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Continued fraction inequalities for the Euler-Mascheroni constant

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Abstract
The aim of this paper is to establish new inequalities for the Euler-Mascheroni constant by the continued fraction method.
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1 Introduction

The Euler-Mascheroni constant was first introduced by Leonhard Euler (1707-1783) in 1734 as the limit of the sequence

$$\gamma(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n. \tag{1.1}$$

There are many famous unsolved problems about the nature of this constant (see, *e.g.*, the survey papers or books of Brent and Zimmermann [1], Dence and Dence [2], Havil [3] and Lagarias [4]). For example, it is a long-standing open problem if the Euler-Mascheroni constant is a rational number. A good part of its mystery comes from the fact that the known algorithms converging to γ are not very fast, at least, when they are compared to similar algorithms for π and e.

The sequence $(\gamma(n))_{n\in\mathbb{N}}$ converges very slowly toward γ, like $(2n)^{-1}$. Up to now, many authors have been preoccupied with improving its rate of convergence (see, *e.g.*, [2, 5–22] and the references therein). We list some main results as follows:

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(n + \frac{1}{2}\right) = \gamma + O\left(n^{-2}\right) \quad \text{(DeTemple [6])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \frac{n^3 + \frac{3}{2}n^2 + \frac{227}{240} + \frac{107}{480}}{n^2 + n + \frac{97}{240}} = \gamma + O\left(n^{-6}\right) \quad \text{(Mortici [13])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5{,}760n^4}\right)$$
$$= \gamma + O\left(n^{-5}\right) \quad \text{(Chen and Mortici [5])}.$$

Recently, Mortici and Chen [14] provided a very interesting sequence,

$$\begin{aligned}\nu(n) &= \sum_{m=1}^{n}\frac{1}{m} - \frac{1}{2}\ln\left(n^2+n+\frac{1}{3}\right)\\ &\quad - \left(\frac{-\frac{1}{180}}{(n^2+n+\frac{1}{3})^2} + \frac{\frac{8}{2{,}835}}{(n^2+n+\frac{1}{3})^3} + + \frac{\frac{5}{1{,}512}}{(n^2+n+\frac{1}{3})^4} + \frac{\frac{592}{93{,}555}}{(n^2+n+\frac{1}{3})^5}\right),\end{aligned}$$

and proved

$$\lim_{n\to\infty} n^{12}\big(\nu(n)-\gamma\big) = -\frac{796{,}801}{43{,}783{,}740}. \tag{1.2}$$

Hence the rate of convergence of the sequence $(\nu(n))_{n\in\mathbb{N}}$ is n^{-12}.

Very recently, by inserting the continued fraction term in (1.1), Lu [9] introduced a class of sequences $(r_k(n))_{n\in\mathbb{N}}$ (see Theorem 1) and showed

$$\frac{1}{72(n+1)^3} < \gamma - r_2(n) < \frac{1}{72n^3}, \tag{1.3}$$

$$\frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}. \tag{1.4}$$

In fact, Lu [9] also found a_4 without proof. In general, the continued fraction method could provide a better approximation than others, and has less numerical computations.

First, we will prove the following theorem.

Theorem 1 *For the Euler-Mascheroni constant, we have the following convergent sequence*:

$$r(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \ddots}}},$$

where $(a_1,a_2,a_4,a_6,a_8,a_{10},a_{12}) = (\frac{1}{2},\frac{1}{6},\frac{3}{5},\frac{79}{126},\frac{7{,}230}{6{,}241},\frac{4{,}146{,}631}{3{,}833{,}346},\frac{306{,}232{,}774{,}533}{179{,}081{,}182{,}865})$, *and* $a_{2k+1} = -a_{2k}$ *for* $1 \le k \le 6$.

Let

$$R_k(n) := \cfrac{a_1}{n + \cfrac{a_2 n}{n + \cfrac{a_3 n}{n + \cfrac{a_4 n}{\ddots \, \frac{}{n + a_k}}}}}$$

(*see the Appendix for their simple expressions*) *and*

$$r_k(n) := \sum_{m=1}^{n}\frac{1}{m} - \ln n - R_k(n).$$

For $1 \le k \le 13$, *we have*

$$\lim_{n\to\infty} n^{k+1}\big(r_k(n)-\gamma\big) = C_k, \tag{1.5}$$

where

$$(C_1,\dots,C_{13}) = \left(-\frac{1}{12},-\frac{1}{72},\frac{1}{120},\frac{1}{200},-\frac{79}{25{,}200},-\frac{6{,}241}{3{,}175{,}200},\frac{241}{105{,}840},\frac{58{,}081}{22{,}018{,}248},\right.$$
$$-\frac{262{,}445}{91{,}974{,}960},-\frac{2{,}755{,}095{,}121}{892{,}586{,}949{,}408},\frac{20{,}169{,}451}{3{,}821{,}257{,}440},$$
$$\left.\frac{406{,}806{,}753{,}641{,}401}{45{,}071{,}152{,}103{,}463{,}200},-\frac{71{,}521{,}421{,}431}{5{,}152{,}068{,}292{,}800}\right).$$

Open problem For every $k \ge 1$, we have $a_{2k+1} = -a_{2k}$.

The main aim of this paper is to improve (1.3) and (1.4). We establish the following more precise inequalities.

Theorem 2 *Let* $r_{10}(n)$, $r_{11}(n)$, C_{10} *and* C_{11} *be defined in Theorem* 1, *then*

$$C_{10}\frac{1}{(n+1)^{11}} < \gamma - r_{10}(n) < C_{10}\frac{1}{n^{11}}, \tag{1.6}$$

$$C_{11}\frac{1}{(n+1)^{12}} < r_{11}(n) - \gamma < C_{11}\frac{1}{n^{12}}. \tag{1.7}$$

Remark 1 In fact, Theorem 2 implies that $r_{10}(n)$ is a strictly increasing function of n, whereas $r_{11}(n)$ is a strictly decreasing function of n. Certainly, it has similar inequalities for $r_k(n)$ $(1 \le k \le 9)$, we leave these for readers to verify. It is also should be noted that (1.4) cannot deduce the monotonicity of $r_3(n)$.

Remark 2 It is worth to point out that Theorem 2 provides sharp bounds for a harmonic sequence which are superior to Theorems 3 and 4 of Mortici and Chen [14].

2 The proof of Theorem 1

The following lemma gives a method for measuring the rate of convergence. This lemma was first used by Mortici [23, 24] for constructing asymptotic expansions or to accelerate some convergences. For proof and other details, see, *e.g.*, [24].

Lemma 1 *If the sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty,+\infty] \tag{2.1}$$

with $s > 1$, *then there exists the limit*

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.2}$$

In the sequel, we always assume $n \ge 2$.

We need to find the value $a_1 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$r_1(n) = \sum_{m=1}^{n}\frac{1}{m} - \ln n - \frac{a_1}{n}, \tag{2.3}$$

here we note $R_1(n) = a_1/n$. To measure the accuracy of this approximation, we usually say that approximation (2.3) is better as $r_1(n) - \gamma$ faster converges to zero. Clearly,

$$r_1(n) - r_1(n+1) = \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} + \frac{a_1}{n+1} - \frac{a_1}{n}. \tag{2.4}$$

It is well known that for $|x| < 1$,

$$\ln(1+x) = \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^m}{m} \quad \text{and} \quad \frac{1}{1-x} = \sum_{m=0}^{\infty} x^m.$$

Developing expression (2.4) into power series expansion in $1/n$, we obtain

$$r_1(n) - r_1(n+1) = \left(\frac{1}{2} - a_1\right)\frac{1}{n^2} + \left(a_1 - \frac{2}{3}\right)\frac{1}{n^3} + \left(\frac{3}{4} - a_1\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{2.5}$$

From Lemma 1, we see that the rate of convergence of the sequence $(r_1(n) - \gamma)_{n\in\mathbb{N}}$ is even higher than the value s satisfying (2.1). By Lemma 1, we have

(i) If $a_1 \neq \frac{1}{2}$, then the rate of convergence of $(r_1(n) - \gamma)_{n\in\mathbb{N}}$ is n^{-1} since

$$\lim_{n\to\infty} n\bigl(r_1(n) - \gamma\bigr) = \frac{1}{2} - a_1 \neq 0.$$

(ii) If $a_1 = \frac{1}{2}$, from (2.5) we have

$$r_1(n) - r_1(n+1) = -\frac{1}{6}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Hence the rate of convergence of $(r_1(n) - \gamma)_{n\in\mathbb{N}}$ is n^{-2} since

$$\lim_{n\to\infty} n^2\bigl(r_1(n) - \gamma\bigr) = -\frac{1}{12}.$$

We also observe that the fastest possible sequence $(r_1(n))_{n\in\mathbb{N}}$ is obtained only for $a_1 = \frac{1}{2}$.

Just as Lu [9] did, we may repeat the above approach to determine a_1 to a_4 step by step. However, the computations become very difficult when $k \geq 5$. In this paper we use *Mathematica* software to manipulate symbolic computations.

Let

$$r_k(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - R_k(n), \tag{2.6}$$

then

$$r_k(n) - r_k(n+1) = \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} + R_k(n+1) - R_k(n). \tag{2.7}$$

It is easy to get the following power series:

$$\ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} = \sum_{m=2}^{\infty} (-1)^m \frac{m-1}{m} \frac{1}{n^m}. \tag{2.8}$$

Hence the key step is to expand $R_k(n+1) - R_k(n)$ into power series in $\frac{1}{n}$. Here we use some examples to explain our method.

Step 1: For example, given a_1 to a_7, find a_8. Define

$$R_8(n) = \cfrac{\frac{1}{2}}{n + \cfrac{\frac{n}{6}}{n + \cfrac{-\frac{n}{6}}{n + \cfrac{\frac{3}{5}n}{n + \cfrac{\frac{-3}{5}n}{n + \cfrac{\frac{79}{126}n}{n + \cfrac{\frac{-79}{126}n}{n+a_8}}}}}}}$$

$$= \frac{-237 + 1{,}405a_8 + 1{,}800n + 1{,}740a_8n - 630n^2 + 3{,}780a_8n^2 + 3{,}780n^3}{6(79a_8 + 600a_8n + 600n^2 + 790a_8n^2 + 1{,}260a_8n^3 + 1{,}260n^4)}. \tag{2.9}$$

By using *Mathematica* software (*Mathematica Program* is very similar to the one given in Remark 3; however, it has a parameter a_8), we obtain

$$\begin{aligned} &R_8(n+1) - R_8(n) \\ &\quad = -\frac{1}{2n^2} + \frac{2}{3n^3} - \frac{3}{4n^4} + \frac{4}{5n^5} - \frac{5}{6n^6} + \frac{6}{7n^7} - \frac{7}{8n^8} \\ &\qquad + \frac{360{,}030 - 6{,}241a_8}{396{,}900n^9} + \frac{-346{,}440 + 24{,}964a_8 + 6{,}241a_8^2}{352{,}800n^{10}} + O\left(\frac{1}{n^{11}}\right). \end{aligned} \tag{2.10}$$

Substituting (2.8) and (2.10) into (2.7), we get

$$\begin{aligned} r_8(n) - r_8(n+1) = {}& \left(-\frac{8}{9} + \frac{360{,}030 - 6{,}241a_8}{396{,}900}\right)\frac{1}{n^9} \\ &+ \left(\frac{9}{10} + \frac{-346{,}440 + 24{,}964a_8 + 6{,}241a_8^2}{352{,}800}\right)\frac{1}{n^{10}} + O\left(\frac{1}{n^{11}}\right). \end{aligned} \tag{2.11}$$

The fastest possible sequence $(r_8(n))_{n\in\mathbb{N}}$ is obtained only for $a_8 = \frac{7{,}230}{6{,}241}$. At the same time, it follows from (2.11) that

$$r_8(n) - r_8(n+1) = \frac{58{,}081}{2{,}446{,}472}\frac{1}{n^{10}} + O\left(\frac{1}{n^{11}}\right), \tag{2.12}$$

the rate of convergence of $(r_8(n) - \gamma)_{n\in\mathbb{N}}$ is n^{-9} since

$$\lim_{n\to\infty} n^9\bigl(r_8(n) - \gamma\bigr) = -\frac{58{,}081}{22{,}018{,}248}.$$

We can use the above approach to find a_k $(3 \le k \le 8)$. Unfortunately, it does not work well for a_9. Since $a_3 = -a_2$, $a_5 = -a_4$ and $a_7 = -a_6$. So, we may conjecture $a_9 = -a_8$. Now let us check it carefully.

Step 2: Check $a_9 = -\frac{7{,}230}{6{,}241}$ to $a_{13} = -\frac{306{,}232{,}774{,}533}{179{,}081{,}182{,}865}$.

Let $a_1, \ldots, a_9$ and $R_9(n)$ be defined in Theorem 1. Applying *Mathematica* software, we obtain

$$\begin{aligned} R_9(n+1) - R_9(n) = {}& -\frac{1}{2n^2} + \frac{2}{3n^3} - \frac{3}{4n^4} + \frac{4}{5n^5} - \frac{5}{6n^6} + \frac{6}{7n^7} - \frac{7}{8n^8} + \frac{8}{9}\frac{1}{n^9} \\ &- \frac{9}{10}\frac{1}{n^{10}} + \frac{736{,}265}{836{,}136}\frac{1}{n^{11}} + O\left(\frac{1}{n^{12}}\right), \end{aligned} \tag{2.13}$$

which is the desired result. Substituting (2.8) and (2.13) into (2.7), we get

$$r_9(n)-r_9(n+1)=-\frac{262{,}445}{9{,}197{,}496}\frac{1}{n^{11}}+O\left(\frac{1}{n^{12}}\right), \tag{2.14}$$

the rate of convergence of $(r_9(n)-\gamma)_{n\in\mathbb{N}}$ is n^{-10} since

$$\lim_{n\to\infty} n^{10}\left(r_9(n)-\gamma\right)=-\frac{262{,}445}{91{,}974{,}960}.$$

Next, we can use Step 1 to find a_{10}, and Step 2 to check a_{11} and a_{12}. It should be noted that Theorem 2 will provide the other proofs for a_{10} and a_{11}. So we omit the details here.

Finally, we check $a_{13}=-\frac{306{,}232{,}774{,}533}{179{,}081{,}182{,}865}$.

$$\begin{aligned}
&R_{13}(n+1)-R_{13}(n)\\
&\quad=-\frac{1}{2n^2}+\frac{2}{3n^3}-\frac{3}{4n^4}+\frac{4}{5n^5}-\frac{5}{6n^6}+\frac{6}{7n^7}-\frac{7}{8n^8}+\frac{8}{9}\frac{1}{n^9}\\
&\qquad-\frac{9}{10}\frac{1}{n^{10}}+\frac{10}{11}\frac{1}{n^{11}}-\frac{11}{12}\frac{1}{n^{12}}+\frac{12}{13}\frac{1}{n^{13}}-\frac{13}{14}\frac{1}{n^{14}}\\
&\qquad+\frac{1{,}903{,}648{,}586{,}623}{2{,}576{,}034{,}146{,}400}\frac{1}{n^{15}}+O\left(\frac{1}{n^{16}}\right).
\end{aligned} \tag{2.15}$$

Substituting (2.8) and (2.15) into (2.7), one has

$$r_{13}(n)-r_{13}(n+1)=-\frac{500{,}649{,}950{,}017}{2{,}576{,}034{,}146{,}400}\frac{1}{n^{15}}+O\left(\frac{1}{n^{16}}\right). \tag{2.16}$$

Since

$$\lim_{n\to\infty} n^{14}\left(r_{13}(n)-\gamma\right)=-\frac{71{,}521{,}421{,}431}{5{,}152{,}068{,}292{,}800},$$

thus the rate of convergence of $(r_{13}(n)-\gamma)_{n\in\mathbb{N}}$ is n^{-14}.

This completes the proof of Theorem 1.

Remark 3 In fact, if the assertion $a_{13}=-\frac{306{,}232{,}774{,}533}{179{,}081{,}182{,}865}$ holds, then the other values a_j $(1\le j\le 12)$ must be true. The following *Mathematica Program* will generate $R_{13}(n+1)-R_{13}(n)$ into power series in $\frac{1}{n}$ with order 16: Normal[Series[$(R_{13}[n+1]-R_{13}[n])$/. $n\to 1/x$, $\{x,0,16\}$]]/. $x\to 1/n$.

Remark 4 It is a very interesting question to find a_k for $k\ge 14$. However, it seems impossible by the above method.

3 The proof of Theorem 2

Before we prove Theorem 2, let us give a simple inequality by the Hermite-Hadamard inequality, which plays an important role in the proof.

Lemma 2 *Let f be twice derivable with f'' continuous. If $f''(x)>0$, then*

$$\int_a^{a+1} f(x)\,dx>f(a+1/2). \tag{3.1}$$

In the sequel, the notation $P_k(x)$ means a polynomial of degree k in x with all of its non-zero coefficients positive, which may be different at each occurrence.

Let us begin to prove Theorem 2. Note $r_{10}(\infty) = 0$, it is easy to see

$$\gamma - r_{10}(n) = \sum_{m=n}^{\infty}\bigl(r_{10}(m+1) - r_{10}(m)\bigr) = \sum_{m=n}^{\infty} f(m), \tag{3.2}$$

where

$$f(m) = \frac{1}{m+1} - \ln\left(1 + \frac{1}{m}\right) - R_{10}(m+1) + R_{10}(m).$$

Let $D_1 = \frac{2{,}755{,}095{,}121}{6{,}762{,}022{,}344}$. By using *Mathematica* software, we have

$$f'(x) + D_1\frac{1}{(x+1)^{13}} = -\frac{P_{19}(x)(x-1) + 1{,}619{,}906{,}998{,}377 \cdots 5{,}270{,}931}{33{,}810{,}111{,}720x(1+x)^{13}P_{10}^{(1)}(x)P_{10}^{(2)}(x)} < 0,$$

and

$$f'(x) + D_1\frac{1}{(x+\frac{1}{2})^{13}} = \frac{P_{22}(x)}{4{,}226{,}263{,}965x(1+x)^2(1+2x)^{13}P_{10}^{(3)}(x)P_{10}^{(4)}(x)} > 0.$$

Hence, we get the following inequalities for $x \geq 1$:

$$D_1\frac{1}{(x+1)^{13}} < -f'(x) < D_1\frac{1}{(x+\frac{1}{2})^{13}}. \tag{3.3}$$

Applying $f(\infty) = 0$, (3.3) and Lemma 2, we get

$$\begin{aligned} f(m) &= -\int_m^{\infty} f'(x)\,dx \leq D_1\int_m^{\infty}\left(x + \frac{1}{2}\right)^{-13}dx \\ &= \frac{D_1}{12}\left(m + \frac{1}{2}\right)^{-12} \leq \frac{D_1}{12}\int_m^{m+1} x^{-12}\,dx. \end{aligned} \tag{3.4}$$

From (3.1) and (3.4) we obtain

$$\begin{aligned} \gamma - r_{10}(n) &\leq \sum_{m=n}^{\infty}\frac{D_1}{12}\int_m^{m+1} x^{-12}\,dx \\ &= \frac{D_1}{12}\int_n^{\infty} x^{-12}\,dx = \frac{D_1}{132}\frac{1}{n^{11}}. \end{aligned} \tag{3.5}$$

Similarly, we also have

$$\begin{aligned} f(m) &= -\int_m^{\infty} f'(x)\,dx \geq D_1\int_m^{\infty}(x+1)^{-13}\,dx \\ &= \frac{D_1}{12}(m+1)^{-12} \geq \frac{D_1}{12}\int_{m+1}^{m+2} x^{-12}\,dx \end{aligned}$$

and

$$\begin{aligned}\gamma - r_{10}(n) &\geq \sum_{m=n}^{\infty} \frac{D_1}{12} \int_{m+1}^{m+2} x^{-12}\,dx \\ &= \frac{D_1}{12} \int_{n+1}^{\infty} x^{-12}\,dx = \frac{D_1}{132} \frac{1}{(n+1)^{11}}. \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) completes the proof of (1.6).

Note $r_{11}(\infty) = 0$, it is easy to deduce

$$r_{11}(n) - \gamma = \sum_{m=n}^{\infty} \big(r_{11}(m) - r_{11}(m+1)\big) = \sum_{m=n}^{\infty} g(m), \tag{3.7}$$

where

$$g(m) = \ln\left(1 + \frac{1}{m}\right) - \frac{1}{m+1} - R_{11}(m) + R_{11}(m+1).$$

We write $D_2 = \frac{20{,}169{,}451}{24{,}495{,}240}$. By using *Mathematica* software, we have

$$-g'(x) - D_2 \frac{1}{(x+1)^{14}} = \frac{P_{18}(x)}{24{,}495{,}240x^3(1+x)^{14}P_8^{(1)}(x)P_8^{(2)}(x)} > 0$$

and

$$\begin{aligned}-g'(x) - D_2 \frac{1}{(x+\frac{1}{2})^{14}} &= -\frac{P_{19}(x)(x-1) + 4{,}622{,}005{,}677{,}839{,}353{,}997{,}724{,}676{,}307{,}741}{6{,}123{,}810x^3(1+x)^3(1+2x)^{14}P_8^{(3)}(x)P_8^{(4)}(x)} \\ &< 0.\end{aligned}$$

Hence, for $x \geq 1$,

$$D_2 \frac{1}{(x+1)^{14}} < -g'(x) < D_2 \frac{1}{(x+\frac{1}{2})^{14}}. \tag{3.8}$$

Applying $g(\infty) = 0$, (3.8) and (3.1), we get

$$\begin{aligned}g(m) &= -\int_m^{\infty} g'(x)\,dx \leq D_2 \int_m^{\infty} \left(x + \frac{1}{2}\right)^{-14} dx \\ &= \frac{D_2}{13} \left(m + \frac{1}{2}\right)^{-13} \leq \frac{D_2}{13} \int_m^{m+1} x^{-13}\,dx.\end{aligned} \tag{3.9}$$

It follows from (3.7) and (3.9) that

$$\begin{aligned}r_{11}(n) - \gamma &\leq \sum_{m=n}^{\infty} \frac{D_2}{13} \int_m^{m+1} x^{-13}\,dx \\ &= \frac{D_2}{13} \int_n^{\infty} x^{-13}\,dx = \frac{D_2}{156} \frac{1}{n^{12}}.\end{aligned} \tag{3.10}$$

Finally,

$$\begin{aligned} g(m) &= -\int_m^\infty g'(x)\,dx \ge D_2 \int_m^\infty (x+1)^{-14}\,dx \\ &= \frac{D_2}{13}(m+1)^{-13} \ge \frac{D_2}{13}\int_{m+1}^{m+2} x^{-13}\,dx \end{aligned}$$

and

$$\begin{aligned} r_{11}(n) - \gamma &\ge \sum_{m=n}^\infty \frac{D_2}{13}\int_{m+1}^{m+2} x^{-13}\,dx \\ &= \frac{D_2}{13}\int_{n+1}^\infty x^{-13}\,dx = \frac{D_2}{156}\frac{1}{(n+1)^{12}}. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11) completes the proof of (1.7).

Remark 5 As an example, we give *Mathematica Program* for the proof of the left-hand side of (3.3):

(i) Together $[D[f[x], \{x, 1\}] + D_1(x+1)^{13}]$;

(ii) Take out the numerator $P[x]$ of the above rational function, then manipulate the program: Apart $[P[x]/(x-1)]$.

Appendix

For the reader's convenience, we rewrite $R_k(n)$ ($k \le 13$) with minimal denominators as follows.

$$\begin{aligned}
R_1(n) &= \frac{1}{2n}, \\
R_3(n) &= \frac{1}{2n} - \frac{1}{12}\frac{1}{n^2}, \\
R_5(n) &= \frac{1}{2n} - \frac{5}{6(1+10n^2)}, \\
R_7(n) &= \frac{1}{2n} - \frac{79}{1{,}200}\frac{1}{n^2} - \frac{147}{400(10+21n^2)}, \\
R_9(n) &= \frac{1}{2n} - \frac{7(871+790n^2)}{20(241+3{,}990n^2+3{,}318n^4)}, \\
R_{11}(n) &= \frac{1}{2n} - \frac{52{,}489}{894{,}348}\frac{1}{n^2} - \frac{1{,}237{,}227{,}621+584{,}280{,}400n^2}{4{,}471{,}740(3{,}549+13{,}020n^2+5{,}302n^4)}, \\
R_{13}(n) &= \frac{1}{2n} - \frac{39{,}577{,}260{,}671+66{,}288{,}226{,}620n^2+15{,}762{,}446{,}700n^4}{1{,}260(20{,}169{,}451+434{,}410{,}620n^2+646{,}328{,}298n^4+150{,}118{,}540n^6)}, \\
R_2(n) &= \frac{3}{6n+1}, \\
R_4(n) &= \frac{13+30n}{6(1+6n+10n^2)}, \\
R_6(n) &= \frac{5(281+348n+756n^2)}{6(79+600n+790n^2+1{,}260n^3)},
\end{aligned}$$

$$R_8(n) = \frac{964{,}337 + 2{,}646{,}000n + 2{,}599{,}730n^2 + 2{,}621{,}220n^3}{20(19{,}039 + 144{,}600n + 315{,}210n^2 + 303{,}660n^3 + 262{,}122n^4)},$$

$$\begin{aligned} R_{10}(n) = {} & \bigl(7\bigl(108{,}237{,}701 + 208{,}886{,}046n + 523{,}341{,}290n^2 \\ & + 210{,}464{,}400n^3 + 230{,}000{,}760n^4\bigr)\bigr) \\ & /\bigl(20\bigl(12{,}649{,}849 + 107{,}768{,}934n + 209{,}431{,}110n^2 \\ & + 395{,}365{,}320n^3 + 174{,}158{,}502n^4 + 161{,}000{,}532n^5\bigr)\bigr), \end{aligned}$$

$$\begin{aligned} R_{12}(n) = {} & \bigl(3{,}604{,}759{,}235{,}968{,}501 + 11{,}032{,}319{,}618{,}513{,}046n \\ & + 17{,}366{,}281{,}558{,}290{,}420n^2 + 19{,}958{,}033{,}982{,}902{,}400n^3 \\ & + 7{,}661{,}417{,}445{,}218{,}460n^4 + 4{,}964{,}130{,}389{,}017{,}800n^5\bigr) \\ & /\bigl(1{,}260\bigl(1{,}058{,}674{,}313{,}539 + 9{,}019{,}254{,}081{,}474n \\ & + 22{,}801{,}779{,}033{,}180n^2 + 33{,}088{,}387{,}754{,}520n^3 + 33{,}925{,}126{,}033{,}722n^4 \\ & + 13{,}474{,}242{,}079{,}452n^5 + 7{,}879{,}572{,}046{,}060n^6\bigr)\bigr). \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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