (a,b). *If* $f'(x)/g'(x)$ *is increasing* (*decreasing*) *on* (a,b), *then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2 (See [16, Lemma 1.1]) *Suppose that the power series* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $g(x) = \sum_{n=0}^{\infty} b_n x^n$ *have the radius of convergence* $r > 0$ *and* $a_n, b_n > 0$ *for all* $n \geq 0$. *If the sequence* $\{a_n/b_n\}$ *is* (*strictly*) *increasing* (*decreasing*) *for all* $n \geq 0$, *then the function* $f(x)/g(x)$ *is also* (*strictly*) *increasing* (*decreasing*) *on* $(0, r)$.

Lemma 2.3 (See [12, Theorem 4.1]) *If* $a > b$, *then*

$$N(b,a) > N(a,b).$$

Lemma 2.4 *The function*

$$\varphi_1(t) = \frac{\sinh(2t) - 4\sinh(t) + 2t}{\sinh(2t) - 2\sinh(t)}$$

is strictly increasing from $(0, \infty)$ *onto* $(2/3, 1)$.

Proof Making use of power series expansion we get

$$\varphi_1(t) = \frac{\sum_{n=1}^{\infty} \frac{2^{2n+1}-4}{(2n+1)!} t^{2n+1}}{\sum_{n=1}^{\infty} \frac{2^{2n+1}-2}{(2n+1)!} t^{2n+1}} = \frac{\sum_{n=0}^{\infty} \frac{2^{2n+3}-4}{(2n+3)!} t^{2n}}{\sum_{n=0}^{\infty} \frac{2^{2n+3}-2}{(2n+3)!} t^{2n}}. \tag{2.1}$$

Let

$$a_n = \frac{2^{2n+3}-4}{(2n+3)!}, \qquad b_n = \frac{2^{2n+3}-2}{(2n+3)!}. \tag{2.2}$$

Then

$$a_n > 0, \qquad b_n > 0, \tag{2.3}$$

and $a_n/b_n = 1 - 1/(2^{2n+2}-1)$ is strictly increasing for all $n \geq 0$.

Note that

$$\varphi_1(0^+) = \frac{a_0}{b_0} = \frac{2}{3}, \qquad \varphi_1(\infty) = \lim_{n\to\infty} \frac{a_n}{b_n} = 1. \tag{2.4}$$

Therefore, Lemma 2.4 follows easily from Lemma 2.2 and (2.1)-(2.4) together with the monotonicity of the sequence $\{a_n/b_n\}$. □

Lemma 2.5 *The function*

$$\varphi_2(t) = \frac{2t - \sin(2t)}{\sin t(1-\cos t)}$$

is strictly increasing from $(0, \pi/2)$ *onto* $(8/3, \pi)$.

Proof Let $f_1(t) = 2t - \sin(2t)$ and $g_1(t) = \sin t(1-\cos t)$. Then simple computations lead to

$$\varphi_2(t) = \frac{f_1(t) - f_1(0)}{g_1(t) - g_1(0)} \tag{2.5}$$

and $f_1'(t)/g_1'(t) = 4[1 - 1/(2 + 1/\cos t)]$ is strictly increasing on $(0, \pi/2)$.

Note that

$$\varphi_2(0^+) = \lim_{t\to 0^+} \frac{f_1'(t)}{g_1'(t)} = \frac{8}{3}, \qquad \varphi_2(\pi/2) = \pi. \tag{2.6}$$

Therefore, Lemma 2.5 follows from Lemma 2.1, (2.5), (2.6), and the monotonicity of $f_1'(t)/g_1'(t)$. □

Lemma 2.6 *The function*

$$\varphi_3(t) = \frac{\sinh(t)\cosh(t) - t}{[\sinh(t)\cosh(t) + t](\cosh(t) - 1)}$$

is strictly decreasing from $(0, \infty)$ *onto* $(0, 2/3)$.

Proof Simple computations lead to

$$\begin{aligned}\varphi_3(t) &= \frac{2\sinh(2t) - 4t}{\sinh(3t) + 4t\cosh(t) + \sinh(t) - 2\sinh(2t) - 4t} \\ &= \frac{\sum_{n=0}^{\infty} \frac{2^{2n+4}}{(2n+3)!} t^{2n}}{\sum_{n=0}^{\infty} \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{(2n+3)!} t^{2n}}.\end{aligned} \tag{2.7}$$

Let

$$a_n = \frac{2^{2n+4}}{(2n+3)!}, \qquad b_n = \frac{3^{2n+3} - 2^{2n+4} + 8n + 13}{(2n+3)!}. \tag{2.8}$$

Then

$$a_n > 0, \qquad b_n > 0, \tag{2.9}$$

and

$$\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\frac{2^{2n+4}(5 \times 3^{2n+3} - 24n - 31)}{(3^{2n+5} - 2^{2n+6} + 8n + 21)(3^{2n+3} - 2^{2n+4} + 8n + 13)} < 0 \tag{2.10}$$

for all $n \geq 0$.

Note that

$$\varphi_3(0^+) = \frac{a_0}{b_0} = \frac{2}{3}, \qquad \varphi_3(\infty) = \lim_{n\to\infty} \frac{a_n}{b_n} = 0. \tag{2.11}$$

Therefore, Lemma 2.6 follows easily from (2.7)-(2.11) and Lemma 2.2. □

Lemma 2.7 *The function*

$$f(t) = 9\cos t + \frac{t}{\sin t}$$

is strictly decreasing on the interval $(0, \pi/2)$.

Proof Let $f_2(t) = 9\sin t\cos t + t$ and $g_2(t) = \sin t$. Then simple computations lead to

$$\begin{aligned} f(t) &= \frac{f_2(t) - f_2(0)}{g_2(t) - g_2(0)}, \\ \frac{f_2'(t)}{g_2'(t)} &= \frac{18\cos^2 t - 8}{\cos t}, \end{aligned} \tag{2.12}$$

and

$$\left[\frac{f_2'(t)}{g_2'(t)}\right]' = -\frac{2\sin t(9\cos^2 t + 4)}{\cos^2 t} < 0 \tag{2.13}$$

for $t \in (0, \pi/2)$.

Therefore, Lemma 2.7 follows easily from (2.12) and (2.13) together with Lemma 2.1. □

Lemma 2.8 *The function*

$$\varphi_4(t) = \frac{\sin t \cos t - t}{(t + \sin t \cos t)(1 - \cos t)}$$

is strictly decreasing from $(0, \pi/2)$ *onto* $(-1, -2/3)$.

Proof Let $f_3(t) = \sin t \cos t - t$ and $g_3(t) = (t + \sin t \cos t)(1 - \cos t)$. Then simple computations lead to

$$\varphi_4(t) = \frac{f_3(t)}{g_3(t)} = \frac{f_3(t) - f_3(0)}{g_3(t) - g_3(0)}, \tag{2.14}$$

$$\frac{f_3'(t)}{g_3'(t)} = \frac{f_3'(t) - f_3'(0)}{g_3'(t) - g_3'(0)}, \tag{2.15}$$

and

$$\frac{f_3''(t)}{g_3''(t)} = \frac{4}{4 - (9 \cos t + \frac{t}{\sin t})}. \tag{2.16}$$

Note that

$$\varphi_4(0^+) = \lim_{t \to 0^+} \frac{f_3''(t)}{g_3''(t)} = -\frac{2}{3}, \qquad \varphi_4\left(\frac{\pi}{2}\right) = -1. \tag{2.17}$$

Therefore, Lemma 2.8 follows from Lemma 2.1 and Lemma 2.7 together with (2.14)-(2.17). □

3 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1 It follows from (1.1)-(1.3) as we clearly see that

$$\begin{aligned}
N_{AH}(a,b) &= \frac{1}{2}\left[A(a,b) + \frac{H^2(a,b)}{S_{AH}(a,b)}\right] = \frac{1}{2}A(a,b)\left[1 + \left(1 - v^2\right)^2 \frac{p}{\tanh(p)}\right] \\
&= \frac{1}{2}A(a,b)\left[1 + \frac{p}{\tanh(p)\cosh^2(p)}\right] = \frac{1}{2}A(a,b)\left[1 + \frac{2p}{\sinh(2p)}\right], \\
N_{HA}(a,b) &= \frac{1}{2}\left[H(a,b) + \frac{A^2(a,b)}{S_{HA}(a,b)}\right] = \frac{1}{2}A(a,b)\left[\left(1 - v^2\right) + \frac{q}{\sin q}\right] \\
&= \frac{1}{2}A(a,b)\left[\cos q + \frac{q}{\sin q}\right], \\
N_{CA}(a,b) &= \frac{1}{2}\left[C(a,b) + \frac{A^2(a,b)}{S_{CA}(a,b)}\right] = \frac{1}{2}A(a,b)\left[\left(1 + v^2\right) + \frac{r}{\sinh(r)}\right] \\
&= \frac{1}{2}A(a,b)\left[\cosh(r) + \frac{r}{\sinh(r)}\right], \\
N_{AC}(a,b) &= \frac{1}{2}\left[A(a,b) + \frac{C^2(a,b)}{S_{AC}(a,b)}\right] = \frac{1}{2}A(a,b)\left[1 + \left(1 + v^2\right)^2 \frac{s}{\tan(s)}\right] \\
&= \frac{1}{2}A(a,b)\left[1 + \frac{s}{\tan(s)\cos^2 s}\right] = \frac{1}{2}A(a,b)\left[1 + \frac{2s}{\sin(2s)}\right].
\end{aligned}$$

Inequalities (1.8) follow easily from $H(a,b) < A(a,b) < C(a,b)$ and Lemma 2.3 together with the fact that $N_{KL}(a,b)$ is a mean of $K(a,b)$ and $L(a,b)$ for $K(a,b), L(a,b) \in \{H(a,b), A(a,b), C(a,b)\}$. □

Proof of Theorem 1.2 Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$, $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0, \log(2+\sqrt{3}))$, and $s \in (0,\pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$. Then from (1.4)-(1.7) we have

$$\begin{aligned}\frac{N_{AH}(a,b) - H(a,b)}{A(a,b) - H(a,b)} &= \frac{[1 + 2p/\sinh(2p)]/2 - (1-v^2)}{v^2}\\ &= \frac{[1 + 2p/\sinh(2p)]/2 - 1/\cosh(p)}{1 - 1/\cosh(p)} = \varphi_1(p), \end{aligned}\tag{3.1}$$

$$\begin{aligned}\frac{N_{HA}(a,b) - H(a,b)}{A(a,b) - H(a,b)} &= \frac{[\cos q + q/\sin q]/2 - (1-v^2)}{v^2}\\ &= \frac{[\cos q + q/\sin q]/2 - \cos q}{1 - \cos q} = \frac{1}{4}\varphi_2(q), \end{aligned}\tag{3.2}$$

$$\begin{aligned}\frac{N_{CA}(a,b) - A(a,b)}{C(a,b) - A(a,b)} &= \frac{[\cosh(r) + r/\sinh(r)]/2 - 1}{v^2}\\ &= \frac{[\cosh(r) + r/\sinh(r)]/2 - 1}{\cosh(r) - 1} = \frac{1}{2}\varphi_1(r), \end{aligned}\tag{3.3}$$

$$\begin{aligned}\frac{N_{AC}(a,b) - A(a,b)}{C(a,b) - A(a,b)} &= \frac{[1 + 2s/\sin(2s)]/2 - 1}{v^2}\\ &= \frac{[1 + 2s/\sin(2s)]/2 - 1}{\sec(s) - 1} = \frac{1}{4}\varphi_2(s), \end{aligned}\tag{3.4}$$

where the functions φ_1 and φ_2 are defined as in Lemmas 2.4 and 2.5, respectively.

Note that

$$\varphi_1\big[\log(2+\sqrt{3})\big] = \sqrt{3}\log(2+\sqrt{3})/6 \tag{3.5}$$

and

$$\varphi_2\left(\frac{\pi}{3}\right) = \frac{8\sqrt{3}\pi - 18}{9}. \tag{3.6}$$

Therefore, inequality (1.9) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 1/3$ and $\beta_1 \geq 1/2$ follows from (3.1) and Lemma 2.4, inequality (1.10) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 2/3$ and $\beta_2 \geq \pi/4$ follows from (3.2) and Lemma 2.5, inequality (1.11) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 1/3$ and $\beta_3 \geq \sqrt{3}\log(2+\sqrt{3})/6$ follows from (3.3) and (3.5) together with Lemma 2.4, and inequality (1.12) holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 2/3$ and $\beta_4 \geq (4\sqrt{3}\pi - 9)/18$ follows from (3.4) and (3.6) together with Lemma 2.5. □

Proof of Theorem 1.3 Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b) \in (0,1)$, $p \in (0,\infty)$, $q \in (0,\pi/2)$, $r \in (0, \log(2+\sqrt{3}))$, and $s \in (0,\pi/3)$ be the parameters such that $1/\cosh(p) = \cos(q) = 1 - v^2$, $\cosh(r) = \sec(s) = 1 + v^2$. Then from (1.4)-(1.7) we have

$$\frac{1/N_{AH}(a,b)-1/A(a,b)}{1/H(a,b)-1/A(a,b)}=\frac{\frac{2}{1+2p/\sinh(2p)}-1}{\frac{1}{1-v^2}-1}=\frac{\frac{2\sinh(2p)}{2p+\sinh(2p)}-1}{\cosh(p)-1}=\varphi_3(p), \tag{3.7}$$

$$\frac{1/N_{HA}(a,b)-1/A(a,b)}{1/H(a,b)-1/A(a,b)}=\frac{\frac{2}{\cos(q)+q/\sin(q)}-1}{\frac{1}{1-v^2}-1}=\frac{\frac{2\sin(q)-\sin(q)\cos(q)-q}{\sin(q)\cos(q)+q}}{\frac{1-\cos(q)}{\cos(q)}}=1+\varphi_4(q), \tag{3.8}$$

$$\frac{1/N_{CA}(a,b)-1/C(a,b)}{1/A(a,b)-1/C(a,b)}=\frac{\frac{2}{\cosh(r)+r/\sinh(r)}-\frac{1}{1+v^2}}{1-\frac{1}{1+v^2}}=\varphi_3(r), \tag{3.9}$$

and

$$\frac{1/N_{AC}(a,b)-1/C(a,b)}{1/A(a,b)-1/C(a,b)}=1+\varphi_4(s), \tag{3.10}$$

where the functions φ_3 and φ_4 are defined as in Lemmas 2.6 and 2.8, respectively.

Note that

$$\varphi_3\bigl[\log(2+\sqrt{3})\bigr]=\frac{2\sqrt{3}-\log(2+\sqrt{3})}{2\sqrt{3}+\log(2+\sqrt{3})} \tag{3.11}$$

and

$$\varphi_4\left(\frac{\pi}{3}\right)=-\frac{8\pi-6\sqrt{3}}{4\pi+3\sqrt{3}}. \tag{3.12}$$

Therefore, Theorem 1.3 follows easily from (3.7)-(3.12) together with Lemmas 2.6 and 2.8. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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